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Regularity of spectral fractional Dirichlet and Neumann problems

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Consider the fractional powers $(A_{\mathrm{Dir}})^a$ and $(A_{\mathrm{Neu}})^a$ of the Dirichlet and Neumann realizations of a second-order strongly elliptic differential operator A on a smooth bounded subset Ω of $\mathbb{R}^n$. Recalling the results on complex powers and complex interpolation of domains of elliptic boundary value problems by Seeley in the 1970's, we demonstrate how they imply regularity properties in full scales of H_p^s-Sobolev spaces and Hölder spaces, for the solutions of the associated equations. Extensions to nonsmooth situations for low values of s are derived by use of recent results on H^∞-calculus. We also include an overview of the various Dirichlet- and Neumann-type boundary problems associated with the fractional Laplacian.

1 Introduction

There is currently a great interest in fractional powers of the Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $a > 0$, and derived operators associated with a subset of $\mathbb{R}^n$. The fractional Laplacian $(-\Delta)^a$ can be described as the pseudodifferential operator

$$u \mapsto (-\Delta)^a u = \mathcal{F}^{-1}\left(|\xi|^{2a}\hat{u}(\xi)\right) = \mathrm{Op}\left(|\xi|^{2a}\right)u, \tag{1.1}$$

with symbol $|\xi|^{2a}$, see also (6.1) below. Let Ω be a bounded C^∞-smooth subset of $\mathbb{R}^n$. Since $(-\Delta)^a$ is nonlocal, it is not obvious how to define boundary value problems for it on Ω, and in fact there are several interesting choices.

One choice for a Dirichlet realization on Ω is to take the power $(-\Delta_{\mathrm{Dir}})^a$ defined from the Dirichlet realization $-\Delta_{\mathrm{Dir}}$ of $-\Delta$ by spectral theory in the Hilbert space $L_2(\Omega)$; let us call it "the spectral Dirichlet fractional Laplacian", following a suggestion of Bonforte, Sire and Vazquez [8].

Another very natural choice is to take the Friedrichs extension of the operator $r^+(-\Delta)^a|_{C_0^\infty(\Omega)}$ (where r^+ denotes restriction to Ω); let us denote it $(-\Delta)^a_{\mathrm{Dir}}$ and call it "the restricted Dirichlet fractional Laplacian", following [8].

Both choices enter in nonlinear PDE; $(-\Delta)^a_{\mathrm{Dir}}$ is moreover important in probability theory. The operator $-\Delta$ can be replaced by a variable-coefficient strongly elliptic second-order operator A (not necessarily symmetric).

For the restricted Dirichlet fractional Laplacian, detailed regularity properties of solutions of $(-\Delta)^a_{\mathrm{Dir}}u = f$ in Hölder spaces and H_p^s Sobolev spaces have just recently been shown, in Ros-Oton and Serra [36]–[38], Grubb [25], [26].

For the spectral Dirichlet fractional Laplacian, regularity properties in H_p^s-spaces have been known for many years, as a consequence of Seeley's work [41], [42]; we shall account for this below in Sections 2 and 3. Further results have recently been presented by Caffarelli and Stinga in [12], treating domains with limited smoothness and obtaining certain Hölder estimates of Schauder type. See also Cabré and Tan [9] Thm. 1.9, for the case $a = \frac{1}{2}$.

In Section 4 we show how similar regularity properties of the spectral Neumann fractional Laplacian $(-\Delta_{\mathrm{Neu}})^a$ follow from Seeley's results. Also for this case, [12] has recently shown Hölder estimates of Schauder type under weaker smoothness hypotheses.

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In Section 5, we first briefly discuss extensions to more general scales of function spaces. Next, for generalizations to nonsmooth domains, we show how a recent result of Denk, Dore, Hieber, Prüss and Venni [16], on the existence of H^∞-calculi for boundary problems, can be combined with more recent results of Yagi [48], [49], to extend the regularity properties of Sections 3 and 4 to suitable nonsmooth situations for small s, leading to new results.

Finally, Section 6 gives a brief overview of the many kinds of boundary problems associated with $(-\Delta)^a$, expanding the references given above. This includes several other Neumann-type problems.

A primary purpose of the present note is to put forward some direct consequences of Seeley [41], [42] for the spectral fractional Laplacians. One of the main results is that when A is second-order strongly elliptic and B stands for either a Dirichlet or a Neumann condition, and $0 < a < 1$, then for solutions of

$$(A_B)^a u = f, \tag{1.2}$$

$f \in H_p^s(\Omega)$ for an $s \geq 0$ implies $u \in H_p^{s+2a}(\Omega)$ *if and only if* f itself satisfies all those boundary conditions of the form $BA^k f = 0$ $(k \in \mathbb{N}_0)$ that have a meaning on $H_p^s(\Omega)$. Consequences are also drawn for C^∞-solutions and for solutions where f is in $L_\infty(\Omega)$ or a Hölder space. We think this is of interest not just as a demonstration of early results, but also in showing how far one can reach, as a model for less smooth situations.

Section 5 shows one such generalization to nonsmooth domains and coefficients.

2 Seeley's results on complex interpolation

Let A be a strongly elliptic second-order differential operator on $\mathbb{R}^n$ with C^∞-coefficients. (The following theory extends readily to $2m$-order systems with normal boundary conditions as treated in Seeley [41], [42] and Grubb [23], but we restrict the attention to the second-order scalar case to keep notation and explanations simple.)

Let Ω be a C^∞-smooth bounded open subset of $\mathbb{R}^n$, and let A_B denote the realization of A in $L_2(\Omega)$ with domain $\{u \in H^2(\Omega) \mid Bu = 0\}$; here $Bu = 0$ stands for either the Dirichlet condition $\gamma_0 u = 0$ or a suitable Neumann-type boundary condition. In details,

$$Bu = \gamma_0 B_j u, \quad \text{where} \quad j = 0 \text{ or } j = 1; \tag{2.1}$$

here $B_0 = I$, and B_1 is a first-order differential operator on $\mathbb{R}^n$ such that $\{A, \gamma_0 B_1\}$ together form a strongly elliptic boundary value problem. Then A_B is lower bounded with spectrum in a sectorial region $V = \{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| \leq C(\operatorname{Re}\lambda - b)\}$. Our considerations in the following are formulated for the case where A_B is bijective. Seeley's papers also show how to handle a finite-dimensional 0-eigenspace.

The complex powers of A_B can be defined by spectral theory in $L_2(\Omega)$ in the cases where A_B is selfadjoint, but Seeley has shown in [41] how the powers can be defined more generally in a consistent way, acting in L_p-based Sobolev spaces $H_p^s(\Omega)$ $(1 < p < \infty)$, by a Cauchy integral of the resolvent around the spectrum

$$(A_B)^z = \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^z (A_B - \lambda)^{-1}\, d\lambda. \tag{2.2}$$

Here $H_p^s(\mathbb{R}^n)$ is the set of distributions u (functions if $s \geq 0$) such that $(1-\Delta)^{s/2} u \in L_p(\mathbb{R}^n)$, and $H_p^s(\Omega) = r^+ H_p^s(\mathbb{R}^n)$ (denoted $\overline{H}_p^s(\Omega)$ in [25], [26]), where r^+ stands for restriction to Ω. The general point of view is that the resolvent is constructed as an integral operator (found here by pseudodifferential methods) that can be applied to various function spaces, e.g. when p varies. The different realizations coincide on their common domains, so the labels $(A_B - \lambda)^{-1}$ and $(A_B)^z$ are used without indication of the actual spaces, which are understood from the context (this is standard terminology).

The formula (2.2) has a good meaning for $\operatorname{Re} z < 0$; extensions to other values of z are defined by compositions with integer powers of A_B. As shown in [41], [42], one has in general that $(A_B)^{z+w} = (A_B)^z (A_B)^w$, and the operators $(A_B)^z$ consitute a holomorphic semigroup in $L_p(\Omega)$ for $\operatorname{Re} z \leq 0$. This is based on the fundamental estimates of the resolvent shown in [40]. For $\operatorname{Re} z > 0$, the $(A_B)^z$ define unbounded operators in $L_p(\Omega)$, with domains $D_p((A_B)^z) = (A_B)^{-z}(L_p(\Omega))$. Note in particular that

$$(A_B)^{-z} : D_p((A_B)^w) \xrightarrow{\sim} D_p((A_B)^{z+w}) \quad \text{for} \quad \operatorname{Re} z, \operatorname{Re} w > 0. \tag{2.3}$$

We can of course not repeat the full analysis of Seeley here. An abstract framework for similar constructions of powers of operators in general Banach spaces is given in Amann [3], [4].

The domains in $L_p(\Omega)$ of the positive powers of A_B will now be explained for the cases $j = 0, 1$ in (2.1).

The domain of the realization A_B of A in $L_p(\Omega)$ with boundary condition $Bu = 0$ is

$$D_p(A_B) = \{u \in H^2_p(\Omega) \mid Bu = 0\}. \tag{2.4}$$

In [42], Seeley showed that for $0 < a < 1$, the domain of $(A_B)^a$ (the range of $(A_B)^{-a}$ applied to $L_p(\Omega)$) equals the complex interpolation space between $L_p(\Omega)$ and $\{u \in H^2_p(\Omega) \mid Bu = 0\}$ of the appropriate order. He showed moreover that this is the space of functions $u \in H^{2a}_p(\Omega)$ satisfying $Bu = 0$ if $2a > j + \frac{1}{p}$, and the space of functions $u \in H^{2a}_p(\Omega)$ with no extra condition if $2a < j + \frac{1}{p}$. He gives the special description for the case $2a = j + \frac{1}{p}$:

$$D_p\left((A_B)^{\frac{1}{2}(j+\frac{1}{p})}\right) = \left\{u \in H_p^{j+\frac{1}{p}}(\Omega) \mid B_j u \in \dot{H}_p^{\frac{1}{p}}(\overline{\Omega})\right\}; \tag{2.5}$$

one can say that $B_j u$ vanishes at $\partial\Omega$ in a generalized sense. (It is also recalled in Triebel [T95], Thm. 4.3.3.) We here use a notation of [25], [26], [30] where $\dot{H}^t_p(\overline{\Omega})$ stands for the space of functions in $H^t_p(\mathbb{R}^n)$ with support in $\overline{\Omega}$.

Let us define:

Definition 2.1 The spaces $H^s_{p,B,A}(\Omega)$ are defined by:

$$\begin{aligned}
&H^s_{p,B,A}(\Omega) = H^s_{p,B}(\Omega) = H^s_p(\Omega) \text{ for } 0 \le s < j + \tfrac{1}{p},\\
&H^s_{p,B,A}(\Omega) = H^s_{p,B}(\Omega) = \{u \in H^s_p(\Omega) \mid Bu = 0\} \text{ for } j < s - \tfrac{1}{p} < j + 2,\\
&H^s_{p,B,A}(\Omega) = \{u \in H^s_p(\Omega) \mid Bu = BAu = \dots = BA^k u = 0\}\\
&\qquad\qquad \text{for } j + 2k < s - \tfrac{1}{p} < j + 2(k+1),\\
&H^s_{p,B,A}(\Omega) = \left\{u \in H^s_p(\Omega) \mid BA^l u = 0 \text{ for } l < k,\ B_j A^k u \in \dot{H}_p^{\frac{1}{p}}(\overline{\Omega})\right\}\\
&\qquad\qquad \text{when } s - \tfrac{1}{p} = j + 2k,
\end{aligned} \tag{2.6}$$

where $k \in \mathbb{N}_0$.

Note that in the first three statements, $H^s_{p,B,A}(\Omega)$ consists of the functions in $H^s_p(\Omega)$ satisfying those boundary conditions $BA^l u = 0$ for which $j + 2l < s - \frac{1}{p}$ (i.e., those that are well-defined on $H^s_p(\Omega)$). The definition in the fourth statement, although slightly complicated, is included here primarily in order that we can use the notation $H^s_{p,B,A}(\Omega)$ freely without exceptional parameters.

The spaces $H^s_{p,B}(\Omega)$ were defined in Seeley [42] (in Grisvard [22] for $p = 2$); we have added the definitions for $s > 2$ (they can be called extrapolation spaces, as in [3], [4]). In the L_2-case, the extra requirement in (2.5) can be replaced by $d^{-\frac{1}{2}} B_j u \in L_2(\Omega)$, where $d(x)$ is the distance from x to $\partial\Omega$.

With this notation, Seeley's works show:

Theorem 2.2 *When $0 < a < 1$, $D_p((A_B)^a)$ equals the space $\left[L_p(\Omega), H^2_{p,B}(\Omega)\right]_a$ obtained by complex interpolation between $L_p(\Omega)$ and $H^2_{p,B}(\Omega)$.*

For all $a > 0$, $D_p((A_B)^a) = H^{2a}_{p,B,A}(\Omega)$.

P r o o f . The first statement is a direct quotation from [42]. So is the second statement for $0 < a \le 1$, and it follows for $a = a' + k$, $0 < a' \le 1$ and $k \in \mathbb{N}$, by using (2.3) with $w = a'$, $z = k$. □

Observe the general homeomorphism property that follows from this theorem in view of formula (2.3):

Corollary 2.3 *For $a > 0$, $(A_B)^a$ defines homeomorphisms:*

$$(A_B)^a : H^{s+2a}_{p,B,A}(\Omega) \xrightarrow{\sim} H^s_{p,B,A}(\Omega), \quad \textit{for all} \quad s \ge 0. \tag{2.7}$$

The characterization of the interpolation space was given (also for $2m$-order operators) by Grisvard in the case of scalar elliptic operators in L_2 Sobolev spaces in [22], in terms of real interpolation. Seeley's result for $1 < p < \infty$ is shown for general elliptic operators in vector bundles, with normal boundary conditions.

3 Consequences for the Dirichlet problem

Let $B = \gamma_0$, denoted γ for brevity. Corollary 2.3 already shows how the regularity of u and $f = (A_\gamma)^a u$ are related, when the functions are known on beforehand to lie in the special spaces in (2.6). But we can also discuss cases where f is just given in a general Sobolev space. Namely, we have as a generalization of the remarks at the end of [42]:

Theorem 3.1 *Let $0 < a < 1$. Let $f \in H_p^s(\Omega)$ for some $s \geq 0$, and assume that $u \in D_p((A_\gamma)^a)$ is a solution of*

$$(A_\gamma)^a u = f. \tag{3.1}$$

1° If $s < \frac{1}{p}$, then $u \in H_{p,\gamma}^{s+2a}(\Omega)$.

2° Let $\frac{1}{p} < s < 2 + \frac{1}{p}$. Then $u \in H_{p,\gamma}^{\frac{1}{p}+2a-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. Moreover, $u \in H_p^{s+2a}(\Omega)$ if and only if $\gamma f = 0$, and then in fact $u \in H_{p,\gamma}^{s+2a}(\Omega)$.

P r o o f. 1°. When $s < \frac{1}{p}$, we can simply use that $u = (A_\gamma)^{-a} f$, where $(A_\gamma)^{-a}$ defines a homeomorphism from $H_p^s(\Omega)$ to $H_{p,\gamma}^{s+2a}(\Omega)$ in view of (2.7).

2°. We first note that since $s > \frac{1}{p} > \frac{1}{p} - \varepsilon$, all $\varepsilon > 0$, the preceding result shows that $u \in H_{p,\gamma}^{\frac{1}{p}+2a-\varepsilon}(\Omega)$ for all $\varepsilon > 0$.

Now if $\gamma f = 0$, then $f \in H_{p,\gamma}^s(\Omega)$ by (2.6). Hence $u \in H_{p,\gamma}^{s+2a}(\Omega)$ since $(A_\gamma)^{-a}$ defines a homeomorphism from $H_{p,\gamma}^s(\Omega)$ to $H_{p,\gamma}^{s+2a}(\Omega)$ according to (2.7).

Conversely, let $u \in H_p^{s+2a}(\Omega)$. Then since we know already that $u \in H_{p,\gamma}^{\frac{1}{p}+2a-\varepsilon}(\Omega)$, we see that $\gamma u = 0$ (taking $\varepsilon < 2a$). Then by (2.6), $u \in H_{p,\gamma}^\sigma(\Omega)$ for $\frac{1}{p} + 2a < \sigma < \min\{s + 2a, 2 + \frac{1}{p}\}$; such σ exist since $a < 1$. Hence $f \in H_{p,\gamma}^{\sigma-2a}(\Omega)$ with $\sigma - 2a > \frac{1}{p}$ and therefore has $\gamma f = 0$. □

Point 2° in the theorem shows that f may have to be provided with a nontrivial boundary condition in order for the best possible regularity to hold for u. This is in contrast to the case where $a = 1$, where it is known that for u satisfying $-\Delta u = f$ with $\gamma u = 0$, $f \in H_p^s(\Omega)$ always implies $u \in H_p^{s+2}(\Omega)$.

The case $s = \frac{1}{p}$ can be included in 2° when we use the generalized boundary condition in (2.4); details are given for the general case in Theorem 3.2 2° below.

The importance of a boundary condition on f for optimal regularity of u is also demonstrated in the results of Caffarelli and Stinga [12] (and Cabré and Tan [9]).

By induction, we can extend the result to higher s:

Theorem 3.2 *Let $0 < a < 1$. Let $u \in D_p((A_\gamma)^a)$ be the solution of* (3.1) *with $f \in H_p^s(\Omega)$ for some $s \geq 0$. One has for any $k \in \mathbb{N}_0$:*

1° If $2k + \frac{1}{p} < s < 2k + 2 + \frac{1}{p}$, and $\gamma A^l f = 0$ for $l = 0, 1, \dots, k$ (i.e., $f \in H_{p,\gamma,A}^s(\Omega)$), then $u \in H_{p,\gamma,A}^{s+2a}(\Omega)$. On the other hand, if $u \in H_p^{s+2a}(\Omega)$, then necessarily $\gamma A^l f = 0$ for $l = 0, 1, \dots, k$ (and hence $f \in H_{p,\gamma,A}^s(\Omega)$ and $u \in H_{p,\gamma,A}^{s+2a}(\Omega)$).

2° Let $s = 2k + \frac{1}{p}$. If $f \in H_{p,\gamma,A}^s(\Omega)$, then $u \in H_{p,\gamma,A}^{s+2a}(\Omega)$. On the other hand, if $u \in H_p^{s+2a}(\Omega)$, then necessarily $f \in H_{p,\gamma,A}^s(\Omega)$ and $u \in H_{p,\gamma,A}^{s+2a}(\Omega)$.

P r o o f. Statement 1° was shown for $k = 0$ in Theorem 3.12°. We proceed by induction: Assume that the statement holds for $k \leq k_0 - 1$. Now show it for k_0:

If $\gamma A^l f = 0$ for $l \leq k_0$, then $f \in H_{p,\gamma,A}^s(\Omega)$ by (2.6). Hence $u \in H_{p,\gamma,A}^{s+2a}(\Omega)$ since $(A_\gamma)^{-a}$ defines a homeomorphism from $H_{p,\gamma,A}^s(\Omega)$ to $H_{p,\gamma,A}^{s+2a}(\Omega)$ according to (2.7).

Conversely, let $u \in H_p^{s+2a}(\Omega)$. Note that since $s > \frac{1}{p} + 2k_0 > \frac{1}{p} + 2k_0 - \varepsilon$, all $\varepsilon > 0$, the result for $k_0 - 1$ shows that $u \in H_{p,\gamma,A}^{\frac{1}{p}+2k_0+2a-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. Then, taking $\varepsilon < 2a$, we see that $\gamma A^l u = 0$ for $l \le k_0$. Now in view of (2.6), $u \in H_{p,\gamma,A}^{\sigma}(\Omega)$ for $\frac{1}{p} + 2k_0 + 2a < \sigma < \min\{s + 2a, 2 + 2k_0 + \frac{1}{p}\}$; such σ exist since $a < 1$. Hence $f \in H_{p,\gamma,A}^{\sigma-2a}(\Omega)$ with $\sigma - 2a > 2k_0 + \frac{1}{p}$; therefore it has $\gamma A^l f = 0$ for $l \le k_0$.

The first part of statement 2° follows immediately from (2.7). For the second part, let $u \in H_p^{s+2a}(\Omega)$, $s = 2k + \frac{1}{p}$. Since $s > 2k + \frac{1}{p} - \varepsilon$, we see by application of 1° with $s' = 2k + \frac{1}{p} - \varepsilon$ that $u \in H_{p,\gamma,A}^{2k+\frac{1}{p}-\varepsilon+2a}(\Omega)$. For $\varepsilon < 2a$ this shows that $\gamma A^l u = 0$ for $l \le k$. Now $s + 2a = 2k + \frac{1}{p} + 2a$ also lies in $\left]2k + \frac{1}{p}, 2k + 2 + \frac{1}{p}\right[$ (since $a < 1$) so in fact $u \in H_{p,\gamma,A}^{s+2a}(\Omega)$, and $f \in H_{p,\gamma,A}^{s}(\Omega)$. □

Briefly expressed, the theorem shows that in order to have optimal regularity, namely the improvement from f lying in an H_p^s-space to u lying in an H_p^{s+2a}-space, it is necessary and sufficient to impose all the boundary conditions for the space $H_{p,\gamma,A}^s(\Omega)$ on f.

In the following, we assume throughout that $0 < a < 1$. (Results for higher a can be deduced from the present results by use of elementary mapping properties for integer powers, and are left to the reader.) As a first corollary, we can describe C^∞-solutions. Define

$$C_{\gamma,A}^{\infty}(\overline{\Omega}) = \left\{u \in C^\infty(\overline{\Omega}) \mid \gamma A^k u = 0 \text{ for all } k \in \mathbb{N}_0\right\}. \tag{3.2}$$

Corollary 3.3 *The operator $(A_\gamma)^a$ defines a homeomorphism of $C_{\gamma,A}^\infty(\overline{\Omega})$ onto itself.*

Moreover, if $u \in H_{p,\gamma,A}^{2a}(\Omega) \cap C^\infty(\overline{\Omega})$ for some p, then $(A_\gamma)^a u \in C^\infty(\overline{\Omega})$ implies $u \in C_{\gamma,A}^\infty(\overline{\Omega})$ (and hence $(A_\gamma)^a u \in C_{\gamma,A}^\infty(\overline{\Omega})$).

P r o o f . Fix p. We first note that

$$C_{\gamma,A}^{\infty}(\overline{\Omega}) = \bigcap_{s\ge 0} H_{p,\gamma,A}^s(\Omega). \tag{3.3}$$

Here the inclusion "$\subset$" follows from the observation

$$\left\{u \in C^\infty(\overline{\Omega}) \mid \gamma A^l u = 0 \text{ for } l \le k\right\} \subset H_{p,\gamma,A}^{2k+\frac{1}{p}-\varepsilon}(\Omega),$$

by taking the intersection over all k. The other inclusion follows from

$$H_{p,\gamma,A}^{2k+\frac{1}{p}-\varepsilon}(\Omega) \subset \left\{u \in C^N(\overline{\Omega}) \mid N < 2k + \tfrac{1}{p} - \varepsilon - \tfrac{n}{p},\ \gamma A^l u = 0 \text{ for } 2l \le N\right\},$$

by taking intersections for $k \to \infty$.

The fact that $(A_\gamma)^a$ maps $H_{p,\gamma,A}^s(\Omega)$ homeomorphically to $H_{p,\gamma,A}^{s-2a}(\Omega)$ for all $s \ge 2a$ now implies that $(A_\gamma)^a$ maps $C_{\gamma,A}^\infty(\overline{\Omega})$ to $C_{\gamma,A}^\infty(\overline{\Omega})$ with inverse $(A_\gamma)^{-a}$.

Next, let $u \in H_{p,\gamma}^{2a}(\Omega) \cap C^\infty(\overline{\Omega})$. If $(A_\gamma)^a u \in C^\infty(\overline{\Omega})$, then Theorem 3.2 can be applied with arbitrarily large k, showing that $u \in C_{\gamma,A}^\infty(\overline{\Omega})$, and hence $(A_\gamma)^a u \in C_{\gamma,A}^\infty(\overline{\Omega})$. □

Remark 3.4 It follows that for each $1 < p < \infty$, the eigenfunctions of $(A_\gamma)^a$ (with domain $H_{p,\gamma}^{2a}(\Omega)$) belong to $C_{\gamma,A}^\infty(\overline{\Omega})$; they are the same for all p. In particular, when A_γ is selfadjoint in $L_2(\Omega)$, the eigenfunctions of $(A_\gamma)^a$ defined by spectral theory (that are the same as those of A_γ) are the eigenfunctions also in the L_p-settings.

Finally, let us draw some conclusions for regularity properties when $f \in L_\infty(\Omega)$ or is in a Hölder space. As in [26], we denote by $C^\alpha(\overline{\Omega})$ the space of functions that are continuously differentiable up to order α when $\alpha \in \mathbb{N}_0$, and are in the Hölder class $C^{k,\sigma}(\overline{\Omega})$ when $\alpha = k + \sigma$, $k \in \mathbb{N}_0$ and $0 < \sigma < 1$. Recall that the Hölder-Zygmund spaces $B_{\infty,\infty}^s(\overline{\Omega})$, also denoted $C_*^s(\overline{\Omega})$, coincide with $C^s(\overline{\Omega})$ when $s \in \mathbb{R}_+ \setminus \mathbb{N}$, and there is the Sobolev embedding property

$$H_p^s(\Omega) \subset C_*^{s-\frac{n}{p}}(\overline{\Omega}) \quad \text{for all} \quad s > \tfrac{n}{p}.$$

(Embedding and trace mapping properties for Besov-Triebel-Lizorkin spaces $F^s_{p,q}$ and $B^s_{p,q}$ are compiled e.g. in Johnsen [32], Sect. 2.3, 2.6; note that $H^s_p = F^s_{p,2}$.) Recall also that $C^k(\overline{\Omega}) \subset C^{k-1,1}(\overline{\Omega}) \subset C^k_*(\overline{\Omega}) \subset C^{k-0}(\overline{\Omega})$ for $k \in \mathbb{N}$. Here we use the notation $C^{\alpha-0} = \bigcap_{\varepsilon>0} C^{\alpha-\varepsilon}$ (it is applied similarly to H^s_p-spaces).

Corollary 3.5 1° *Let $f \in L_p(\Omega)$ with $\frac{n}{p} < 2a$. If $2a - \frac{n}{p} \neq 1$, resp. $= 1$, then the solution u of* (3.1) *is in $C^{2a-\frac{n}{p}}(\overline{\Omega})$, resp. $C^1_*(\overline{\Omega})$, with $\gamma u = 0$.*

2° *If $f \in L_\infty(\Omega)$, then the solution u of* (3.1) *is in $C^{2a-0}(\overline{\Omega})$ with $\gamma u = 0$.*

P r o o f. 1°. When $f \in L_p(\Omega)$, then $u \in H^{2a}_{p,\gamma}(\Omega) \subset H^{2a}_p(\Omega)$ by Theorem 3.1 1°. Now when $p > \frac{n}{2a}$, Sobolev embedding gives that $u \in C^{2a-\frac{n}{p}}(\overline{\Omega})$, except when $2a - \frac{n}{p} = 1$, where it gives $u \in C^1_*(\overline{\Omega})$. Since a fortiori $p > \frac{1}{2a}$, we see from (2.6) that $\gamma u = 0$ in $H^{2a}_p(\overline{\Omega})$, hence in $C^{2a-\frac{n}{p}}(\overline{\Omega})$ resp. $C^1_*(\overline{\Omega})$.

2°. When $f \in L_\infty(\Omega)$, then $f \in L_p(\Omega)$ for all $1 < p < \infty$. Using 1° and letting $p \to \infty$, we conclude that $u \in C^{2a-0}(\overline{\Omega})$. □

Corollary 3.6 *Let $k \in \mathbb{N}_0$, and let $2k < \alpha < 2k + 2$. If $f \in C^\alpha(\overline{\Omega})$ with $\gamma A^l f = 0$ for $l \leq k$, then the solution u of* (3.1) *satisfies:*

$$u \in C^{\alpha+2a-0}(\overline{\Omega}) \text{ with } \begin{cases} \gamma A^l u = 0 \text{ for } l \leq k & \text{if } \alpha + 2a \leq 2k + 2, \\ \gamma A^l u = 0 \text{ for } l \leq k + 1 & \text{if } \alpha + 2a > 2k + 2. \end{cases} \tag{3.4}$$

P r o o f. When $f \in C^\alpha(\overline{\Omega})$, then $f \in H^{\alpha-\varepsilon}_p(\Omega)$ for all p, all $\varepsilon > 0$. For ε so small that $\alpha - \varepsilon > 2k$, we see from (2.6) that since $\gamma A^l f = 0$ for $l \leq k$, $f \in H^{\alpha-\varepsilon}_{p,\gamma,A}(\Omega)$. Then it follows from (2.7) that $u \in H^{\alpha+2a-\varepsilon}_{p,\gamma}(\Omega)$.

If $\alpha + 2a > 2k + 2$, we have for ε so small that $\alpha + 2a - \varepsilon > 2k + 2$, and then $\frac{1}{p}$ sufficiently small, that u satisfies the boundary conditions $\gamma A^l u = 0$ for $l \leq k + 1$. For $p \to \infty$, this implies that $u \in C^{\alpha+2a-0}(\overline{\Omega})$ satisfying these boundary conditions.

If $\alpha + 2a \leq 2k + 2$, we have for ε in a small interval $]0, \varepsilon_0[$ that $2k < \alpha + 2a - \varepsilon < 2k + 2$, and then for all p sufficiently small, that u satisfies the boundary conditions $\gamma A^l u = 0$ for $l \leq k$. For $p \to \infty$, this implies that $u \in C^{\alpha+2a-0}(\overline{\Omega})$ satisfying those boundary conditions. □

The regularity results of Caffarelli and Stinga [12] are concerned with cases assuming much less smoothness of the domain and coefficients, getting results in Hölder spaces of low order (< 2). See also Section 5.

The above results deduced from [42] explain the role of boundary conditions on f. The results in Hölder spaces resemble the results of [12] for the values of α considered there, however with a loss of sharpness (the "-0") in some of the estimates in Corollary 3.6.

4 Consequences for Neumann-type problems

The proofs are analogous for a Neumann-type boundary operator B ($j = 1$ in (2.1)ff.).

Theorem 4.1 *Let $0 < a < 1$. Let $u \in D_p((A_B)^a)$ be the solution of*

$$(A_B)^a u = f, \tag{4.1}$$

where $f \in H^s_p(\Omega)$ for some $s \geq 0$.

1° *If $s < 1 + \frac{1}{p}$, then $u \in H^{s+2a}_{p,B}(\Omega)$.*

One has for any $k \in \mathbb{N}_0$:

2° *If $2k + 1 + \frac{1}{p} < s < 2k + 3 + \frac{1}{p}$, and $BA^l f = 0$ for $l = 0, 1, \dots, k$ (i.e., $f \in H^s_{p,B,A}(\Omega)$), then $u \in H^{s+2a}_{p,B,A}(\Omega)$.*

On the other hand, if $u \in H^{s+2a}_p(\Omega)$, then necessarily $BA^l f = 0$ for $l = 0, 1, \dots, k$ (and hence $f \in H^s_{p,B,A}(\Omega)$ and $u \in H^{s+2a}_{p,B,A}(\Omega)$).

3° *Let $s = 2k + 1 + \frac{1}{p}$. If $f \in H^s_{p,B,A}(\Omega)$, then $u \in H^{s+2a}_{p,B,A}(\Omega)$. On the other hand, if $u \in H^{s+2a}_p(\Omega)$, then necessarily $f \in H^s_{p,B,A}(\Omega)$ and $u \in H^{s+2a}_{p,B,A}(\Omega)$.*

Define

$$C^{\infty}_{B,A}(\overline{\Omega}) = \{u \in C^{\infty}(\overline{\Omega}) \mid BA^k u = 0 \text{ for all } k \in \mathbb{N}_0\}. \tag{4.2}$$

Corollary 4.2 *The operator $(A_B)^a$ defines a homeomorphism of $C^{\infty}_{B,A}(\overline{\Omega})$ onto itself.*

Moreover, if $u \in H^{2a}_{p,B,A}(\Omega) \cap C^{\infty}(\overline{\Omega})$ for some p, then $(A_B)^a u \in C^{\infty}(\overline{\Omega})$ implies $u \in C^{\infty}_{B,A}(\overline{\Omega})$ (and hence $(A_B)^a u \in C^{\infty}_{B,A}(\overline{\Omega})$).

Corollary 4.3 1° *Let $f \in L_p(\Omega)$ with $\frac{n}{p} < 2a$. If $2a - \frac{n}{p} \neq 1$, resp. $= 1$, then the solution u of* (4.1) *is in $C^{2a-\frac{n}{p}}(\overline{\Omega})$, resp. $C^1_*(\overline{\Omega})$, with $Bu = 0$ if $2a - \frac{n}{p} > 1$.*

2° *If $f \in L_\infty(\Omega)$, then the solution u of (4.1) is in $C^{2a-0}(\overline{\Omega})$, with $Bu = 0$ precisely when $a > \frac{1}{2}$.*

Proof. 1° It is seen as in Corollary 3.5 that $u \in C^{2a-\frac{n}{p}}(\overline{\Omega})$ resp. $C^1_*(\overline{\Omega})$. If $2a - \frac{n}{p} > 1$, then a fortiori $2a - \frac{1}{p} > 1$, and $Bu = 0$ in $H^{2a}_p(\Omega)$; this carries over to the space we embed in.

2°. When $f \in L_\infty(\Omega)$, then $f \in L_p(\Omega)$ for all $1 < p < \infty$, so we have 1° for all p. Letting $p \to \infty$, we conclude that $u \in C^{2a-0}(\overline{\Omega})$, and $Bu = 0$ is assured if $2a > 1$. When $a \le \frac{1}{2}$, then $2a \le 1 < 1 + \frac{1}{p}$ for all p, so $H^{2a}_{p,B}(\Omega) = H^{2a}_p(\Omega)$ for all p; no boundary condition is imposed. □

Corollary 4.4 *Let $k \in \mathbb{N}_0$, and let $\alpha \ge 0$ satisfy $2k - 1 < \alpha < 2k + 1$.*

If $f \in C^{\alpha}(\overline{\Omega})$ with $BA^l f = 0$ for $l \le k - 1$, then the solution u of (4.1) *satisfies:*

$$u \in C^{\alpha+2a-0}(\overline{\Omega}) \text{ with } \begin{cases} BA^l u = 0 \text{ for } l \le k-1 & \text{if } \alpha + 2a \le 2k+1, \\ BA^l u = 0 \text{ for } l \le k & \text{if } \alpha + 2a > 2k+1. \end{cases} \tag{4.3}$$

In the case of $(-\Delta_{\mathrm{Neu}})^a$ considered on a connected set Ω, there is a one-dimensional nullspace consisting of the constants (that are of course in $C^{\infty}(\overline{\Omega})$). This case is included in the above results by a trick found in [41]: Replace $-\Delta$ by

$$A = -\Delta + E_0, \quad E_0 u = \frac{1}{\mathrm{vol}(\Omega)} \int_{\Omega} u(x)\, dx; \tag{4.4}$$

note that E_0 is a projection onto the constants, orthogonal in $L_2(\Omega)$ (it is also a pseudodifferential operator of order $-\infty$). Here $\Delta E_0 = 0$ and $\gamma_1 E_0 = 0$, where $\gamma_1 u = \partial_n u|_{\partial\Omega}$. With $B = \gamma_1$, $(A_{\gamma_1})^a$ equals $(-\Delta_{\gamma_1})^a + E_0$ and is invertible, and the above results apply to it and lead to similar regularity results for $(-\Delta_{\gamma_1})^a$ itself (note that $\gamma_1 A^k u = \gamma_1(-\Delta)^k u$).

5 Further developments

5.1 More general function spaces

The above theorems in L_p Sobolev spaces are likely to extend to a large number of other scales of function spaces. Notably, it seems possible to extend them to the scale of Besov spaces $B^s_{p,q}$ with $1 \le p \le \infty$, $1 \le q < \infty$, since the decisive complex interpolation properties of domains of elliptic realizations have been shown by Guidetti in [G91].

It is not at the moment clear to the author whether the scale $B^s_{\infty,\infty} = C^s_*$ of Hölder-Zygmund spaces, or the scale of "small" Hölder-Zygmund spaces c^s_* (obtained by closure in C^s_*-spaces of the compactly supported smooth functions), cf. e.g. Escher and Seiler [19], can be or has been included for these boundary value problems. (It was possible to include C^s_* in the regularity study for the restricted fractional Laplacian in [25] using Johnsen [32].) Such an extension would allow removing the "-0" in some formulas in Corollaries 3.6 and 4.4 above.

Let us mention for cases *without* boundary conditions, that the continuity of classical pseudodifferential operators on $\mathbb{R}^n$ (such as $(-\Delta)^a$ and its parametrices) in Hölder-Zygmund spaces has been known for many years, cf. e.g. Yamazaki [50] for a more general result and references to earlier contributions. On this point, [12] refers to Caffarelli and Silvestre [10].

5.2 Nonsmooth situations

It is of great interest to treat the problems also when the set Ω and the coefficients of A have only limited smoothness. One of the common strategies is to transfer the results known for constant-coefficient operators on $\mathbb{R}^n_+$ to variable-coefficient operators by perturbation arguments, and to sets Ω by local coordinates. (This strategy is used in [12].) The pseudodifferential theory in smooth cases is in fact set up to incorporate the perturbation arguments in a systematic and more informative calculus. For nonsmooth cases, we remark that there do exist pseudodifferential theories requiring only limited smoothness in x, cf. [2] and other works of Abels listed there. Applications to the present problems await development.

Another point of view comes forward in the efforts to establish so-called maximal regularity, H^∞-calculus and R-boundedness properties for operators generating semigroups; see e.g. Denk, Hieber and Prüss [17] for results, references to the vast literature, and an overview of the theory. Fractional powers of boundary problems entered in this theory at an early stage, starting with Seeley's results, but are not so much in focus in the latest developments, that are primarily aimed towards solvability of parabolic problems.

However, there is an interesting result by Yagi [48] that is relevant for the present purposes. He considers an operator

$$A = -\sum_{j,k=1,\dots,n} \partial_j a_{jk}(x)\partial_k + c(x), \text{ with } \sum_{j,k=1,\dots,n} a_{jk}(x)\xi_j\xi_j \ge c_0|\xi|^2, \tag{5.1}$$

$a_{jk} = a_{kj}$ real in $C^1(\overline{\Omega})$, $c(x)$ real bounded ≥ 0 and $c_0 > 0$, on a bounded C^2-domain $\Omega \subset \mathbb{R}^n$. Define

$$H^s_{p,\gamma}(\Omega) = \begin{cases} H^s_p(\Omega) \text{ for } 0 \le s < \frac{1}{p}, \\ \{u \in H^s_p(\Omega) \mid \gamma u = 0\} \text{ for } \frac{1}{p} < s \le 2. \end{cases} \tag{5.2}$$

Since $A = -\sum_{j,k}(a_{jk}\partial_j\partial_k + (\partial_j a_{jk})\partial_k) + c$ with $a_{jk} \in C^1$ and $\partial a_{jk} \in C^0$, it follows from Denk, Dore, Hieber, Prüss and Venni [16] Thm. 2.3, for $1 < p < \infty$, that the Dirichlet realization A_γ of A in $L_p(\Omega)$ with domain

$$D_p(A_\gamma) = H^2_{p,\gamma}(\Omega),$$

admits a bounded H^∞-calculus in $L_p(\Omega)$. We here use that for $p = 2$, A_γ is selfadjoint in $L_2(\Omega)$ with a positive lower bound (since Ω is bounded), hence the constant μ_ϕ in the theorem can be taken equal to 0. We also observe that the definitions of the operators for various p are consistent (and they all have the same eigenvector system).

Combined with the existence of an H^∞-calculus, Theorem 5.2 of [48] then shows:

Theorem 5.1 *Let $1 < p < \infty$. For $0 \le a \le 1$, the fractional powers $(A_\gamma)^a$ in $L_p(\Omega)$ have domains*

$$D_p((A_\gamma)^a) = \begin{cases} H^{2a}_p(\Omega) & \text{if } 0 \le 2a < \frac{1}{p}, \\ H^{2a}_{p,\gamma}(\Omega) & \text{if } \frac{1}{p} < 2a \le 2,\ 2a \ne 1 + \frac{1}{p}. \end{cases} \tag{5.3}$$

([48] does not describe the excepted cases $s = \frac{1}{p}, 1 + \frac{1}{p}$.)

With this statement we can repeat the proof of Theorem 3.1 in cases where $s \le 2 - 2a$, obtaining:

Theorem 5.2 *(Recall the smoothness assumptions: Ω is C^2 and the a_{jk} are in $C^1(\overline{\Omega})$, $c \in L_\infty(\Omega)$.) Let $0 < a < 1$. Let $f \in H^s_p(\Omega)$ for some $s \in [0, 2-2a]$, and assume that $u \in D_p((A_\gamma)^a)$ is a solution of*

$$(A_\gamma)^a u = f. \tag{5.4}$$

Assume that s and $s + 2a$ are different from $\frac{1}{p}$ and $1 + \frac{1}{p}$.

1° If $s < \frac{1}{p}$, then $u \in H^{s+2a}_{p,\gamma}(\Omega)$.

2° Let $\frac{1}{p} < s \le 2 - 2a$. Then $u \in H^{\frac{1}{p}+2a-\varepsilon}_{p,\gamma}(\Omega)$ for all $\varepsilon > 0$. Moreover, $u \in H^{s+2a}_p(\Omega)$ if and only if $\gamma f = 0$, and then in fact $u \in H^{s+2a}_{p,\gamma}(\Omega)$.

P r o o f . We first note that by the general properties of fractional powers,

$$(A_\gamma)^a : D_p\left((A_\gamma)^{t+a}\right) \xrightarrow{\sim} D_p((A_\gamma)^t), \quad \text{for} \quad t \geq 0; \tag{5.5}$$

this covers part of the statements in view of Theorem 5.1.

1° follows from (5.5), since $H_p^s(\Omega) = D_p\big((A_\gamma)^{s/2}\big)$ for $s < \frac{1}{p}$ and $D_p\big((A_\gamma)^{s/2+a}\big) = H_{p,\gamma}^{s+2a}(\Omega)$, by (5.3).

For 2°, we first note that since $s > \frac{1}{p} > \frac{1}{p} - \varepsilon$, all $\varepsilon > 0$, the preceding result shows that $u \in H_{p,\gamma}^{\frac{1}{p}+2a-\varepsilon}(\Omega)$ for all $\varepsilon > 0$.

Now if $\gamma f = 0$, then $f \in H_{p,\gamma}^s(\Omega)$ by (5.1), which equals $D_p\big((A_\gamma)^{s/2}\big)$ by (5.3), and hence $u \in D_p\big((A_\gamma)^{s/2+a}\big) = H_{p,\gamma}^{s+2a}(\Omega)$ in view of (5.5) and (5.3).

Conversely, let $u \in H_p^{s+2a}(\Omega)$. Then since we know already that $u \in H_{p,\gamma}^{\frac{1}{p}+2a-\varepsilon}(\Omega)$, we see that $\gamma u = 0$ (taking $\varepsilon < 2a$). Then by (5.3), $u \in H_{p,\gamma}^\sigma(\Omega)$ for $\frac{1}{p} + 2a < \sigma < \min\left\{s + 2a, 2 + \frac{1}{p}\right\}$; such σ exists since $a < 1$. Hence $f \in H_{p,\gamma}^{\sigma-2a}(\Omega)$ with $\sigma - 2a > \frac{1}{p}$ and therefore has $\gamma f = 0$. □

Case 2° is of course only relevant when $a < 1 - \frac{1}{2p}$.

Now one can draw corollaries exactly as in Corollaries 3.5 and 3.6:

Corollary 5.3 *Let u be a solution of* (5.4).

1° *Let $f \in L_p(\Omega)$ with $\frac{n}{p} < 2a$, $2a \notin \left\{\frac{1}{p}, 1 + \frac{1}{p}\right\}$. If $2a - \frac{n}{p} \neq 1$, resp. $= 1$, then the solution u of (2.1) is in $C^{2a-\frac{n}{p}}(\overline{\Omega})$, resp. $C_*^1(\overline{\Omega})$, with $\gamma u = 0$.*

2° *If $f \in L_\infty(\Omega)$, then $u \in C^{2a-0}(\overline{\Omega})$ with $\gamma u = 0$.*

3° *If $f \in C^\alpha(\overline{\Omega})$ with $\gamma f = 0$ for some $\alpha \in]0, 2 - 2a]$, then $u \in C^{\alpha+2a-0}(\overline{\Omega})$ with $\gamma u = 0$.*

In the cases $2a = \frac{1}{p}$ or $1 + \frac{1}{p}$ in 1°, one has at least that $u \in H_{p,\gamma}^{2a-0}(\Omega)$, from which one concludes $u \in C^{2a-\frac{n}{p}-0}(\overline{\Omega})$ with $\gamma u = 0$.

It would be natural to generalize the results of Yagi to boundary problems for higher-order operators A, including integer powers of A_γ (the latter would make it possible to consider larger s in Theorem 5.2 under increased smoothness requirements), but to our knowledge, no such efforts seem to have been made so far.

In the book of Yagi [49], Chapter 16, there are shown similar results for the Neumann problem; here $c(x) \geq c_1 > 0$ in (5.1) and the boundary operator is the conormal derivative

$$Bu = \sum_{j,k=1,\ldots,n} \nu_j \gamma(a_{jk}\partial_k u),$$

where $\nu = \{\nu_1, \ldots, \nu_n\}$ is the normal to $\partial\Omega$. We define

$$H_{p,B}^s(\Omega) = \begin{cases} H_p^s(\Omega) \text{ for } 0 \leq s < 1 + \frac{1}{p}, \\ \left\{u \in H_p^s(\Omega) \mid Bu = 0\right\} \text{ for } 1 + \frac{1}{p} < s \leq 2. \end{cases} \tag{5.6}$$

It follows from [16] Thm. 2.3, for $1 < p < \infty$, that the Neumann realization A_B of A in $L_p(\Omega)$ with domain $D_p(A_B) = H_{p,B}^2(\Omega)$ admits a bounded H^∞-calculus in $L_p(\Omega)$. Then Thm. 16.11 of [49] implies that the fractional powers $(A_B)^a$ in $L_p(\Omega)$ for $0 < a < 1$ have domains

$$D_p((A_B)^a) = \begin{cases} H_p^{2a}(\Omega) & \text{if } 0 \leq 2a < 1 + \frac{1}{p}, \\ H_{p,B}^{2a}(\Omega) & \text{if } 1 + \frac{1}{p} < 2a \leq 2. \end{cases} \tag{5.7}$$

We can now extend the results in Section 3 to this nonsmooth situation, when $s \leq 2 - 2a$, $\alpha \leq 2 - 2a$. The proofs are the same as there, only used in the applicable range.

Theorem 5.4 *Let $0 < a < 1$. Let $f \in H_p^s(\Omega)$ for some $s \in [0, 2 - 2a]$, and assume that $u \in D_p((A_B)^a)$ is a solution of*

$$(A_B)^a u = f. \tag{5.8}$$

Assume that s and $s + 2a$ are different from $1 + \frac{1}{p}$.

1° *If* $s < 1 + \frac{1}{p}$, *then* $u \in H_{p,B}^{s+2a}(\Omega)$.

2° *Let* $1 + \frac{1}{p} < s \le 2 - 2a$. *Then* $u \in H_{p,B}^{1+\frac{1}{p}+2a-\varepsilon}(\Omega)$ *for all* $\varepsilon > 0$. *Moreover,* $u \in H_p^{s+2a}(\Omega)$ *if and only if* $Bf = 0$, *and then in fact* $u \in H_{p,B}^{s+2a}(\Omega)$.

Here 2° is only relevant when $a < \frac{1}{2} - \frac{1}{2p}$.

Corollary 5.5 *Let u be a solution of* (5.8).

1° *Let* $f \in L_p(\Omega)$ *with* $\frac{n}{p} < 2a$, $2a \neq 1 + \frac{1}{p}$. *If* $2a - \frac{n}{p} \neq 1$, *resp.* $= 1$, *then the solution u of* (5.8) *is in* $C^{2a-\frac{n}{p}}(\overline{\Omega})$, *resp.* $C_*^1(\overline{\Omega})$, *with* $Bu = 0$ *if* $2a - \frac{n}{p} > 1$.

2° *If* $f \in L_\infty(\Omega)$, *then* $u \in C^{2a-0}(\overline{\Omega})$, *with* $Bu = 0$ *precisely when* $a > \frac{1}{2}$.

3° *If* $f \in C^\alpha(\overline{\Omega})$ *for some* $\alpha \in]0, 2-2a]$, *with* $Bf = 0$ *if* $\alpha > 1$, *then* $u \in C^{\alpha+2a-0}(\overline{\Omega})$, *with* $Bu = 0$ *if* $\alpha + 2a > 1$.

For $2a = 1 + \frac{1}{p}$ in 1° one gets $C^{2a-\frac{n}{p}-0}(\overline{\Omega})$ instead of $C^{2a-\frac{n}{p}}(\overline{\Omega})$.

Remark 5.6 Whereas the results in Theorems 5.2 and 5.4 for general s are new, those in 1° and 3° of Corollaries 5.3 and 5.5 are comparable to the results of Caffarelli and Stinga [12]. The smoothness assumptions there are up to 1 step weaker than ours. On the other hand, for 1°, the case $2a = \frac{n}{p}$ is not addressed in [12], and the validity of the boundary conditions in the standard sense for u is not discussed. For 3°, our result misses the best Hölder space for u by an ε, but we treat f in the full range $\alpha \le 2 - 2a$, not assuming $\alpha < 1$ on beforehand.

One can moreover deduce results in L_2 Sobolev spaces for more rough domains (Lipschitz or convex) from [49].

6 Overview of boundary problems associated with the fractional Laplacian

For the convenience of the reader, we here go through various boundary value problems associated with $(-\Delta)^a$, $0 < a < 1$. For the problems considered in Sections 6.1 and 6.2, one can consider generalizations where $-\Delta$ is replaced by a variable-coefficient second-order strongly elliptic differential operator. More generally, one can replace $(-\Delta)^a$ by an elliptic pseudodifferential operator P of order $2a$ having the so-called a-transmission property at $\partial\Omega$, cf. [25], [26].

In much of the recent literature, $(-\Delta)^a$ is presented in the form

$$(-\Delta)^a u(x) = c_{n,a}\, \mathrm{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2a}}\, dy. \tag{6.1}$$

This is sometimes generalized by replacing $|y|^{-n-2a}$ by other nonnegative functions $K(y)$, satisfying $K(-y) = K(y)$ and homogeneous of degree $-n-2a$. (Cf. e.g. [37], [38] and their references; in the case where K is C^∞ outside 0, this defines an operator of the type P mentioned above.) More generally, K can be subject to estimates comparing with $|y|^{-n-2a}$.

6.1 The restricted Dirichlet and Neumann fractional Laplacians

The properties of the restricted Dirichlet fractional Laplacian $(-\Delta)^a_{\mathrm{Dir}}$ defined in the introduction were studied e.g. in Blumenthal and Getoor [6], Landkof [34], Hoh and Jacob [29], Kulczycki [33], Chen and Song [14], Jakubowski [31], Silvestre [44], Caffarelli and Silvestre [11], Frank and Geisinger [21], Ros-Oton and Serra [36], [37], Felsinger, Kassmann and Voigt [20], Grubb [25], [26], Bonforte, Sire and Vazquez [8], Servadei and Valdinoci [43], Binlin, Molica Bisci and Servadei [5], and many more papers referred to in these works (see in particular the list in [43]).

The operator acts like $r^+(-\Delta)^a$ applied to functions supported in $\overline{\Omega}$. The domain in $L_2(\Omega)$ is for $a < \frac{1}{2}$ equal to $\dot{H}_2^{2a}(\overline{\Omega})$ (the $H_2^{2a}(\mathbb{R}^n)$-functions supported in $\overline{\Omega}$), and has for $a \geq \frac{1}{2}$ been described in exact form in [25], [26] by

$$D_2\left((-\Delta)^a_{\mathrm{Dir}}\right) = H_2^{a(2a)}(\overline{\Omega}) = \Lambda_+^{(-a)} e^+ \overline{H}_2^a(\Omega). \tag{6.2}$$

Here $\Lambda_+^{(\mu)}$ is a so-called order-reducing operator of order $\mu \in \mathbb{C}$ that preserves support in $\overline{\Omega}$, e^+ extends by zero on $\mathbb{R}^n \setminus \Omega$, and $\overline{H}_p^s(\Omega)$ is the sharper notation for $H_p^s(\Omega)$ used in [25], [26]. Hörmander's spaces $H_p^{\mu(s)}(\overline{\Omega})$ are defined there in general by

$$H_p^{\mu(s)}(\overline{\Omega}) = \Lambda_+^{(-\mu)} e^+ \overline{H}_p^{s-\operatorname{Re}\mu}(\Omega), \qquad \text{for} \quad s - \operatorname{Re}\mu > -1 + 1/p. \tag{6.3}$$

The operator $(-\Delta)^a_{\mathrm{Dir}}$ represents the *homogeneous* Dirichlet problem, and there is an associated well-posed *nonhomogeneous Dirichlet problem* defined on a larger space:

$$\begin{cases} r^+(-\Delta)^a u = f \text{ on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}, \\ \gamma_{a-1,0} u = \varphi \text{ on } \partial\Omega, \end{cases} \tag{6.4}$$

where $\gamma_{a-1,0}u = c_0\big(d^{1-a}u\big)|_{\partial\Omega}$ with $d(x) = \operatorname{dist}(x, \partial\Omega)$. When $f \in H_p^{s-2a}(\Omega)$, the solutions are in spaces $H_p^{(a-1)(s)}(\overline{\Omega})$, which allow a blowup of u (of the form d^{a-1}) at $\partial\Omega$, see [25], [26] and also Abatangelo [1]. The solutions with $\varphi = 0$ are exactly the solutions of the homogeneous Dirichlet problem, lying in $H_p^{a(s)}(\overline{\Omega})$ and behaving like d^a at the boundary.

Likewise, one can define a well-posed *nonhomogeneous Neumann problem* (cf. [25])

$$\begin{cases} r^+(-\Delta)^a u = f \text{ on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}, \\ \gamma_{a-1,1} u = \psi \text{ on } \partial\Omega, \end{cases} \tag{6.5}$$

where $\gamma_{a-1,1}u = c_1\partial_n\big(d(x)^{1-a}u\big)|_{\partial\Omega}$; it has solutions in $H_p^{(a-1)(s)}(\overline{\Omega})$. There is then a *homogeneous Neumann problem*, with $\psi = 0$ in (6.5); its solutions for $f \in H_p^{s-2a}(\Omega)$ lie in a closed subset of $H_p^{(a-1)(s)}(\overline{\Omega})$.

These boundary conditions are *local*; one can also impose *nonlocal* pseudodifferential boundary conditions prescribing $\gamma_0 Bu$ with a pseudodifferential operator B, see [25], Section 4A.

The problems (6.4) and (6.5) are sometimes considered with the condition $\operatorname{supp} u \subset \overline{\Omega}$ replaced by prescription of a nontrivial value g of u on $\mathbb{R}^n \setminus \overline{\Omega}$. It is accounted for e.g. in [25] how such problems are reduced to the case where $g = 0$ as in (6.4), (6.5).

6.2 The spectral Dirichlet and Neumann fractional Laplacians

Fractional powers of realizations of the Laplacian and other elliptic operators have been considered for many years. In the case of a selfadjoint operator in $L_2(\Omega)$, there is an operator-theoretical definition by spectral theory. More general, not necessarily selfadjoint cases can be included, when the powers are defined by a Dunford integral as in (2.2). Moreover, this representation allows a discussion of the analytical structure. The structure of powers of differential operators acting on a manifold without boundary, was cleared up by Seeley [39], who showed that they are classical pseudodifferential operators. The case of realizations A_B on a manifold with boundary was described by Seeley in [41], [42], based on [40]. The resulting operators $(A_B)^a$ have been further analyzed in the book [24], Section 4.4, from which follows that they are sums of a truncated pseudodifferential term $r^+A^a e^+$ and a generalized singular Green operator, having its importance at the boundary; here e^+ denotes extension by zero (on $\mathbb{R}^n \setminus \Omega$). (The detailed analysis of the singular Green term is complicated.) Fractional powers are of interest in differential geometry e.g. for the determination of topological constants such as residues or indices.

The operators have been considered more recently for questions arising in nonlinear PDE. Stinga and Torrea [45], Cabré and Tan [9] for $a = \frac{1}{2}$, and Caffarelli and Stinga [12] for both $(-\Delta_{\mathrm{Dir}})^a$ and $(-\Delta_{\mathrm{Neu}})^a$, show how the spectral fractional Laplacians can be defined on a bounded domain by a generalization of the Caffarelli-Silvestre extension [10] to cylindrical situations. The paper of Servadei and Valdinoci [43], which compares the eigenvalues of $(-\Delta_{\rm Dir})^a$ and $(-\Delta)^a_{\rm Dir}$, contains an extensive list of references to the recent literature, to which we refer. See also Bonforte, Sire and Vazquez [8], Capella, Davila, Dupaigne and Sire [13], and their references.

The regularity analyses of [9], [12] were preceded by that of [41], [42] accounted for above.

It should be noted that the operators $(-\Delta)^a_{\rm Dir}$ and $(-\Delta_{\rm Dir})^a$ are both selfadjoint positive in $L_2(\Omega)$, but they act differently, and their domains differ when $a \ge \frac12$.

For the spectral Dirichlet and Neumann fractional Laplacians there have not been formulated nonhomogeneous boundary problems. In constrast, the restricted Dirichlet and Neumann fractional Laplacians allow nonhomogeneous boundary conditions.

6.3 Two other Neumann cases

For completeness, we moreover mention two further choices of operators associated with the fractional Laplacian and a set Ω, namely operators defined from the sesquilinear forms

$$p_0(u,v) = \frac12 c_{n,a}\int_{\Omega\times\Omega}\frac{(u(x)-u(y))(\bar v(x)-\bar v(y))}{|x-y|^{n+2a}}\,dx\,dy,$$
$$p_1(u,v) = \frac12 c_{n,a}\int_{\mathbb{R}^{2n}\setminus(\complement\Omega\times\complement\Omega)}\frac{(u(x)-u(y))(\bar v(x)-\bar v(y))}{|x-y|^{n+2a}}\,dx\,dy.$$

It is known that $\big(p_0(u,u)+\|u\|^2\big)^{\frac12}$ is equivalent with the norm on $H_2^a(\Omega)$. By a variational construction, p_0 with domain $H_2^a(\Omega)$ gives rise to a selfadjoint operator P_0 in $L_2(\Omega)$, sometimes called "the regional fractional Laplacian". To see how it acts, we note that one has from (6.1), for suitable functions U, V on $\mathbb{R}^n$,

$$((-\Delta)^a U, V)_{\mathbb{R}^n} = \frac12 c_{n,a}\int_{\mathbb{R}^{2n}}\frac{(U(x)-U(y))(\overline V(x)-\overline V(y))}{|x-y|^{n+2a}}\,dx\,dy$$

(the factor $\frac12$ comes in since V appears twice); hence for u, v given on Ω,

$$\begin{aligned}((-\Delta)^a e^+u, e^+v)_{\mathbb{R}^n} &\\ = \frac12 c_{n,a}\int_{\mathbb{R}^{2n}}\frac{(e^+u(x)-e^+u(y))(e^+\bar v(x)-e^+\bar v(y))}{|x-y|^{n+2a}}\,dx\,dy &\\ = p_0(u,v)+\frac12 c_{n,a}\int_{x\in\Omega,y\in\complement\Omega}\frac{u(x)\bar v(x)}{|x-y|^{n+2a}}\,dx\,dy+\frac12 c_{n,a}\int_{y\in\Omega,x\in\complement\Omega}\frac{u(y)\bar v(y)}{|x-y|^{n+2a}}\,dx\,dy &\\ = p_0(u,v)+(wu,v)_\Omega,\ \text{ where } w(x)=c_{n,a}\int_{y\in\complement\Omega}\frac{1}{|x-y|^{n+2a}}\,dx\,dy. &\end{aligned}$$

It follows that the operator P_0 acts like $u\mapsto r^+(-\Delta)^a e^+u - wu$; observe that the function w has a singularity at $\partial\Omega$ (balancing the singularity of the first term). This case appears e.g. in Lieb and Yau [35], Chen and Kim [15], Bogdan, Burdzy and Chen [7]. For $\frac12 < a < 1$, it is shown in Guan [27] how P_0 represents a Neumann condition $(d^{2-2a}\partial_n u)|_{\partial\Omega}=0$. Nonhomogeneous Neumann and Robin problems for the regional fractional Laplacian are studied in Warma [47].

The other choice p_1 has recently been introduced in Dipierro, Ros-Oton and Valdinoci in [18] (formulated for real functions), where it is shown how it defines an operator $r^+(-\Delta)^a$ applied to functions on $\mathbb{R}^n$ satisfying a special condition viewed as a "nonlocal Neumann condition", relating the behavior in $\mathbb{R}^n\setminus\Omega$ to that in Ω. Here one can also define nonhomogeneous nonlocal Neumann problems.

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