Boundary problems for fractional Laplacians, and Hörmander's μ-transmission property

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1. The fractional Laplacian

The fractional Laplacian $(-\Delta)^a$, $0 < a < 1$, is currently of great interest in probability, finance, mathematical physics and differential geometry. It is a pseudodifferential operator (ψdo) of order $2a$:

$$(-\Delta)^a u = \mathrm{Op}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)), \quad \hat{u}(\xi) = \mathcal{F}u = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx.$$

An example is the Dirichlet-to-Neumann operator for the Laplacian on $\mathbb{R}^n \times \mathbb{R}_+$, equal to $c(-\Delta)^{\frac{1}{2}}$.

Strangely, ψdo methods have not been used much for these operators recently. Caffarelli and Silvestre CPDE '07 showed that $(-\Delta)^a$ is the Dirichlet-to-Neumann operator for the degenerate elliptic equation

$$\nabla_{x,y} \cdot (y^a \nabla_{x,y} v(x,y)) = 0 \text{ on } \mathbb{R}^n \times \mathbb{R}_+,$$

with Dirichlet data $v(x,0)$ and Neumann data $y^a \partial_y v(x,y)|_{y=0}$; this allows an analysis via local (differential) operators. Also integral operator methods and potential theory are used. Moreover, nonlinear questions involving $(-\Delta)^a$ are studied.

Let $\Omega \subset \mathbb{R}^n$, smooth bounded. Definitions on Ω are not obvious, since $(-\Delta)^a$ is nonlocal. We shall use the notation (omitting p when $p = 2$)

$$H_p^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\},$$
$$\dot{H}_p^s(\overline{\Omega}) = \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}, \quad \overline{H}_p^s(\Omega) = r^+ H_p^s(\mathbb{R}^n).$$

Here $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$; r^+ restricts to Ω, e^+ extends by zero on $\complement\Omega$. (The notation with $\dot{H}$ and $\overline{H}$ stems from Hörmander's books '63 and '85.)

There are several ways to define a homogeneous Dirichlet problem for $(-\Delta)^a$ on Ω:

One way is to consider the operator $r^+(-\Delta)^a|_{C_0^\infty(\Omega)}$ in $L_2(\Omega)$. It is positive symmetric, and we can take its Friedrichs extension $(-\Delta)^a_{\mathrm{Dir}}$, positive selfadjoint in $L_2(\Omega)$. Since the domain of the associated sesquilinear form equals $\dot{H}^a(\overline{\Omega})$, the domain of the Friedrichs extension is

$$D((-\Delta)^a_{\mathrm{Dir}}) = \{u \in \dot{H}^a(\overline{\Omega}) \mid r^+(-\Delta)^a u \in L_2(\Omega)\}.$$

This is called the *restricted* Dirichlet fractional Laplacian.

Another choice is the *spectral* Dirichlet fractional Laplacian $(-\Delta_{\mathrm{Dir}})^a$ defined by spectral theory in $L_2(\Omega)$; here the domain is the interpolation space between $\overline{H}^2(\Omega) \cap \dot{H}^1(\overline{\Omega})$ and $L_2(\Omega)$. This is a subset of $\overline{H}^{2a}(\Omega)$, well-known from Lions-Magenes '68, Seeley '71 and '72.

From now on we consider the restricted fractional Laplacian.

It is easy to see that $D((-\Delta)^a_{\mathrm{Dir}}) = \dot{H}^{2a}(\overline{\Omega})$ for $a < \frac{1}{2}$, but for $a \geq \frac{1}{2}$, the domain has not been precisely described until recently. It equals a space $H^{a(2a)}(\overline{\Omega})$ introduced by Hörmander, *not* a subset of $\overline{H}^{2a}(\overline{\Omega})$ for $a \geq \frac{1}{2}$.

Let $(-\Delta)^a_{\mathrm{Dir}} u = f$. Ros-Oton and Serra (arXiv 2012) showed, when Ω is $C^{1,1}$, that

$$f \in L_\infty(\Omega) \implies u \in d^a C^\alpha(\overline{\Omega}), \quad d(x) = \operatorname{dist}(x, \partial\Omega),$$

for some $\alpha > 0$. (With a slight improvement on α if f is more smooth.) They stated that they did not know of other results on boundary regularity for $(-\Delta)^a_{\mathrm{Dir}}$ in the literature.

However, this can be much improved by a method developed from a lecture note of Hörmander at IAS Princeton '65. In fact we can show (G arXiv '13 and '14)

$$f \in L_\infty(\Omega) \implies u \in d^a C^a(\overline{\Omega}), \quad a \neq \tfrac{1}{2}, \tag{1}$$

$$f \in C^t(\overline{\Omega}) \implies u \in d^a C^{a+t}(\overline{\Omega}), \text{ for } t > 0,\ a+t, 2a+t \notin \mathbb{N}. \tag{2}$$

In the exceptional cases, ε is subtracted from the Hölder exponent.

2. Pseudodifferential methods

Recall that a function $p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ is called a *classical symbol* when it is expanded in a series of functions $p_j(x,\xi)$ homogeneous of degree $m-j$ in ξ for $|\xi| \geq 1$ (here the order $m \in \mathbb{C}$).
It defines the operator $P = P(x,D) = \mathrm{Op}(p(x,\xi))$ by

$$(Pu)(x) = \mathcal{F}^{-1}(p(x,\xi)\hat{u}(\xi)).$$

P is elliptic when the principal symbol p_0 is invertible.

Boutet de Monvel '66, '71, introduced a calculus of boundary value problems for ψdo's on $\Omega \subset \mathbb{R}^n$, when P has the so-called *transmission property*, and m is integer (this excludes $(-\Delta)^a$).

Hörmander presented in the lecture note '65 and in his book '85 (with different notation):

Definition 1. *Let* $d(x) > 0$ *on* Ω, $d \in C^\infty(\overline{\Omega})$, *proportional to* $\mathrm{dist}(x, \partial\Omega)$ *near* $\partial\Omega$.
For $\mathrm{Re}\,\mu > -1$, $\mathcal{E}_\mu(\overline{\Omega})$ *consists of the functions* u *of the form*

$$u(x) = \begin{cases} d(x)^\mu v(x) \text{ for } x \in \Omega, \text{ with } v \in C^\infty(\overline{\Omega}), \\ 0 \text{ for } x \in \complement\Omega. \end{cases}$$

Generalized to $\operatorname{Re}\mu \le -1$ by taking distribution derivatives.

Definition 2. *A classical ψdo of order $m \in \mathbb{C}$ is said to have the* μ-transmission property *at $\partial\Omega$ (for short: to be* of type μ*), when*

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -N) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, N),$$

for all indices; here $x \in \partial\Omega$ and N denotes the interior normal at x.

It is a kind of twisted symmetry condition on the normal N to Ω. Boutet de Monvel's transmission property is the case $\mu = 0$, $m \in \mathbb{Z}$.

$(-\Delta)^a$ fits in; it has order $m = 2a$ and *even parity*, therefore $\mu = a$.

With these definitions, Hörmander showed:

Theorem 3. *r^+P maps $\mathcal{E}_\mu(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$* **if and only if** *the symbol has the μ-transmission property for $x \in \partial\Omega$.*

The Princeton '65 lecture note contains much more, namely a *solvability theory* in L_2 Sobolev spaces for operators of type μ, which in addition have a certain factorization property of the principal symbol.

3. Solvability with homogeneous boundary conditions

The starting point was some Doklady notes by Vishik and Eskin '64, based on a factorization property of (scalar) ψdo symbols. Let us define:

Definition 4. *P (of order m) has the factorization index μ_0 when, in local coordinates where Ω is replaced by $\mathbb{R}^n_+$ with coordinates (x', x_n),*

$$p_0(x', 0, \xi', \xi_n) = p_-(x', \xi', \xi_n)p_+(x', \xi', \xi_n),$$

with $p_\pm$ homogeneous in ξ of degrees μ_0 resp. $m - \mu_0$, and $p_\pm$ extending to $\{\operatorname{Im} \xi_n \lessgtr 0\}$ analytically in ξ_n.

Here $Op(p_\pm(x', \xi))$ on $\mathbb{R}^n$ preserve support in $\overline{\mathbb{R}}^n_+$ resp. $\overline{\mathbb{R}}^n_-$.

There is always such a factorization at each x' but we here study the case where μ_0 is constant in x'.

Example: For $(-\Delta)^a$ on $\mathbb{R}^n$ (of order $m = 2a$) we have

$$|\xi|^{2a} = (|\xi'|^2 + \xi_n^2)^a = (|\xi'| - i\xi_n)^a(|\xi'| + i\xi_n)^a,$$

so that $p_\pm = (|\xi'| \pm i\xi_n)^a$, and the factorization index is a.

The operators $\Xi^\mu_\pm = \operatorname{Op}((\langle\xi'\rangle \pm i\xi_n)^\mu)$ play a great role in the theory.

Based on the factorization, Vishik and Eskin showed in '64 (extended to L_p by Shargorodsky '94, $1 < p < \infty$, $1/p' = 1 - 1/p$):

Theorem 5. *When P is elliptic of order m and has the factorization index μ_0, then*

$$r^+ P\colon \dot{H}_p^s(\overline{\Omega}) \to \overline{H}_p^{s - \operatorname{Re} m}(\Omega)$$

is a Fredholm operator for $\operatorname{Re}\mu_0 - 1/p' < s < \operatorname{Re}\mu_0 + 1/p$.

Note that s runs in a small interval $]\operatorname{Re}\mu_0 - 1/p', \operatorname{Re}\mu_0 + 1/p[$. The problem was then to find the correct solution space for higher s.
For this, Hörmander introduced for $p = 2$ a particular space combining the $\dot{H}$ and the $\overline{H}$ definitions:

Definition 6. *For $\mu \in \mathbb{C}$ and $s > \operatorname{Re}\mu - 1/p'$, the space $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ is defined by*

$$H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) = \Xi_+^{-\mu} e^+ \overline{H}_p^{s - \operatorname{Re}\mu}(\mathbb{R}^n_+).$$

Here $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \subset \mathcal{S}'(\mathbb{R}^n)$, *supported in* $\overline{\mathbb{R}}^n_+$. Note the jump at $x_n = 0$ in $e^+ \overline{H}_p^{s - \operatorname{Re}\mu}(\mathbb{R}^n_+)$.

Proposition 7. *Let* $s > \operatorname{Re}\mu - 1/p'$. *Then*

$$\Xi_+^{-\mu} e^+ \colon \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+) \to H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \text{ has the inverse}$$
$$r^+\Xi_+^{\mu} \colon H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+),$$

and $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ *is a Banach space with the norm*

$$\|u\|_{\mu(s)} = \|r^+\Xi_+^{\mu} u\|_{\overline{H}_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+)}.$$

One has that $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \supset \dot{H}_p^s(\overline{\mathbb{R}}^n_+)$, and elements of $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ are locally in H_p^s on $\mathbb{R}^n_+$, but they are not in general H_p^s up to the boundary. The definition generalizes to $\Omega \subset \mathbb{R}^n$ by use of local coordinates.

These are Hörmander's μ-spaces, very important since they turn out to be the correct solution spaces.

The spaces $H_p^{\mu(s)}$ replace the $\mathcal{E}_\mu$ in a Sobolev space context, in fact one has:

Proposition 8. *Let* $\overline{\Omega}$ *be compact, and let* $s > \operatorname{Re}\mu - 1/p'$. *Then*

$$\mathcal{E}_\mu(\overline{\Omega}) \subset H_p^{\mu(s)}(\overline{\Omega}) \text{ densely, and } \bigcap_s H_p^{\mu(s)}(\overline{\Omega}) = \mathcal{E}_\mu(\overline{\Omega}).$$

We can now state the basic theorems:

Theorem 9. *When P is of order m and type μ, r^+P maps $H_p^{\mu(s)}(\overline{\Omega})$ continuously into $\overline{H}_p^{s-\operatorname{Re} m}(\Omega)$ for all $s > \operatorname{Re}\mu - 1/p'$.*

Theorem 10. *Let P be elliptic of order m, with factorization index μ_0, and of type μ_0 (mod 1). Let $s > \operatorname{Re}\mu_0 - 1/p'$. The* **homogeneous Dirichlet problem**

$$r^+Pu = f, \quad \operatorname{supp} u \subset \overline{\Omega},$$

considered for $u \in \dot{H}_p^{\operatorname{Re}\mu_0 - 1/p' + \varepsilon}(\overline{\Omega})$, satifies:

$$f \in \overline{H}_p^{s-\operatorname{Re} m}(\Omega) \implies u \in H_p^{\mu_0(s)}(\overline{\Omega}).$$

Moreover, the mapping

$$r^+P\colon H_p^{\mu_0(s)}(\overline{\Omega}) \to \overline{H}_p^{s-\operatorname{Re} m}(\Omega) \quad (4)$$

is Fredholm.

The proofs in Hörmander's 1965 notes (for $p=2$) are long and difficult. One of the difficulties is that the $\Xi_\pm^\mu$ are not truly ψdo's in n variables, the derivatives of the symbols $(\langle\xi'\rangle \pm i\xi_n)^\mu$ do not decrease for $|\xi| \to \infty$ in the required way.

More recently we have found (G '90) a modified choice of symbol that gives true ψdo's $\Lambda_\pm^{(\mu)}$ with the same holomorphic extension properties for $\operatorname{Im}\xi_n \lesseqgtr 0$; they can be used instead of $\Xi_\pm^\mu$, also for $p \neq 2$.
This allows a *reduction* of some of the considerations to cases where the Boutet de Monvel calculus (extended to H_p^s in G '90) can be made useful.

In fact, when we for Theorem 9 introduce

$$Q = \Lambda_-^{(\mu_0 - m)} P \Lambda_+^{(-\mu_0)},$$

we get a ψdo of order 0 and type 0, with factorization index 0; then

$$r^+ Pu = f, \text{ with } \operatorname{supp} u \subset \overline{\Omega},$$

can be transformed to the equation

$$r^+ Qv = g, \text{ where } v = \Lambda_+^{(\mu_0)} u,\ g = r^+ \Lambda_-^{(\mu_0 - m)} e^+ f.$$

A closer analysis shows that $Q_+ = r^+ Q e^+$ is elliptic in the Boutet de Monvel calculus *without extra trace or Poisson operators*, so using a parametrix of it we can construct a parametrix for the original problem. This leads to the regularity of solutions and Fredholm property for all $s > \operatorname{Re}\mu_0 - 1/p'$.

Since $\bigcap_s H_p^{\mu(s)}(\overline{\Omega}) = \mathcal{E}_\mu(\overline{\Omega})$, and $\bigcap_s \overline{H}_p^{s-\operatorname{Re} m}(\Omega) = C^\infty(\overline{\Omega})$, one finds as a corollary when $s \to \infty$:

Corollary 11. *Let P and u be as in Theorem 9. If $r^+ Pu \in C^\infty(\overline{\Omega})$, then $u \in \mathcal{E}_{\mu_0}(\overline{\Omega})$. Moreover, the mapping*

$$r^+ P \colon \mathcal{E}_{\mu_0}(\overline{\Omega}) \to C^\infty(\overline{\Omega})$$

is Fredholm.

One can furthermore show that the finite dimensional kernel and cokernel (a complement of the range) of the mapping in Corollary 11 serve as kernel and cokernel also in the mappings for other spaces in Theorem 10.

NB! The functions in $\mathcal{E}_{\mu_0}$ have the behavior $u(x) = d(x)^{\mu_0} v(x)$ at $\partial\Omega$ with $v \in C^\infty(\overline{\Omega})$; they are *not in C^∞ themselves*, when $\mu_0 \notin \mathbb{N}_0$!

• The spaces $H_p^{\mu(s)}(\overline{\Omega})$ are in general very different for different μ. However, for $M \in \mathbb{N}$, there is the inclusion $H_p^{(\mu+M)(s)}(\overline{\Omega}) \subset H_p^{\mu(s)}(\overline{\Omega})$.

• The results extend to Triebel-Lizorkin spaces $F^s_{p,q}$ and Besov spaces $B^s_{p,q}$ (using Johnsen '96). In particular they extend to the Hölder-Zygmund spaces $B^s_{\infty,\infty}$, also denoted C^s_*, and equal to the Hölder spaces C^s when $s \in \mathbb{R}_+ \setminus \mathbb{N}$. This leads to the regularity results in Hölder spaces mentioned at the start.

• The results extend to *strongly elliptic systems*.

On $\mathcal{E}_\mu(\overline{\mathbb{R}}^n_+) = x_n^\mu C^\infty(\overline{\mathbb{R}}^n_+)$ we have by Taylor expansion when $M \geq 1$:

$$u(x) = x_n^\mu u_0(x') + x_n^{\mu+1} u_1(x') + \cdots + x_n^{\mu+M-1} u_{M-1}(x') + x_n^{\mu+M} v_M(x),$$

where $x_n^{\mu+M} v_M(x) \in \mathcal{E}_{\mu+M}(\overline{\mathbb{R}}^n_+)$. Here, with $\gamma_j v = (\partial^j_{x_n} v)|_{x_n=0}$,

$$u_j = c_j \gamma_j (x_n^{-\mu} u), \text{ denoted } \gamma_{\mu,j} u.$$

The definition can be extended by continuity to $H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ for sufficiently large s:

Theorem 12. *For $s > \operatorname{Re}\mu + M - 1/p'$,*

$$\varrho_{\mu,M} = \{\gamma_{\mu,0}, \dots, \gamma_{\mu,M-1}\} \colon H_p^{\mu(s)}(\overline{\mathbb{R}}^n_+) \to \prod\nolimits_{j<M} B_p^{s-\operatorname{Re}\mu-j-1/p}(\mathbb{R}^{n-1}).$$

The mapping is surjective, with kernel $H_p^{(\mu+M)(s)}(\overline{\mathbb{R}}^n_+)$.

A similar statement holds for $\overline{\Omega}$, with x_n replaced by $d(x)$.

In other words, there is a bijection

$$\varrho_{\mu,M}\colon H_p^{\mu(s)}(\overline{\Omega})/H_p^{(\mu+M)(s)}(\overline{\Omega}) \overset{\sim}{\to} \prod\nolimits_{j<M} B_p^{s-\operatorname{Re}\mu-j-1/p}(\partial\Omega).$$

Recall from Theorem 10 the Fredholm map

$$r^+P\colon H_p^{\mu_0(s)}(\overline{\Omega}) \to \overline{H}_p^{s-\operatorname{Re} m}(\Omega).$$

Taking $\mu = \mu_0 - M$ (so that $H_p^{\mu(s)}/H_p^{(\mu+M)(s)} = H_p^{(\mu_0-M)(s)}/H_p^{\mu_0(s)}$), we put the two maps together and find for $s > \operatorname{Re}\mu_0 - 1/p'$:

Corollary 13. *For each $M \in \mathbb{N}$, there is a Fredholm map:*

$$\{r^+P, \varrho_{\mu_0-M,M}\}\colon H_p^{(\mu_0-M)(s)}(\overline{\Omega}) \to \overline{H}_p^{s-\operatorname{Re} m}(\Omega)\times\prod\nolimits_{j<M} B_p^{s-\operatorname{Re}\mu_0+M-j-1/p}(\partial\Omega).$$

(Typical vector space idea!)

In Hölder-Zygmund spaces, we get the Fredholm map for $s > \operatorname{Re}\mu_0 - 1$,

$$\{r^+P, \varrho_{\mu_0-M,M}\}\colon C_*^{(\mu_0-M)(s)}(\overline{\Omega}) \to \overline{C}_*^{s-\operatorname{Re} m}(\Omega)\times\prod\nolimits_{j<M} C_*^{s-\operatorname{Re}\mu_0+M-j}(\partial\Omega).$$

The maps represent *nonhomogeneous Dirichlet problems.*

The example $P_a=(-\Delta)^a$. Here $\mu_0=a$. The first nonhomogeneous boundary condition comes in for $M=1$, i.e., $\mu=a-1$, $\varrho_{a-1,1}u=\gamma_{a-1,0}u=c_0\gamma_0(d^{1-a}u)$. Then we find bijectiveness of

$$
\begin{aligned}
\{r^+P_a,\gamma_{a-1,0}\}&\colon H_p^{(a-1)(s)}(\overline{\Omega})\overset{\sim}{\to}\overline{H}_p^{s-2a}(\Omega)\times B_p^{s-a+1-1/p}(\partial\Omega)\\
\{r^+P_a,\gamma_{a-1,0}\}&\colon C_*^{(a-1)(s)}(\overline{\Omega})\overset{\sim}{\to}\overline{C}_*^{s-2a}(\Omega)\times C_*^{s-a+1}(\partial\Omega).\quad (5)
\end{aligned}
$$

This solves the nonhomogeneous Dirichlet problem

$$r^+P_au=f\text{ in }\Omega,\quad \operatorname{supp}u\subset\overline{\Omega},\quad \gamma_{a-1,0}u=\varphi.$$

The domains can be described further by

Theorem 14.

$$
\begin{aligned}
&H_p^{\mu(s)}(\overline{\Omega})\subset e^+d(x)^\mu\overline{H}_p^{s-\operatorname{Re}\mu}(\Omega)+\dot{H}_p^{s\,(-\varepsilon)}(\overline{\Omega}),\ \textit{if } s>\operatorname{Re}\mu+1/p,\\
&C_*^{\mu(s)}(\overline{\Omega})\subset e^+d(x)^\mu\overline{C}_*^{s-\operatorname{Re}\mu}(\Omega)+\dot{C}_*^{s\,(-\varepsilon)}(\overline{\Omega}),\ \textit{if } s>\operatorname{Re}\mu,
\end{aligned}
$$

where $(-\varepsilon)$ *is active for* $s-\operatorname{Re}\mu\in 1/p+\mathbb{N}$ *resp.* $\mathbb{N}$.

For solutions by (5) this means that $u=e^+d^{a-1}u_0+u_1$, with $u_0\in C^{s-a+1}(\overline{\Omega})$ and $u_1\in\dot{C}^s(\overline{\Omega})$. Here $d^{a-1}u_0\neq 0$ when $\varphi\neq 0$, and it *blows up* at $\partial\Omega$.

A related observation was made by Abatangelo arXiv '13.

When $\varphi = 0$, we get the homogeneous Dirichlet problem considered earlier, having unique solutions in $H_p^{a(s)}(\overline{\Omega})$ resp. $C_*^{a(s)}(\overline{\Omega})$, not blowing up at $\partial\Omega$ but containing a factor d^a.

• It is also possible to define a *nonhomogeneous Neumann problem*

$$r^+ P_a u = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}, \quad \gamma_{a-1,1} u \equiv c_1 \gamma_1 (d^{1-a} u) = \psi.$$

It is Fredholm solvable in similar spaces, and has a blow-up of the solutions at $\partial\Omega$.

• The preceding boundary conditions are *local*, even though they are associated with a nonlocal ψdo.

• There also exist *nonlocal* boundary conditions. As a natural example, consider the problem

$$r^+ P u = f \text{ on } \Omega, \quad \gamma_0 B u = \varphi \text{ at } \partial\Omega,$$

where B is a ψdo. Here we can show that if B is of the same type μ as P, and is of order μ+integer, then there are principal symbol criteria for well-posedness. (Here we use a reduction to a problem in the Boutet de Monvel calculus.)

• There is also another type of nonlocal "boundary condition" that interests the probability people, in the form of integral conditions reaching out into $\complement\Omega$.

5. Some spectral results

One of the advantages of this systematic theory is that it allows variable-coefficient operators. For example, all that is said above for $P_a = (-\Delta)^a$ holds also when $-\Delta$ is replaced by a strongly elliptic second order differential operator A with C^∞-coefficients, except that the bijectiveness is replaced by the Fredholm property.

In the following, let P_a be this more general A^a, and let $P_{a,\mathrm{Dir}}$ be the L_2-realization of $r^+ P_a$ with domain $H^{a(2a)}(\overline{\Omega})$; then

$$P_{a,\mathrm{Dir}} \colon H^{a(2a)}(\overline{\Omega}) \to L_2(\Omega) \text{ is Fredholm.}$$

It is known in the case $P_a = (-\Delta)^a$ that there is a spectral asymptotic formula

$$s_j(P_{a,\mathrm{Dir}}) j^{-2a/n} \to C(P_{a,\mathrm{Dir}}) \text{ for } j \to \infty. \quad (7)$$

Theorem 15. *For A second-order strongly elliptic, the Dirichlet realization of $P_a = A^a$ satisfies* (7).

In the proof, one uses a parametrix for $P_{a,\mathrm{Dir}}$ composed of order-reducing operators and operators in the Boutet de Monvel calculus, applying spectral results for truncated ψdo's and singular Green operators. Related questions were considered recently by Frank and Geisinger.

Another case where the new calculus leads to improvements is in applications where the Dirichlet-to-Neumann operator P_{DN} enters. Recall that for a symmetric second-order strongly elliptic differential operator A with smooth coefficients on a bounded domain Ω, P_{DN} is the map from $\gamma_0 u$ to the conormal derivative $\nu_A u$ when u solves $Au = 0$ in Ω.
It is a first-order elliptic ψdo on the manifold $\Sigma = \partial\Omega$, and we can show that it satisfies:

Theorem 16 P_{DN} *is principally of type* $\frac{1}{2}$ *with factorization index* $\frac{1}{2}$*.*

This can be used e.g. to improve the knowledge of the "mixed problem", where $\Sigma = \Sigma_+ \cup \Sigma_-$ (a smooth partition), and A_{mix} is the realization of A in $L_2(\Omega)$ with domain

$$D(A_{\mathrm{mix}}) = \{u \in \overline{H}^1(\Omega) \mid u \in L_2(\Omega),\, \gamma_0 u = 0 \text{ on } \Sigma_-,\, \nu_A u = 0 \text{ on } \Sigma_+\}.$$

It is known that $D(A_{\mathrm{mix}}) \subset \overline{H}^{\frac{3}{2}-\varepsilon}(\Omega)$ only for $\varepsilon > 0$. Now P_{DN,Σ_+} enters in a detailed description of the structure, and we can show how the elements lying outside of $\overline{H}^{\frac{3}{2}}(\Omega)$ look. Moreover:

Theorem 17.

$$\mu_j((A_{\mathrm{mix}})^{-1} - (A_{\mathrm{Dir}})^{-1}) j^{2/(n-1)} \to C(P_{\mathrm{DN}}, \Sigma_+), \text{ for } j \to \infty.$$