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The mixed boundary value problem, Kreĭn resolvent formulas and spectral asymptotic estimates

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ABSTRACT

For a second-order symmetric strongly elliptic operator A on a smooth bounded open set in $\mathbb{R}^n$, the mixed problem is defined by a Neumann-type condition on a part Σ_+ of the boundary and a Dirichlet condition on the other part Σ_-. We show a Kreĭn resolvent formula, where the difference between its resolvent and the Dirichlet resolvent is expressed in terms of operators acting on Sobolev spaces over Σ_+. This is used to obtain a new Weyl-type spectral asymptotics formula for the resolvent difference (where upper estimates were known before), namely $s_j j^{2/(n-1)} \to C_{0,+}^{2/(n-1)}$, where $C_{0,+}$ is proportional to the area of Σ_+, in the case where A is principally equal to the Laplacian.

The mixed boundary value problem for a second-order strongly elliptic symmetric operator A on a smooth bounded open set $\Omega \subset \mathbb{R}^n$ with boundary Σ, in case of the Laplacian also called the Zaremba problem, is defined by a Neumann-type condition on a part of the boundary Σ_+ and a Dirichlet condition on the other part Σ_-. It does not have the regularity of standard elliptic boundary problems (the L_2-domain is at best in $H^{\frac{3}{2}-\varepsilon}(\Omega)$). It has been analyzed with regards to regularity and mapping properties e.g. in Peetre [45,46], Shamir [52], Eskin [14], Pryde [48], Rempel and Schulze [49], Simanca [53], Harutyunyan and Schulze [33].

We shall here study it from the point of view of extension theory for elliptic operators. There has been a recent revival in the interest for connections between abstract extension theories for operators in Hilbert space (as initiated by Krein [36], Vishik [54], Birman [6], Grubb [22] and others) and interpretations to boundary value problems for partial differential operators. Cf. e.g. Amrein and Pearson [3], Pankrashkin [44], Behrndt and Langer [4], Ryzhov [50], Brown, Marletta, Naboko and Wood [13], Alpay and Behrndt [2], Malamud [41], based on boundary triples theory (as developed from the book of Gorbachuk and Gorbachuk [20] and its sources). Other methods are used in the works of Brown, Grubb and Wood [12,28], Posilicano and Raimondi [47], Gesztesy and Mitrea [16–18] (and their references); see also Grubb [30–32] and Abels, Grubb and Wood [1]. One of the interesting aims has been to derive Kreĭn resolvent formulas that link the resolvent of a general operator with the resolvent of a fixed reference operator by expressing the difference in terms of operators connected to the boundary.

For the mixed problem, a Kreĭn resolvent formula connecting the operator to the Dirichlet realization was worked out in [44], based on boundary triples theory. A different formula results from [22,24], see also [12], Sect. 3.2.5. Observations on the connection with the Neumann realization were given in [41]. An upper bound for the spectral behavior of the resolvent difference was shown by Birman in [6].

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In the present paper we shall work out in detail several Kreĭn resolvent formulas for the mixed problem. The primary result is a formula where the difference between the resolvents for the mixed problem and the Dirichlet problem is expressed explicitly in terms of operators acting over the subset Σ_+; this is based on the universal description from [22] in terms of operators between closed subspaces of the nullspace of the maximal operator. In addition, we show some other explicit formulas related to those of [44]. Mixed problems for $-\Delta$ on creased domains are briefly considered, and we establish a Kreĭn formula for quasi-convex Lipschitz domains as defined in [18].

As an application of our primary formula in the smooth case, we show how it leads to a new result giving a Weyl-type spectral asymptotic estimate for the resolvent difference, with the constant defined by an integral over Σ_+; this sharpens considerably the upper estimates known earlier. The proof draws on various results for nonstandard pseudodifferential operators on Σ.

1. Introduction

On a bounded smooth open subset Ω of $\mathbb{R}^n$ with boundary $\partial\Omega = \Sigma$, consider a second-order symmetric differential operator with real coefficients in $C^\infty(\overline{\Omega})$ and an associated sesquilinear form

$$Au = -\sum\nolimits_{j,k=1}^{n} \partial_j\big(a_{jk}(x)\partial_k u\big) + a_0(x)u, \tag{1.1}$$

$$a(u,v) = \sum\nolimits_{j,k=1}^{n} (a_{jk}\partial_k u, \partial_j v) + (a_0 u, v). \tag{1.2}$$

A is assumed strongly elliptic, i.e., $\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geqslant c_0|\xi|^2$ for $x \in \Omega$, $\xi \in \mathbb{R}^n$, with $c_0 > 0$.

Denote $u|_\Sigma = \gamma_0 u$, and $\sum_j n_j\gamma_0(\partial_j u) = \gamma_1 u$, where $\vec{n} = (n_1,\dots,n_n)$ is the interior unit normal to the boundary. Introduce the conormal derivative ν and a variant χ (Neumann-type boundary operators)

$$\nu u = \sum\nolimits_{j,k=1}^{n} n_j\gamma_0(a_{jk}\partial_k u), \qquad \chi u = \nu u - b\gamma_0 u. \tag{1.3}$$

ν enters in the "halfways Green's formula" (for sufficiently smooth functions)

$$(Au, v)_{L_2(\Omega)} - a(u,v) = (\nu u, \gamma_0 v)_{L_2(\Sigma)}. \tag{1.4}$$

Consider the realizations A_γ, A_ν, A_χ resp. A_{χ,Σ_+} of A defined via sesquilinear forms to represent the respective boundary conditions

$$\begin{aligned}
&\gamma_0 u = 0 \quad \text{on } \Sigma, \text{ the Dirichlet condition,}\\
&\nu u = 0 \quad \text{on } \Sigma, \text{ the Neumann condition,}\\
&\chi u = 0 \quad \text{on } \Sigma, \text{ a Robin (Neumann-type) condition,}\\
&\chi u = 0 \quad \text{on } \Sigma_+, \qquad \gamma_0 u = 0 \quad \text{on } \Sigma \setminus \Sigma_+, \text{ a \textit{mixed condition}};
\end{aligned} \tag{1.5}$$

here b is a bounded measurable real function and Σ_+ is a closed subset of Σ. These realizations are selfadjoint, and by addition of a large constant to a_0 we can obtain that they have positive lower bounds. Their resolvents are compact operators. Note that A_χ equals A_ν for $b = 0$.

For a compact operator B in a Hilbert space H, $s_j(B)$ denotes the j-th eigenvalue of $(B^*B)^{\frac12}$ (the j-th s-number or singular value of B), counted with multiplicities.

Birman showed in [6]:

$$s_j\big(A_\chi^{-1} - A_\gamma^{-1}\big) \quad \text{and} \quad s_j\big(A_{\chi,\Sigma_+}^{-1} - A_\gamma^{-1}\big) \quad \text{are } O\big(j^{-2/(n-1)}\big) \text{ for } j \to \infty; \tag{1.6}$$

also valid for exterior domains. The estimate for $A_\chi^{-1} - A_\gamma^{-1}$ was later improved to an asymptotic estimate (in [24] and [9], the latter including exterior domains):

$$\lim_{j\to\infty} s_j\big(A_\chi^{-1} - A_\gamma^{-1}\big) j^{2/(n-1)} = C_0^{2/(n-1)}, \tag{1.7}$$

for smooth b, where

$$C_0 = \frac{1}{(n-1)(2\pi)^{n-1}} \int_\Sigma \int_{|\xi'|=1} \big(\|\tilde{k}^0\|_{L_2(\mathbb{R}_+)} |p^0|^{1/2}\big)^{n-1} d\omega(\xi')\, dx'; \tag{1.8}$$

this has been extended to nonsmooth b in [31] (the ingredients in the formula are explained around Theorem 2.4 there). For the difference with A_{χ,Σ_+}^{-1} an asymptotic estimate does not seem to have been obtained before; it is one of the aims of the present paper.

In Section 2, we briefly recall some elements of the old extension theory from [22,24]. In Section 3, we show how the method of Birman [6] can be used in combination with later estimates to make a small improvement of his result for mixed problems, valid for nonsmooth b and Σ_+.

In Section 4, we analyze the structure of A_{χ,Σ_+} in terms of the characterization from [22] in more detail, describing the operator $L^\lambda : X \to X^*$ that $A_{\chi,\Sigma_+} - \lambda$ corresponds to when $\lambda \in \varrho(A_\gamma)$ (the resolvent set):

Theorem A. *When b and the subset Σ_+ are smooth, then $X = H_0^{-\frac{1}{2}}(\Sigma_+)$, and L^λ acts like minus the Dirichlet-to-Neumann pseudodifferential operator truncated to Σ_+, $-P^\lambda_{\gamma,\chi,+} = -r^+\chi K^\lambda_\gamma e^+$, with domain $D(L^\lambda) \subset H_0^{1-\varepsilon}(\Sigma_+)$ (any $\varepsilon > 0$); here K^λ_γ is the Poisson operator for the Dirichlet problem for $A - \lambda$.*

For $\lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma)$ there is a Kreĭn resolvent formula:

$$(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = -K^\lambda_{\gamma,X}\big(P^\lambda_{\gamma,\chi,+}\big)^{-1}\big(K^{\bar\lambda}_{\gamma,X}\big)^*. \tag{1.9}$$

Several other Kreĭn-type formulas are shown involving the Poisson operators for the Dirichlet or Neumann problems.

In Section 5, we restrict the attention to operators principally like the Laplacian. Here we use methods for nonstandard pseudodifferential operators to deduce from (1.9):

Theorem B. *When $A = -\Delta + a_0(x)$, then for any $\lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma)$,*

$$\lim_{j\to\infty} s_j\big((A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}\big) j^{2/(n-1)} = C_{0,+}^{2/(n-1)}, \tag{1.10}$$

where $C_{0,+}$ is a constant proportional to the area of Σ_+;

$$C_{0,+} = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma_+}\int_{|\xi'|=1} \big(\|\tilde k^0\|_{L_2(\mathbb{R}_+)}\big|p^0\big|^{1/2}\big)^{n-1}\, d\omega(\xi')\, dx'. \tag{1.11}$$

Remark 3.3 and Section 4.3 give informations on cases where Ω is not smooth.

A general technique for extending the estimates to exterior domains can be found in [30].

2. Preliminaries

2.1. Definition of the operators

The spaces $H^s(\Omega)$, $H^s(\Sigma)$ are the standard Sobolev spaces, with the norm denoted $\|u\|_s$; $H_0^s(\Omega)$ (or $H_0^s(\overline{\Omega})$) stands for the space of distributions in $H^s(\mathbb{R}^n)$ with support in $\overline{\Omega}$. We use the notation $(\cdot,\cdot)_{-s,s}$ for the sesquilinear duality between $H^{-s}(\Sigma)$ and $H^s(\Sigma)$, $s \in \mathbb{R}$; it reduces to the L_2-scalar product when applied to functions in $L_2(\Sigma)$.

It is known e.g. from Lions and Magenes [39] that γ_0 resp. γ_1, ν extend to continuous mappings from $H^s(\Omega) \cap D(A_{\max})$ to $H^{s-\frac{1}{2}}(\Sigma)$ resp. $H^{s-\frac{3}{2}}(\Sigma)$, any $s \geqslant 0$, allowing extensions of Green's formulas. In particular, for $u \in H^1(\Omega) \cap D(A_{\max})$, $v \in H^1(\Omega)$, (1.4) holds with the scalar product in $L_2(\Sigma)$ replaced by the sesquilinear duality between $H^{-\frac{1}{2}}(\Sigma)$ and $H^{\frac{1}{2}}(\Sigma)$.

The realizations of A are the linear operators $\widetilde{A}$ satisfying $A_{\min} \subset \widetilde{A} \subset A_{\max}$, where $A_{\min}$ and $A_{\max}$ act like A with domains $D(A_{\min}) = H_0^2(\Omega)$ resp. $D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}$; $A_{\min}$ is the closure of $A|_{C_0^\infty}$, and $A_{\max} = A^*_{\min}$.

Our assumptions imply that

$$a(u,u) \geqslant c\|u\|^2_{H^1(\Omega)} - k\|u\|^2_{L_2(\Omega)} \quad \text{for } u \in H^1(\Omega), \tag{2.1}$$

with $c > 0$, $k \geqslant 0$. Then the realizations A_γ, etc., can all be defined via variational constructions from sesquilinear forms, namely:

$$\begin{aligned}
&a_\gamma(u,v) = a(u,v) \quad \text{on } D(a_\gamma) = H_0^1(\Omega) \text{ leads to } A_\gamma,\\
&a_\nu(u,v) = a(u,v) \quad \text{on } D(a_\nu) = H^1(\Omega) \text{ leads to } A_\nu,\\
&a_\chi(u,v) = a(u,v) + (b\gamma_0 u, \gamma_0 v)_{L_2(\Sigma)} \quad \text{on } D(a_\chi) = H^1(\Omega) \text{ leads to } A_\chi,\\
&a_{\chi,\Sigma_+}(u,v) = a(u,v) + (b\gamma_0 u, \gamma_0 v)_{L_2(\Sigma_+)} \quad \text{on } D(a_{\chi,\Sigma_+}) = H^1_{\Sigma_+}(\Omega) \text{ leads to } A_{\chi,\Sigma_+};
\end{aligned} \tag{2.2}$$

here

$$H^1_{\Sigma_+}(\Omega) = \big\{u \in H^1(\Omega) \bigm| \operatorname{supp}\gamma_0 u \subset \Sigma_+\big\}. \tag{2.3}$$

The last case (that covers the two preceding cases when $\Sigma_+ = \Sigma$ or $b = 0$) is explained below. Since $\|\gamma_0 u\|^2_{L_2(\Sigma)} \leqslant c'\|u\|^2_{\frac{3}{4}} \leqslant \varepsilon\|u\|_1^2 + C(\varepsilon)\|u\|_0^2$ for any ε, we infer from (2.1) that when K is a constant $\geqslant \operatorname{ess\,sup}|b(x)|$, $a_\chi(u,u) \geqslant a(u,u) - K\|\gamma_0 u\|_0^2$, and hence

$$a_\chi(u,u) \geqslant c_1\|u\|_1^2 - k_1\|u\|_0^2, \quad \text{for } u \in H^1(\Omega),$$

where $c_1 < c$ is close to c and $k_1 \geqslant k$ is a large constant. Then each of the sesquilinear forms in (2.2) satisfies such an inequality on its domain. Defining $\chi_K = \nu + K\gamma_0$ (the case $b = -K$), we also have that $a_{\chi_K}(u,v) = a(u,v) - K(\gamma_0 u, \gamma_0 v)_\Sigma$ satisfies such an inequality. We can (after a fixed choice of the constant K) replace A by $A + k_1$, i.e. add the constant k_1 to the coefficient a_0 in (2.1); then all the resulting sesquilinear forms, including a_{χ_K}, are positive. *For simplicity, $A + k_1$ and $a(u,v) + k_1 \cdot (u,v)$ will in the following again be denoted A and $a(u,v)$.*

We now recall the construction of A_{χ,Σ_+}. The sesquilinear form a_{χ,Σ_+} on $V = H^1_{\Sigma_+}(\Omega)$ in $H = L_2(\Omega)$ defines an operator A_{χ,Σ_+} by

$$\begin{aligned} &D(A_{\chi,\Sigma_+}) = \bigl\{u \in V \bigm| \exists f \in H \text{ such that } a_{\chi,\Sigma_+}(u,v) = (f,v) \text{ for all } v \in V\bigr\}, \\ &A_{\chi,\Sigma_+}u = f. \end{aligned} \tag{2.4}$$

By J.L. Lions' version of the Lax–Milgram lemma, as recalled e.g. in [29], Sect. 12.4, this defines a selfadjoint operator with the same lower bound as a_{χ,Σ_+}. Clearly, A_{χ,Σ_+} extends $A|_{C_0^\infty}$, hence $A_{\min}$, and in view of the selfadjointness is a restriction of $A^*_{\min} = A_{\max}$, so it is a realization of A. By (1.4),

$$(Au,v) - a_{\chi,\Sigma_+}(u,v) = (\nu u, \gamma_0 v)_{-\frac12,\frac12} - (b\gamma_0 u, \gamma_0 v)_{L_2(\Sigma)} = (\chi u, \gamma_0 v)_{-\frac12,\frac12}, \tag{2.5}$$

when $v \in H^1_{\Sigma_+}(\Omega)$. Thus, when $u \in D(A_{\max}) \cap H^1_{\Sigma_+}(\Omega)$, $a_{\chi,\Sigma_+}(u,v) = (Au,v)$ holds for all $v \in H^1_{\Sigma_+}(\Omega)$ precisely when the distribution χu vanishes on the $H^{\frac12}$-functions supported in Σ_+. In this sense, A_{χ,Σ_+} represents the boundary condition $\gamma_0 u = 0$ on $\Sigma \setminus \Sigma_+$, $\chi u = 0$ on Σ_+.

The boundary condition can be made more explicit when Σ_+ is a smooth subset of Σ. We then set $\Sigma_- = \Sigma \setminus \Sigma_+^\circ$, and have that $\Sigma = \Sigma_+ \cup \Sigma_-$, with $\Sigma_+^\circ \cup \Sigma_-^\circ$ dense in Σ. Then for $s \in \mathbb{R}$, we denote by $H_0^s(\Sigma_+)$ the closed subspace of $H^s(\Sigma)$ consisting of the elements with support in Σ_+. Here $C_0^\infty(\Sigma_+^\circ)$ is a dense subspace, and it should be noted that for $s + \frac12 \in \mathbb{N}$, the space is different from the space obtained by closure of $C_0^\infty(\Sigma_+^\circ)$ in $H^s(\Sigma_+^\circ)$. For $s \in \mathbb{R}$, the latter space $H^s(\Sigma_+^\circ)$ consists of the restrictions to Σ_+° of distributions in $H^s(\Sigma)$, provided with the quotient norm. The spaces $H_0^s(\Sigma_+)$ and $H^{-s}(\Sigma_+^\circ)$ are dual with respect to an extension of the L_2 scalar product, for all $s \in \mathbb{R}$.

Lemma 2.1. *When Σ_+ is smooth,*

$$D(A_{\chi,\Sigma_+}) = \bigl\{u \in H^1(\Omega) \cap D(A_{\max}) \bigm| \gamma_0 u \in H_0^{\frac12}(\Sigma_+),\ \chi u = 0 \text{ on } \Sigma_+^\circ\bigr\}. \tag{2.6}$$

Proof. Note first that $\gamma_0 H^1_{\Sigma_+}(\Omega) = H_0^{\frac12}(\Sigma_+)$, since $\gamma_0 H^1(\Omega) = H^{\frac12}(\Sigma)$ and $H_0^{\frac12}(\Sigma_+)$ is the subspace of $H^{\frac12}(\Sigma)$ consisting of the functions supported in Σ_+. Moreover, $C_0^\infty(\Sigma_+^\circ)$ is dense in $H_0^{\frac12}(\Sigma_+)$ and is the image by γ_0 of the space of $C^\infty(\overline{\Omega})$-functions ψ with $\gamma_0\psi$ supported in Σ_+°.

When u is in the right-hand side of (2.6), then

$$\langle \chi u, \gamma_0\psi\rangle = 0 \quad \text{for } \gamma_0\psi \in C_0^\infty\bigl(\Sigma_+^\circ\bigr);$$

hence by the denseness of $C_0^\infty(\Sigma_+^\circ)$ in $H_0^{\frac12}(\Sigma_+)$,

$$(\chi u, \gamma_0 v)_{-\frac12,\frac12} = 0 \quad \text{for } v \in H^1_{\Sigma_+}(\Omega);$$

so $u \in D(A_{\chi,\Sigma_+})$. Conversely, if $u \in D(A_{\chi,\Sigma_+})$, then $u \in D(A_{\max}) \cap H^1_{\Sigma_+}(\Omega)$ implies $\gamma_0 u \in H_0^{\frac12}(\Sigma_+)$, and since χu vanishes on $H^{\frac12}$-functions supported in Σ_+, it vanishes in particular on $C_0^\infty(\Sigma_+^\circ)$, i.e., $\nu u - b\gamma_0 u = 0$ on Σ_+°. $\square$

2.2. Abstract extension theories

We shall now connect the operators with the theory of Kreĭn [36], Vishik [54], Birman [5], Grubb [22,23] (the latter also recalled in [12], the abstract part in [29], Ch. 13). The theory of [22] extends and completes that of [54] by giving a universal description of all adjoint pairs of extensions of a dual pair of injective operators. We here just briefly recall how it describes the extensions $\widetilde{A}$ of a symmetric positive operator $A_{\min}$ with $A_{\min} \subset \widetilde{A} \subset A_{\max} = A^*_{\min}$.

The operators act in a Hilbert space H (in the concrete application, $H = L_2(\Omega)$). Let A_γ be the Friedrichs extension of $A_{\min}$ (in the application it will be the Dirichlet realization), and let $Z = \ker A_{\max}$. Define the decomposition

$$D(A_{\max}) = D(A_\gamma) \dot{+} Z, \quad \text{with notation } u = u_\gamma + u_\zeta, \tag{2.7}$$

where $u_\gamma = \mathrm{pr}_\gamma u = A_\gamma^{-1} A_{\max} u$, $u_\zeta = u - u_\gamma = (1 - \mathrm{pr}_\gamma)u = \mathrm{pr}_\zeta u$. This is used in [22] to show that there is a 1–1 correspondence between the closed realizations $\widetilde{A}$ of A and the closed, densely defined operators between closed subspaces of Z:

$$\widetilde{A} \text{ closed } \leftrightarrow \begin{cases} V, W \subset Z, & \text{closed subspaces,} \\ T : V \to W & \text{closed, densely defined,} \end{cases} \tag{2.8}$$

where $D(T) = \mathrm{pr}_\zeta D(\widetilde{A})$, $X = \overline{D(T)}$, $W = \overline{\mathrm{pr}_\zeta D(\widetilde{A}^*)}$, and $Tu_\zeta = \mathrm{pr}_W(A_{\max} u)$ (here pr_W denotes orthogonal projection onto W). The operator $\widetilde{A}^*$ corresponds similarly to $T^* : W \to V$, and many properties carry over between $\widetilde{A}$ and T. For example, $\widetilde{A}$ is invertible (i.e. bijective) if and only if T is so, and then we have an abstract resolvent formula:

$$\widetilde{A}^{-1} = A_\gamma^{-1} + \mathrm{i}_V T^{-1} \mathrm{pr}_W, \tag{2.9}$$

where i_V denotes the injection $V \hookrightarrow H$.

In particular, $\widetilde{A}$ is selfadjoint if and only if: $V = W$ and $T : V \to V$ is selfadjoint. Then in the invertible case,

$$\widetilde{A}^{-1} = A_\gamma^{-1} + \mathrm{i}_V T^{-1} \mathrm{pr}_V . \tag{2.10}$$

Positivity of $\widetilde{A}$ holds if and only if T is positive.

For the positive selfadjoint operators, there is also a connection between the associated sesquilinear forms. (When S is a positive selfadjoint operator in a Hilbert space H, the associated sesquilinear form s has as its domain $D(s)$ the completion of $D(S)$ in the norm $(Su, u)^{\frac{1}{2}}$, stronger than the H-norm; here $D(s) \subset H$, and the form $s(u, v)$ is the extension by continuity of (Su, v) to $D(s)$. Then S is defined from s by the Lax–Milgram construction.) When $\widetilde{A}$ is positive selfadjoint, corresponding to the positive selfadjoint operator T in V, the associated sesquilinear form $\tilde{a}$ can be written

$$\tilde{a}(u, v) = a_\gamma(u_\gamma, v_\gamma) + t(u_\zeta, v_\zeta) \quad \text{on } D(\tilde{a}) = D(a_\gamma) \dot{+} D(t), \tag{2.11}$$

where t on $D(t) \subset V$ is the sesquilinear form associated with T; the decomposition $u = u_\gamma + u_\zeta$ used here is a continuous extension to $D(a_\gamma) \dot{+} Z$ of the decomposition (2.7) above.

The description of selfadjoint extensions in terms of sesquilinear forms is already found in [36] and [5]; [23] moreover treats nonselfadjoint extensions.

Much of the theory holds unchanged if we replace the "reference operator" A_γ by another selfadjoint positive realization of A, say A_ν (which will in the application be taken as the Neumann realization A_ν). There is again a decomposition

$$D(A_{\max}) = D(A_\nu) \dot{+} Z, \quad \text{say with notation } u = u_\nu + u_{\zeta,1},$$

where $u_\nu = \mathrm{pr}_\nu u = A_\nu^{-1} A_{\max} u$, $u_{\zeta,1} = u - u_\nu = (1 - \mathrm{pr}_\nu)u = \mathrm{pr}_{\zeta,1} u$, and there is a 1–1 correspondence

$$\widetilde{A} \text{ closed } \leftrightarrow \begin{cases} V_1, W_1 \subset Z, & \text{closed subspaces,} \\ T_1 : V_1 \to W_1 & \text{closed, densely defined,} \end{cases} \tag{2.12}$$

where $D(T_1) = \mathrm{pr}_{\zeta,1} D(\widetilde{A})$, $X_1 = \overline{D(T_1)}$, $W_1 = \overline{\mathrm{pr}_{\zeta,1} D(\widetilde{A}^*)}$, and $T_1 u_{\zeta,1} = \mathrm{pr}_{W_1}(A_{\max} u)$; again $\widetilde{A}$ is selfadjoint or invertible if and only if T_1 is so, and in the invertible case,

$$\widetilde{A}^{-1} = A_\nu^{-1} + \mathrm{i}_{V_1} T_1^{-1} \mathrm{pr}_{W_1}. \tag{2.13}$$

However, positivity does not in general carry over between $\widetilde{A}$ and T_1, and the information on associated sesquilinear forms does not generalize to this situation, since those facts depended on A_γ being the Friedrichs extension of $A_{\min}$.

2.3. Concrete boundary conditions. Dirichlet reference operator

We now explain the interpretation to concrete boundary conditions worked out in [22,24]. Along with (1.4) we have the full Green's formula

$$(Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} = (\nu u, \gamma_0 v)_{L_2(\Sigma)} - (\gamma_0 u, \nu v)_{L_2(\Sigma)}, \quad \text{for } u, v \in H^2(\Omega); \tag{2.14}$$

it extends e.g. to $u \in D(A_{\max})$, $v \in H^2(\Omega)$, with the $L_2(\Sigma)$-scalar products replaced by suitable Sobolev space dualities, but it cannot be extended to $u, v \in D(A_{\max})$.

Denote by K_γ resp. K_ν the Poisson operator solving the Dirichlet problem resp. Neumann problem

$$Au = 0 \quad \text{in } \Omega, \text{ with } \gamma_0 u = \varphi, \text{ resp. } \nu u = \psi;$$

they have the mapping properties

$$K_\gamma : H^{s-\frac{1}{2}}(\Sigma) \to H^s(\Omega), \qquad K_\nu : H^{s-\frac{3}{2}}(\Sigma) \to H^s(\Omega), \quad \text{for all } s \in \mathbb{R}.$$

In particular, γ_0 and ν define *homeomorphisms* of Z onto $H^{-\frac{1}{2}}(\Sigma)$ resp. $H^{-\frac{3}{2}}(\Sigma)$, with K_γ resp. K_ν acting as inverses.

Let $\widetilde{A}$ correspond to $T : V \to W$ as in (2.8). Let $X = \gamma_0(V)$, $Y = \gamma_0(W)$, closed subspaces of $H^{-\frac{1}{2}}(\Sigma)$, and introduce the notation for the connecting homeomorphisms

$$\gamma_V : V \overset{\sim}{\to} X, \qquad \gamma_W : W \overset{\sim}{\to} Y. \tag{2.15}$$

By use of these homeomorphisms, $T : V \to W$ is carried over to a map $L : X \to Y^*$:

$$\begin{array}{ccc} V & \xrightarrow[\gamma_V]{\sim} & X \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle L} \\ W & \xleftarrow[\gamma_W^*]{\sim} & Y^* \end{array} \qquad D(L) = \gamma_0 D(T) = \gamma_0 D(\widetilde{A}).$$

In other words,

$$L = \left(\gamma_W^*\right)^{-1} T \gamma_V^{-1}.$$

In the case where $\widetilde{A}$ is invertible, the abstract resolvent formula (2.9) carries over to the formula:

$$\widetilde{A}^{-1} = A_\gamma^{-1} + K_{\gamma,X} L^{-1} (K_{\gamma,Y})^* \tag{2.16}$$

where

$$K_{\gamma,X} = \mathrm{i}_V \gamma_V^{-1} : X \to V \subset H, \qquad (K_{\gamma,Y})^* = \left(\gamma_W^*\right)^{-1} \mathrm{pr}_W : H \to Y^*; \tag{2.17}$$

(2.16) is a *Kreĭn resolvent formula*. In particular, if $V = W = Z$, then $X = Y = H^{-\frac{1}{2}}(\Sigma)$, and (2.16) takes the form

$$\widetilde{A}^{-1} = A_\gamma^{-1} + K_\gamma L^{-1} K_\gamma{}^*, \tag{2.18}$$

where L goes from $D(L) \subset H^{-\frac{1}{2}}(\Sigma)$ to $H^{\frac{1}{2}}(\Sigma)$.

To see how L enters in a concrete boundary condition for $\widetilde{A}$ we define some additional operators, namely the *Dirichlet-to-Neumann* and *Neumann-to-Dirichlet* pseudodifferential operators (ψdo's) $P_{\gamma,\nu}$ and $P_{\nu,\gamma}$, and the associated *reduced trace operators* Γ_ν and Γ_γ:

$$\begin{aligned} &P_{\gamma,\nu} = \nu K_\gamma, \quad \psi \text{ do of order } 1, \qquad \Gamma_\nu = \nu - P_{\gamma,\nu}\gamma_0 : D(A_{\max}) \to H^{\frac{1}{2}}(\Sigma); \\ &P_{\nu,\gamma} = \gamma_0 K_\nu, \quad \psi \text{ do of order } -1, \qquad \Gamma_\gamma = \gamma_0 - P_{\nu,\gamma}\nu : D(A_{\max}) \to H^{\frac{3}{2}}(\Sigma). \end{aligned} \tag{2.19}$$

(We here use the notation of the pseudodifferential boundary operator calculus, initiated by Boutet de Monvel [10] and further developed in [26,27], see also [29].) More generally, $P_{\beta,\beta'}$ denotes the mapping from βu to $\beta' u$, when $u \in Z$ is uniquely determined from βu.

The reduced trace operators are used to establish generalized Green's formulas valid for $u, v \in D(A_{\max})$:

$$\begin{aligned} (Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} &= (\Gamma_\nu u, \gamma_0 v)_{\frac{1}{2},-\frac{1}{2}} - (\gamma_0 u, \Gamma_\nu v)_{-\frac{1}{2},\frac{1}{2}}, \\ (Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} &= (\nu u, \Gamma_\gamma v)_{-\frac{3}{2},\frac{3}{2}} - (\Gamma_\gamma u, \nu v)_{\frac{3}{2},-\frac{3}{2}}. \end{aligned} \tag{2.20}$$

One can then show:

$D(\widetilde{A})$ consists of the functions $u \in D(A_{\max})$ that satisfy:

$$\gamma_0 u \in D(L), \qquad (\Gamma_\nu u, \varphi)_{\frac{1}{2},-\frac{1}{2}} = (L\gamma_0 u, \varphi)_{Y^*,Y} \quad \text{for all } \varphi \in Y. \tag{2.21}$$

The second condition may be rewritten as $\mathrm{i}_Y^* \Gamma_\nu u = L\gamma_0 u$, where $\mathrm{i}_Y^* : H^{\frac{1}{2}}(\Sigma) \to Y^*$ is the adjoint of $\mathrm{i}_Y : Y \hookrightarrow H^{-\frac{1}{2}}(\Sigma)$. By the definition of Γ_ν, this can be written:

$$\mathrm{i}_Y^* \nu u = \left(L + \mathrm{i}_Y^* P_{\gamma,\nu}\right)\gamma_0 u. \tag{2.22}$$

In the case where $X = Y = H^{-\frac{1}{2}}(\Sigma)$, this is simply a Neumann-type condition

$$\nu u = C\gamma_0 u, \quad \text{where } C = L + P_{\gamma,\nu}. \tag{2.23}$$

In the present paper we are more interested in a genuine subspace case, where $X = H_0^{-\frac{1}{2}}(\Sigma_+)$; we return to that below.

2.4. Neumann reference operator

For the abstract theory using A_ν as the reference operator, we get slightly different but analogous formulas:

Let $\widetilde{A}$ correspond to $T_1 : V_1 \to W_1$ as in (2.12). We now set $X_1 = \nu(V_1)$, $Y_1 = \nu(W_1)$, closed subspaces of $H^{-\frac{3}{2}}(\Sigma)$, and denote the connecting homeomorphisms

$$\nu_{V_1} : V_1 \xrightarrow{\sim} X_1, \qquad \nu_{W_1} : W_1 \xrightarrow{\sim} Y_1. \tag{2.24}$$

Now $T_1 : V_1 \to W_1$ is carried over to the map $L_1 : X_1 \to Y_1^*$ defined by

$$L_1 = \big(\nu_{W_1}^*\big)^{-1} T_1 \nu_{V_1}^{-1}. \tag{2.25}$$

In the invertible case, the abstract resolvent formula (2.13) carries over to the formula:

$$\widetilde{A}^{-1} = A_\nu^{-1} + K_{\nu,X_1} L_1^{-1} K_{\nu,Y_1}{}^* \tag{2.26}$$

where $K_{\nu,X_1} = \mathrm{i}_{V_1} \nu_{V_1}^{-1} : X_1 \to V_1 \subset H$, $(K_{\nu,Y_1})^* = (\nu_{W_1}^*)^{-1} \operatorname{pr}_{W_1} : H \to Y_1$; another Kreĭn resolvent formula. In particular, if $V_1 = W_1 = Z$, then $X_1 = Y_1 = H^{-\frac{3}{2}}(\Sigma)$, and (2.26) takes the form

$$\widetilde{A}^{-1} = A_\nu^{-1} + K_\nu L_1^{-1} K_\nu{}^*, \tag{2.27}$$

where L_1 goes from $D(L_1) \subset H^{-\frac{3}{2}}(\Sigma)$ to $H^{\frac{3}{2}}(\Sigma)$.

The interpretation of $\widetilde{A}$ as defined by a boundary condition is here based on the second line of (2.20) and goes as follows: $D(\widetilde{A})$ consists of the functions $u \in D(A_{\max})$ that satisfy the boundary condition

$$\nu u \in D(L_1), \qquad -(\Gamma_\gamma u, \varphi)_{\frac{3}{2},-\frac{3}{2}} = (L_1 \nu u, \varphi)_{Y_1^*, Y_1} \quad \text{for all } \varphi \in Y_1. \tag{2.28}$$

Here the second condition is rewritten as $\mathrm{i}_{Y_1}^* \Gamma_\gamma u = -L_1 \nu u$, or

$$\mathrm{i}_{Y_1}^* \gamma_0 u = \big(-L_1 + \mathrm{i}_{Y_1}^* P_{\nu,\gamma}\big)\nu u. \tag{2.29}$$

In the case where $X_1 = Y_1 = H^{-\frac{3}{2}}(\Sigma)$, this is a "Dirichlet-type" condition

$$\gamma_0 u = C_1 \nu u, \quad \text{where } C_1 = -L_1 + P_{\nu,\gamma}. \tag{2.30}$$

We shall see later that the mixed problem can be written in this form (after a replacement of ν by $\nu + K\gamma_0$, if necessary).

In the above analysis we assumed A_γ resp. A_ν positive, so that $0 \in \varrho(A_\gamma)$ resp. $0 \in \varrho(A_\nu)$. Clearly, by addition of real constants to A this covers the realizations of $A - \lambda$ for $-\lambda$ large positive. The formulation was just chosen for simplicity of notation; the theory of [22] in fact works for any $\lambda \in \varrho(A_\gamma)$ resp. $\lambda \in \varrho(A_\nu)$. For general λ one uses the nullspaces $Z_\lambda = \ker(A_{\max} - \lambda)$ and $Z_{\bar\lambda} = \ker(A_{\max} - \bar\lambda)$. For the various spaces, operators and auxiliary Poisson, pseudodifferential and trace operators, the λ-dependence is indicated by

$$V_\lambda,\ W_{\bar\lambda},\ L^\lambda,\ K_\gamma^\lambda,\ K_\gamma^{\bar\lambda},\ P_{\gamma,\nu}^\lambda,\ P_{\nu,\gamma}^\lambda,\ \Gamma_\nu^\lambda, \quad \text{etc.} \tag{2.31}$$

The λ-dependent formulas are explained in detail in [12] (based on methods from [24]), see also [1] for notation. There is an important point here, namely that $X = \gamma_0 V_\lambda$ and $Y = \gamma_0 W_{\bar\lambda}$ are *independent of* λ. Moreover $D(L^\lambda) = D(L^0)$, and $L^\lambda - L^0$ acts as the *bounded* operator $\mathrm{i}_Y^*(P_{\gamma,\nu}^0 - P_{\gamma,\nu}^\lambda)$. Related statements hold for $L_1^\lambda : X_1 \to Y_1$. The Kreĭn resolvent formulas have the form:

$$\begin{aligned} (\widetilde{A} - \lambda)^{-1} &= (A_\gamma - \lambda)^{-1} + K_{\gamma,X}^\lambda \big(L^\lambda\big)^{-1} \big(K_{\gamma,Y}^{\bar\lambda}\big)^* \quad \text{when } \lambda \in \varrho(A_\gamma) \cap \varrho(\widetilde{A}), \\ (\widetilde{A} - \lambda)^{-1} &= (A_\nu - \lambda)^{-1} + K_{\nu,X_1}^\lambda \big(L_1^\lambda\big)^{-1} \big(K_{\nu,Y_1}^{\bar\lambda}\big)^* \quad \text{when } \lambda \in \varrho(A_\nu) \cap \varrho(\widetilde{A}). \end{aligned} \tag{2.32}$$

Other Kreĭn resolvent formulas have been established e.g. in Malamud and Mogilevskii [40,41], Pankrashkin [44], Behrndt and Langer [4], Alpay and Behrndt [2], Gesztesy and Mitrea [16–18], Brown, Marletta, Naboko and Wood [13], Posilicano and Raimondi [47].

Remark 2.2. The theory recalled above has, in the study of "pure" boundary conditions (of Neumann-type $\nu u = C\gamma_0 u$ or of Dirichlet-type $\gamma_0 u = C_1 \nu u$), much in common with the representations of boundary value problems based on boundary triples theory. It is when *subspaces* V, W of Z occur that our theory differs markedly from the others, which obtain a generalization by allowing *relations* instead of operators.

3. Birman's method revisited

The correspondence (2.8) with A_γ as reference operator is used here. We have that $D(A_\gamma) = H_0^1(\Omega) \cap H^2(\Omega)$ and $D(a_\gamma) = H_0^1(\Omega)$. For A_χ, the decomposition in (2.11) gives $D(a_\chi) = H^1(\Omega) = H_0^1(\Omega) \dot{+} Z^1$, where $Z^1 = Z \cap H^1(\Omega)$. The corresponding operator T_χ is defined from the sesquilinear form t_χ obtained by restricting a_χ to Z^1 in Z; T_χ is a selfadjoint unbounded positive operator in Z with domain dense in Z^1. For the mixed problem, $D(a_{\chi,\Sigma_+}) = H_0^1(\Omega) \dot{+} Z^1_{\Sigma_+}$, where $Z^1_{\Sigma_+} = Z \cap H^1_{\Sigma_+}(\Omega)$ (cf. (2.3)); the corresponding operator T_{χ,Σ_+} is a selfadjoint operator in $Z_{\Sigma_+} = \overline{Z^1_{\Sigma_+}}$ ($L_2(\Omega)$-closure) with domain dense in $Z^1_{\Sigma_+}$.

There are bounded, in fact compact, inverses T_χ^{-1} on Z, resp. T^{-1}_{χ,Σ_+} on Z_{Σ_+}.

When a general T is derived from the form $t = \tilde{a}|_{D(t)}$ and T^{-1} is compact nonnegative, then the eigenvalues are determined by the minimum-maximum principle from Rayleigh quotients:

$$\mu_j\big(T^{-1}\big) = \min_{U \subset D(t), \dim U = j-1} \ \max_{z \perp U, z \in D(t)\setminus\{0\}} \frac{\|z\|_0^2}{\tilde{a}(z,z)}. \tag{3.1}$$

This principle was used in Birman [6] to reduce the proof of upper estimates of the $\mu_j(T^{-1})$ for each of the boundary conditions (1.5) to simpler cases where it could be found by computation.

We shall here show how the principle leads to a lim sup estimate for the mixed problem. Consider a_{χ,Σ_+} and the Robin case $a_{\chi K}$ (where b is replaced by $-K$, cf. Section 2.1). Let the corresponding operators and forms defined on subspaces of Z be denoted T_{χ,Σ_+} and $T_{\chi K}$, resp. t_{χ,Σ_+} and $t_{\chi K}$. Here $D(t_{\chi K}) = Z^1$, and $D(t_{\chi,\Sigma_+}) = Z^1_{\Sigma_+} \subset Z^1$. Then

$$\begin{aligned}
\mu_j\big(T^{-1}_{\chi,\Sigma_+}\big) &= \min_{U \subset Z^1_{\Sigma_+}, \dim U = j-1} \ \max_{z \perp U, z \in Z^1_{\Sigma_+}\setminus\{0\}} \frac{\|z\|_0^2}{a(z,z) + (b\gamma_0 z, \gamma_0 z)_{\Sigma_+}} \\
&\leqslant \min_{U \subset Z^1, \dim U = j-1} \ \max_{z \perp U, z \in Z^1\setminus\{0\}} \frac{\|z\|_0^2}{a(z,z) + (b\gamma_0 z, \gamma_0 z)_{\Sigma_+}} \\
&\leqslant \min_{U \subset Z^1, \dim U = j-1} \ \max_{z \perp U, z \in Z^1\setminus\{0\}} \frac{\|z\|_0^2}{a(z,z) - K\|\gamma_0 z\|^2_{L_2(\Sigma)}} \\
&= \mu_j\big(T^{-1}_{\chi K}\big).
\end{aligned} \tag{3.2}$$

Birman showed in [6] that $\mu_j(T^{-1}_{\chi K})$, and hence also $\mu_j(T^{-1}_{\chi,\Sigma_+})$, is $O(j^{-2/(n-1)})$ for $j \to \infty$. It is noteworthy that this included the mixed problem.

In the finer asymptotic estimate (1.7)–(1.8), $p^0(x',\xi')$ denotes the principal symbol of $P_{\gamma,\nu}$ and $\tilde{k}^0(x',\xi',\xi_n)$ is the principal symbol-kernel of K_γ; the derivation of the formula is explained in [31], Th. 2.4. Applying (1.7)–(1.8) to $T_{\chi K}$ we can now get a lim sup estimate using (3.2):

Proposition 3.1. *The nonzero eigenvalues of $A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma$ satisfy, with C_0 from* (1.8),

$$\limsup_{j\to\infty} \mu_j\big(A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma\big) j^{2/(n-1)} \leqslant C_0^{2/(n-1)}. \tag{3.3}$$

Proof. From (1.7) with $b = -K$ follows in view of (3.2):

$$\begin{aligned}
\limsup_{j\to\infty} \mu_j\big(T^{-1}_{\chi,\Sigma_+}\big) j^{2/(n-1)} &\leqslant \limsup_{j\to\infty} \mu_j\big(T^{-1}_{\chi K}\big) j^{2/(n-1)} \\
&= \lim\nolimits_{j\to\infty} \mu_j\big(A^{-1}_{\chi K} - A^{-1}_\gamma\big) j^{2/(n-1)} \\
&= C_0^{2/(n-1)};
\end{aligned} \tag{3.4}$$

we have here applied formula (2.10) with $\widetilde{A} = A_{\chi K}$. Similarly, $A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma$ and T^{-1}_{χ,Σ_+} have the same nonzero eigenvalues, so the result follows. $\square$

We also get a spectral estimate for the eigenvalues of A^{-1}_{χ,Σ_+} itself:

Corollary 3.2. *The eigenvalues of A_{χ,Σ_+} satisfy*:

$$\mu_j\big(A^{-1}_{\chi,\Sigma_+}\big) - C_A^{2/n} j^{-2/n} \quad \text{is } O\big(j^{-(1+1/(n+1))2/n}\big) \text{ for } j \to \infty, \tag{3.5}$$

where

$$C_A = (2\pi)^{-n} \int_{x\in\Omega,\, a^0(x,\xi)<1} dx\, d\xi. \tag{3.6}$$

Proof. It is known (cf. e.g. [34], Sect. 29.3) that the spectrum of A_γ satisfies the asymptotic estimate

$$\mu_j\big(A_\gamma^{-1}\big) - C_A^{2/n} j^{-2/n} \quad \text{is } O\big(j^{-3/n}\big) \text{ for } j \to \infty, \tag{3.7}$$

with C_A defined by (3.6) (the spectral estimate is formulated for the counting function in [34], but carries over to the above form, cf. e.g. [27], Lemma A.5). We shall apply a perturbation result to this estimate, using (3.3) and (2.10) with $\widetilde{A} = A_{\chi,\Sigma_+}$.

Recall from [26], Prop. 6.1 (or [27], Lemma A.6), that when B and B' are compact operators satisfying for $j \to \infty$, with $p > q > 0$, $p > r > 0$, $c_0 \geqslant 0$,

$$s_j(B) - c_0^{1/p} j^{-1/p} \quad \text{is } O\big(j^{-1/q}\big), \qquad s_j\big(B'\big) \quad \text{is } O\big(j^{-1/r}\big), \tag{3.8}$$

then $B + B'$ satisfies

$$s_j\big(B + B'\big) - c_0^{1/p} j^{-1/p} \quad \text{is } O\big(j^{-1/q'}\big), \text{ with } q' = \max\left\{q,\, p\frac{r+1}{p+1}\right\}. \tag{3.9}$$

We apply the result here with $B = A_\gamma^{-1}$ and $B' = A_{\chi,\Sigma_+}^{-1} - A_\gamma^{-1}$, so that $p = n/2$, $q = n/3$, $r = (n-1)/2$. This gives

$$q' = \max\left\{\frac{n}{3}, \frac{n}{2} \cdot \frac{\frac{n-1}{2}+1}{\frac{n}{2}+1}\right\} = \frac{n}{2} \cdot \frac{n+1}{n+2};$$

here $1/q' = 2/n \cdot (n+2)/(n+1) = (1 + 1/(n+1))2/n$. $\square$

Note that these results hold when $b \in L_\infty(\Sigma)$ and Σ_+ is any closed subset of Σ.

Remark 3.3. Concerning nonsmooth choices of Ω, let us mention that the basic hypothesis of Birman in [6] is that Σ is piecewise C^2. This allows edges or creases, cf. Section 4.3 below. Moreover, in the recent translation to English of that historical paper, the translator M. Solomyak states in a supplementing comment to Section 2.1 on page 50 that the result is valid for Lipschitz domains (at least when $n \geqslant 3$; the reservation for $n = 2$ seems to be concerned with cutoffs in exterior domains).

4. Kreĭn resolvent formulas for the mixed problem

4.1. A formula relative to the Dirichlet problem

We assume in Sections 4.1, 4.2 and 5 that Σ_+ is smooth. First we show a Kreĭn resolvent formula for A_{χ,Σ_+} linked with A_γ. For simplicity of notation, we do the main calculations in the case $\lambda = 0$ (where the indexation by λ is left out); then at the end we account for the consequences in situations with other values of λ.

Recall from Section 2.3 that in the analysis with A_γ as the reference operator, A_{χ,Σ_+} corresponds to $L : X \to X^*$, where $D(L) = \gamma_0 D(A_{\chi,\Sigma_+})$ and X is its closure in $H^{-\frac12}(\Sigma)$. It is seen from (2.6) that $D(L)$ is a subset of $H_0^{\frac12}(\Sigma_+)$, and it contains $C_0^\infty(\Sigma_+^\circ)$ in view of the surjectiveness of $\{\gamma_0, \nu\}$ from $H^2(\Omega)$ to $H^{\frac32}(\Sigma) \times H^{\frac12}(\Sigma)$. Then in fact its closure X in $H^{-\frac12}(\Sigma)$ satisfies

$$X = H_0^{-\frac12}(\Sigma_+), \quad \text{and hence} \quad X^* = H^{\frac12}\big(\Sigma_+^\circ\big). \tag{4.1}$$

We note that the injection $\mathrm{i}_X : X \hookrightarrow H^{-\frac12}(\Sigma)$ and its adjoint satisfy:

$$\mathrm{i}_X = e_{\Sigma_+^\circ} : H_0^{-\frac12}(\Sigma_+) \hookrightarrow H^{-\frac12}(\Sigma), \qquad (\mathrm{i}_X)^* = r_{\Sigma_+^\circ} : H^{\frac12}(\Sigma) \to H^{\frac12}\big(\Sigma_+^\circ\big),$$

where $e_{\Sigma_+^\circ}$ is a well-defined extension of the operator that extends functions on Σ_+° by zero on Σ_-, and $r_{\Sigma_+^\circ}$ denotes restriction to Σ_+°. We denote $e_{\Sigma_+^\circ} = e^+$ and $r_{\Sigma_+^\circ} = r^+$ for short. Since A_{χ,Σ_+} is bijective, so is L, from $D(L)$ to $H^{\frac12}(\Sigma_+^\circ)$.

When $u \in D(A_{\chi,\Sigma_+})$, we see from (2.6), (1.3) that νu equals $b\gamma_0 u$ on Σ_+° in the distribution sense, hence since $\Gamma_\nu = \nu - P_{\gamma,\nu}\gamma_0$,

$$\langle \Gamma_\nu u, \bar\zeta\rangle = \big\langle (b - P_{\gamma,\nu})\gamma_0 u, \bar\zeta\big\rangle \quad \text{for } \zeta \in C_0^\infty\big(\Sigma_+^\circ\big). \tag{4.2}$$

Since $\gamma_0 u \in H^{\frac{1}{2}}(\Sigma)$, which is mapped to $H^{-\frac{1}{2}}(\Sigma)$ by $P_{\gamma,\nu}$, and multiplication by b preserves $L_2(\Sigma)$, we have that $(b - P_{\gamma,\nu})\gamma_0 u \in H^{-\frac{1}{2}}(\Sigma)$.

The operator L satisfies, by (2.21),

$$(L\gamma_0 u, \varphi)_{X^*,X} = (\Gamma_\nu u, \varphi)_{\frac{1}{2},-\frac{1}{2}} \quad \text{for all } \varphi \in X;$$

in particular, when (4.1) and (4.2) are taken into account,

$$(L\gamma_0 u, \zeta)_{H^{\frac{1}{2}}(\Sigma_+^\circ), H_0^{-\frac{1}{2}}(\Sigma_+)} = \langle (b - P_{\gamma,\nu})\gamma_0 u, \bar{\zeta}\rangle, \quad \text{for } \zeta \in C_0^\infty(\Sigma_+^\circ),$$

so

$$L\gamma_0 u = r^+(b - P_{\gamma,\nu})\gamma_0 u, \quad \text{for } u \in D(A_{\chi,\Sigma_+}).$$

Thus L acts as

$$L\varphi = r^+(b - P_{\gamma,\nu})e^+\varphi, \quad \text{for } \varphi \in D(L). \tag{4.3}$$

This shows the form of L. We need deeper theories to say more about the domain. Here we shall use the study of mixed problems in Shamir [52]; in Section 5 we also use Eskin [14]. Some smoothness is needed for this; for convenience we take $b \in C^\infty(\Sigma)$.

Proposition 4.1. *When Σ_+ is smooth, the operator L acts as in* (4.3)*. When also b is smooth, it satisfies*

$$D(L) \subset H_0^{1-\varepsilon}(\Sigma_+), \quad \text{any } \varepsilon > 0, \tag{4.4}$$

and L^{-1} maps $H^{\frac{1}{2}}(\Sigma_+^\circ)$ into $H_0^{1-\varepsilon}(\Sigma_+)$.

Proof. We see from [52] that $D(A_{\chi,\Sigma_+}) \subset H^{\frac{3}{2}-\varepsilon}(\Omega)$, as follows: First Shamir shows this in Th. 3.1 of [52] for the constant-coefficient case of $-\Delta + \alpha^2$ on a half-space with mixed Dirichlet and Neumann boundary conditions. Subsequently the statement is extended to variable coefficients and bounded domains in the proof of Lemma 5.1 of [52] (when we recall that the domain is a priori contained in $H^1(\Omega)$). Since $\gamma_0 H^{\frac{3}{2}-\varepsilon}(\Omega) = H^{1-\varepsilon}(\Sigma)$, it follows by the definition of L that $D(L) \subset H_0^{1-\varepsilon}(\Sigma_+)$. Since L is surjective onto $H^{\frac{1}{2}}(\Sigma_+^\circ)$, the last statement follows. □

There is a simple example mentioned in [52] of a harmonic function $u(x_1, x_2) = \operatorname{Im}(x_2 + ix_1)^{\frac{1}{2}}$ on $\mathbb{R}_+ \times \mathbb{R}$ satisfying the mixed condition on $\{x_1 = 0\}$, namely $\gamma_0 u = 0$ for $x_2 > 0$, $\gamma_1 u = 0$ for $x_2 < 0$. It is not in $H^{\frac{3}{2}}$ in a neighborhood of 0. This shows that $D(A_{\chi,\Sigma_+})$ is not in general contained in $H^{\frac{3}{2}}(\Omega)$, so the regularity cannot be improved.

Now consider the Kreĭn resolvent formula (2.16) for this choice of L and X; by the selfadjointness, $Y = X$. Recall that $K_{\gamma,X} = \mathrm{i}_V \gamma_V^{-1} : X \to L_2(\Omega)$, where V is the subspace of $Z = \ker(A_{\max})$ that is mapped to X by γ_0. Since γ_V^{-1} acts like K_γ from the space $X = H_0^{-\frac{1}{2}}(\Sigma_+)$ to V, we can also write

$$K_{\gamma,X} = \mathrm{i}_V K_\gamma e^+ : H_0^{-\frac{1}{2}}(\Sigma_+) \to L_2(\Omega), \quad \text{and then } K_{\gamma,X}^* = r^+ K_\gamma^* \operatorname{pr}_V : L_2(\Omega) \to H^{\frac{1}{2}}(\Sigma_+^\circ),$$

whereby the formula takes the form

$$A_{\chi,\Sigma_+}^{-1} - A_\gamma^{-1} = \mathrm{i}_V K_\gamma e^+ L^{-1} r^+ K_\gamma^* \operatorname{pr}_V = \mathrm{i}_V K_\gamma e^+ \big(r^+(b - P_{\gamma,\nu})e^+\big)^{-1} r^+ K_\gamma^* \operatorname{pr}_V. \tag{4.5}$$

The λ-dependent version is found by replacing A by $A - \lambda$ in the various defining formulas, as explained at the end of Section 2. Since $\chi = \nu - b\gamma_0$, we have that

$$P_{\gamma,\chi}^\lambda = \chi K_\gamma^\lambda = P_{\gamma,\nu}^\lambda - b, \tag{4.6}$$

a notation we shall now use. Moreover, using the standard abbreviation for a truncated operator $r^+ Q e^+ = Q_+$, we can write $r^+ P_{\gamma,\chi}^\lambda e^+ = P_{\gamma,\chi,+}^\lambda$. Then the result in the λ-dependent formulation is:

Theorem 4.2. *Let Σ_+ and b be smooth. Then*

$$L^\lambda \varphi = -P_{\gamma,\chi,+}^\lambda \varphi \quad \text{for } \varphi \in D(L),\ \lambda \in \varrho(A_\gamma),$$

$$(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = -K_{\gamma,X}^\lambda \big(P_{\gamma,\chi,+}^\lambda\big)^{-1} \big(K_{\gamma,X}^{\bar{\lambda}}\big)^* \quad \text{for } \lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma), \tag{4.7}$$

where $V_\lambda = K_\gamma^\lambda(X)$, $K_{\gamma,X}^\lambda = \mathrm{i}_{V_\lambda} \gamma_{V_\lambda}^{-1} = \mathrm{i}_{V_\lambda} K_\gamma^\lambda e^+$, $(K_{\gamma,X}^{\bar{\lambda}})^* = (\gamma_{V_{\bar{\lambda}}}^{-1})^* \operatorname{pr}_{V_{\bar{\lambda}}} = r^+(K_\gamma^{\bar{\lambda}})^* \operatorname{pr}_{V_{\bar{\lambda}}}$.

The inverse of $P_{\gamma,\chi}^\lambda$ is $P_{\chi,\gamma}^\lambda$, when it exists. It is important to observe that $(P_{\gamma,\chi,+}^\lambda)^{-1}$ is *not* the same as $P_{\chi,\gamma,+}^\lambda$; this is part of the difficulty treated in Section 5.

4.2. Other Kreĭn resolvent formulas

Next, if we work instead with a Neumann realization as the reference operator, we can show a different formula containing full Poisson operators.

Consider again the boundary condition

$$\gamma_0 u = 0 \quad \text{on } \Sigma_-, \qquad \nu u = b\gamma_0 u \quad \text{on } \Sigma_+. \tag{4.8}$$

If b has a bounded inverse f, we can set $f_+ = 1_{\Sigma_+} f$ and write condition (4.8) as one equation, a Dirichlet-type condition

$$\gamma_0 u = f_+ \nu u. \tag{4.9}$$

Here $\gamma_0 u$ is a function of νu, so that the operator A_ν can be used in a simple way as the reference operator.

Actually, it only takes a small modification to obtain invertibility of the coefficient in general: If b does not have a bounded inverse, we can replace νu by

$$\nu' u = \nu u + K\gamma_0 u, \tag{4.10}$$

where K is chosen $>$ ess sup $|b(x)|$ (as in Section 2.1); then the condition (4.8) takes the form

$$\gamma_0 u = 0 \quad \text{on } \Sigma_-, \qquad \nu' u = b'\gamma_0 u \quad \text{on } \Sigma_+, \tag{4.11}$$

where $b' = b + K$ does have a bounded inverse. Note that $\chi' = \nu' - b'\gamma_0 = \chi$ by cancellation. In Green's formula (2.14) we get ν replaced by ν' by adding the term $(K\gamma_0 u, \gamma_0 u) - (\gamma_0 u, K\gamma_0 u)$ (equal to 0) to the right-hand side, and the sesquilinear form is adapted to these formulas by addition of the first-order terms $\sum_{j=1}^n [(Kn_j\partial_j u, v)_\Omega + (u, K\partial_j(n_j v))_\Omega]$, giving the form

$$a'(u,v) = a(u,v) + \sum_{j=1}^n \big[(Kn_j\partial_j u, v) + \big(u, K\partial_j(n_j v)\big)\big] \quad \text{on } H^1(\Omega).$$

Here the n_j are extended smoothly to the interior of Ω, vanishing outside a small neighborhood of Σ. The operators defined from a' on various spaces between $H^1(\Omega)$ and $H_0^1(\Omega)$ still act like A, since $(u, K\partial_j(n_j\varphi)) = -(Kn_j\partial_j u, \varphi)$ for $\varphi \in C_0^\infty(\Omega)$. The "halfways Green's formula" is here

$$(Au, v) - a'(u,v) = \big(\nu' u, \gamma_0 v\big)_{L_2(\Sigma)}, \tag{4.12}$$

since $\sum_{j=1}^n [(Kn_j\partial_j u, v)_\Omega + (u, K\partial_j(n_j v))_\Omega] = -(K\sum n_j^2\gamma_0 u, \gamma_0 v)_\Sigma = -(K\gamma_0 u, \gamma_0 v)_\Sigma$. The forms in the scheme (2.2) are now replaced by

$$\begin{aligned}
&a'_\gamma(u,v) = a'(u,v) \quad \text{on } H_0^1(\Omega), \quad \text{leading to } A_\gamma,\\
&a'_{\nu'}(u,v) = a'(u,v) \quad \text{on } D(a_{\nu'}) = H^1(\Omega), \quad \text{leading to } A_{\nu'},\\
&a'_{\chi'}(u,v) = a'(u,v) + \big(b'\gamma_0 u, \gamma_0 v\big)_{L_2(\Sigma)} \quad \text{on } D(a_{\chi'}) = H^1(\Omega), \quad \text{leading to } A_\chi,\\
&a'_{\chi',\Sigma_+}(u,v) = a'(u,v) + \big(b'\gamma_0 u, \gamma_0 v\big)_{L_2(\Sigma_+)} \quad \text{on } D\big(a'_{\chi',\Sigma_+}\big) = H^1_{\Sigma_+}(\Omega)\\
&\qquad = \big\{u \in H^1(\Omega) \mid \operatorname{supp} u \subset \Sigma_+\big\}, \quad \text{leading to } A_{\chi,\Sigma_+};
\end{aligned} \tag{4.13}$$

here $A_{\nu'}$ is the realization of A under the boundary condition $\nu' u = 0$, whereas the choices with b' still give the boundary condition $\nu u = b\gamma_0 u$ on Σ resp. Σ_+, since $b' = b + K$, $\nu' = \nu + K\gamma_0$. With $K_{\nu'}$, $P_{\gamma,\nu'}$ and $P_{\nu',\gamma}$ defined as in Section 4.1 with ν replaced by ν', and

$$\Gamma'_\gamma = \gamma_0 - P_{\nu',\gamma}\nu',$$

we have the generalized Green's formula valid for $u, v \in D(A_{\max})$:

$$(Au, v)_{L_2(\Omega)} - (u, Av)_{L_2(\Omega)} = \big(\nu' u, \Gamma'_\gamma v\big)_{-\frac{3}{2},\frac{3}{2}} - \big(\Gamma'_\gamma u, \nu' v\big)_{\frac{3}{2},-\frac{3}{2}}. \tag{4.14}$$

In the following, we assume that the forms in (4.13) all have positive lower bound. We set $f = (b')^{-1}$ so that the mixed boundary condition (4.11) can be written

$$\gamma_0 u = f_+ \nu' u, \tag{4.15}$$

where $f_+ = 1_{\Sigma_+} f$, as accounted for above.

We now describe A_{χ,Σ_+} in terms of the correspondence (2.12) and its interpretation in Section 2.4, with ν replaced by ν'.

Here X_1 is the full space $H^{-\frac{3}{2}}(\Sigma)$, which is seen as follows: When $\psi \in C_0^\infty(\Sigma_+^\circ) \cup C_0^\infty(\Sigma_-^\circ)$, then $f_+\psi \in C_0^\infty(\Sigma_+^\circ)$, and there exists $u \in C^\infty(\overline{\Omega})$ such that $\nu' u = \psi$, $\gamma_0 u = f_+\psi$; this u satisfies (4.9). So $C_0^\infty(\Sigma_+^\circ) \cup C_0^\infty(\Sigma_-^\circ) \subset D(L_1)$. It is known that $C_0^\infty(\Sigma_+^\circ) \cup C_0^\infty(\Sigma_-^\circ)$ is dense in $H^s(\Sigma)$ for $s < \frac{1}{2}$. In particular $C_0^\infty(\Sigma_+^\circ) \cup C_0^\infty(\Sigma_-^\circ)$ is dense in $H^{-\frac{3}{2}}(\Sigma)$, so we conclude that $X_1 = H^{-\frac{3}{2}}(\Sigma)$. Since A_{χ,Σ_+} is selfadjoint, also $Y_1 = H^{-\frac{3}{2}}(\Sigma)$.

Thus the realization A_{χ,Σ_+} with domain (2.6) corresponds to an operator $L_1 : H^{-\frac{3}{2}}(\Sigma) \to H^{\frac{3}{2}}(\Sigma)$ with domain $D(L_1) = \nu' D(A_{\chi,\Sigma_+})$; the latter lies in $H^{-\frac{1}{2}}(\Sigma)$ since $D(A_{\chi,\Sigma_+}) \subset H^1(\Omega)$. It follows by comparison of (2.30) with (4.15) that L_1 acts as

$$L_1 = -f_+ + P_{\nu',\gamma}. \tag{4.16}$$

Since A_{χ,Σ_+} is bijective, so is L_1.

Then the Kreĭn resolvent formula reads

$$A_{\chi,\Sigma_+}^{-1} - A_{\nu'}^{-1} = K_{\nu'} L_1^{-1} K_{\nu'}^* = K_{\nu'}(P_{\nu',\gamma} - f_+)^{-1} K_{\nu'}^*, \quad \text{for } \lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_{\nu'}). \tag{4.17}$$

It may look a little more useful than (4.5), since the operators surrounding L_1^{-1} are a full Poisson operator and trace operator in the pseudodifferential boundary operator calculus, but it poses again the question of a detailed understanding of the term in the middle, defined on $H^{\frac{3}{2}}(\Sigma)$. This may not be any easier than our treatment in Section 4.1, since the principal part of L_1 is the 0-order multiplication by $-f_+$ which vanishes on Σ_-, and $P_{\nu',\gamma}$ is of order -1.

We can replace A by $A - \lambda$ in the various defining formulas and obtain:

Theorem 4.3. *Let Σ_+ and b be smooth. Define ν' by* (4.10)ff. *and $f = (b + K)^{-1}$. Then*

$$(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_{\nu'} - \lambda)^{-1} = K_{\nu'}^\lambda \big(P_{\nu',\gamma}^\lambda - f_+\big)^{-1} \big(K_{\nu'}^{\bar\lambda}\big)^*, \quad \text{for } \lambda \in \varrho(A_{\nu'}) \cap \varrho(A_{\chi,\Sigma_+}). \tag{4.18}$$

Formula (4.18) can even be turned into a resolvent difference formula where the surrounding Poisson operator and trace operator come from the Dirichlet problem, by use of the fact that $P_{\gamma,\nu'}^\lambda$ and $P_{\nu',\gamma}^\lambda$ are inverses of one another, and

$$K_{\nu'}^\lambda = K_\gamma^\lambda P_{\nu',\gamma}^\lambda, \qquad \big(P_{\nu',\gamma}^{\bar\lambda}\big)^* = P_{\nu',\gamma}^\lambda, \tag{4.19}$$

then. Namely, insertion in (4.18) gives:

$$(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_{\nu'} - \lambda)^{-1} = K_\gamma^\lambda P_{\nu',\gamma}^\lambda \big(P_{\nu',\gamma}^\lambda - f_+\big)^{-1} P_{\nu',\gamma}^\lambda \big(K_\gamma^{\bar\lambda}\big)^*. \tag{4.20}$$

This can be added to the well-known formula

$$(A_{\nu'} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = K_\gamma^\lambda \big(-P_{\gamma,\nu'}^\lambda\big)^{-1} \big(K_\gamma^{\bar\lambda}\big)^* = -K_\gamma^\lambda P_{\nu',\gamma}^\lambda \big(K_\gamma^{\bar\lambda}\big)^*$$

((2.32) with $\widetilde{A} = A_{\nu'}$, hence $L^\lambda = -P_{\gamma,\nu'}^\lambda$), to give a formula for the resolvent difference with the Dirichlet realization, having another structure than (4.7):

$$\begin{aligned}(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} &= K_\gamma^\lambda P_{\nu',\gamma}^\lambda \big[\big(P_{\nu',\gamma}^\lambda - f_+\big)^{-1} P_{\nu',\gamma}^\lambda - 1\big]\big(K_\gamma^{\bar\lambda}\big)^* \\ &= K_\gamma^\lambda P_{\nu',\gamma}^\lambda \big(P_{\nu',\gamma}^\lambda - f_+\big)^{-1} f_+ \big(K_\gamma^{\bar\lambda}\big)^*, \quad \text{for } \lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma) \cap \varrho(A_{\nu'}).\end{aligned} \tag{4.21}$$

The last formula in (4.21) has a similar flavor as the formula found by Pankrashkin in [44], Sect. 4.3.

If b itself is invertible, the formulas will be valid with $f = b^{-1}$, ν' replaced by ν. We have shown:

Corollary 4.4. *Under the hypotheses of Theorem* 4.3*, we have the formulas in* (4.21) *for the difference with the Dirichlet resolvent, when $\lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma) \cap \varrho(A_{\nu'})$.*

If b itself is invertible, there are the formulas with $f = b^{-1}$:

$$(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\nu - \lambda)^{-1} = K_\nu^\lambda \big(P_{\nu,\gamma}^\lambda - f_+\big)^{-1} \big(K_\nu^{\bar\lambda}\big)^*, \tag{4.22}$$

$$(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = K_\gamma^\lambda P_{\nu,\gamma}^\lambda \big(P_{\nu,\gamma}^\lambda - f_+\big)^{-1} f_+ \big(K_\gamma^{\bar\lambda}\big)^*, \tag{4.23}$$

where (4.22) *holds for $\lambda \in \varrho(A_\nu) \cap \varrho(A_{b,\Sigma_+})$,* (4.23) *holds for $\lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma) \cap \varrho(A_\nu)$.*

Remark 4.5. The analysis in Proposition 4.1ff. showed that $D(A_{\chi,\Sigma_+}) \subset H^{\frac{3}{2}-\varepsilon}(\Omega)$ but is not in general contained in $H^{\frac{3}{2}}(\Omega)$. Thus those results in Malamud [41], Sect. 6, that concern second-order realizations with domain contained in $H^{\frac{3}{2}}(\Omega)$ (i.e., with $\gamma_0 u$ and $\nu u \in L_2(\Sigma)$), will not in general apply to the mixed problem.

Remark 4.6. If we instead of (4.9) consider a boundary condition

$$\gamma_0 u = g\nu u, \tag{4.24}$$

where g is an arbitrary C^∞-function on Σ, we can carry an analysis through, showing that if the corresponding realization $\widetilde{A}$ is bijective and selfadjoint, then it corresponds to an operator L_1 from $H^{-\frac{3}{2}}(\Sigma)$ to $H^{\frac{3}{2}}(\Sigma)$, with domain dense in $H^{-\frac{3}{2}}(\Sigma)$ and acting like $P_{\nu,\gamma} - g$, such that there are Kreĭn formulas

$$\begin{aligned} \widetilde{A}^{-1} - A_\nu^{-1} &= K_\nu L_1^{-1} K_\nu{}^* = K_\nu (P_{\nu,\gamma} - g)^{-1} K_\nu{}^*, \\ \widetilde{A}^{-1} - A_\gamma^{-1} &= K_\gamma P_{\nu,\gamma} (P_{\nu,\gamma} - g)^{-1} g K_\gamma{}^*, \end{aligned} \tag{4.25}$$

and λ-dependent variants. But again, the operator $L_1^{-1} = (P_{\nu,\gamma} - g)^{-1}$ is nonstandard in the calculus of ψdo's, since $P_{\nu,\gamma}$ is elliptic of order -1 whereas g defines an operator of order 0 and can vanish on large subsets of Σ.

4.3. Nonsmooth domains

We here include some observations on cases where the set Ω is not smooth. An interesting variant of the Zaremba problem is where $\Sigma = \Sigma_+ \cup \Sigma_-$ with Σ_+ and Σ_- *meeting at an angle* $< \pi$. Then there is the perhaps surprising fact that the realization A_{ν,Σ_+} of $-\Delta$ with Dirichlet condition $\gamma_0 u = 0$ on Σ_- and Neumann condition $\nu u = 0$ on Σ_+ ($\nu = \gamma_1$) can have a *better* regularity than when Ω is smooth. Here is an example:

Example 4.7. Let Ω' be a smooth bounded set that is symmetric in x_1 around $x_1 = 0$; i.e., is preserved under the mapping $J_1 : (x_1, x_2, \dots, x_n) \to (-x_1, x_2, \dots, x_n)$. Let $\Omega = \{x \in \Omega' \mid x_1 > 0\}$. Then the solutions of the mixed problem for $-\Delta$ on Ω with $\Sigma_- = \partial\Omega' \cap \{x_1 \geqslant 0\}$, $\Sigma_+ = \overline{\Omega'} \cap \{x_1 = 0\}$, are the restrictions to Ω of those solutions to the Dirichlet problem for Ω' that are invariant under J_1. (This observation enters in a prominent way in the discussion of isospectral domains for mixed problems by Levitin, Parnovski and Polterovich [38].) Here the domain of the Dirichlet realization of $-\Delta$ on Ω' is in $H^2(\Omega')$, hence $D(A_{\nu,\Sigma_+}) \subset H^2(\Omega)$ (observe that both operators are bijective when defined by the variational construction). In this case Σ_+ and Σ_- meet at an angle $\pi/2$. Related results are found for polygonal domains, cf. Grisvard [21].

More generally, consider the case where Ω is such that Σ_+ and Σ_- meet at an angle $< \pi$, in the way described in Brown [11]; such domains are by some authors called *creased domains*. It is shown there that the solutions $u \in H^1(\Omega)$ of

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \gamma_0 u = \varphi \quad \text{on } \Sigma_-, \qquad \gamma_1 u = \psi \quad \text{on } \Sigma_+, \tag{4.26}$$

with $\varphi \in H^1(\Sigma_-^\circ)$, $\psi \in L_2(\Sigma_+)$, have $\gamma_0(\nabla u) \in L_2(\Sigma)$; in particular $\gamma_1 u \in L_2(\Sigma)$. Here Σ just needs to be Lipschitz, in such a way that $\partial\Sigma_+$ is Lipschitz in Σ (we refer to [11] for the precise description).

To apply this to A_{ν,Σ_+}, we restrict to quasi-convex domains Ω. They are defined by Gesztesy and Mitrea in [18] as a special case of Lipschitz domains including convex domains, which allow showing solvability and regularity theorems for the Dirichlet and Neumann problems for $-\Delta$ on Ω in larger scales of Sobolev-type spaces than in Jerison and Kenig [35]; the work builds on Mitrea, Taylor and Vasy [43] and Mazya, Mitrea and Shaposhnikova [42].

Theorem 4.8. *Assume that Ω is bounded, open and quasi-convex as defined in* [18]. *Assume moreover that Ω is creased, in the way that the boundary Σ equals $\Sigma_+ \cup \Sigma_-$, where Σ_+ and Σ_- meet at an angle $< \pi$, as described in* [11]. *The realization A_{ν,Σ_+} of $-\Delta$ with Neumann condition on Σ_+, Dirichlet condition on Σ_- then has $D(A_{\nu,\Sigma_+}) \subset H^{\frac{3}{2}}(\Omega)$.*

Proof. We can assume that $\Sigma_- \neq \emptyset$. To describe a solution in $H^1(\Omega)$ of

$$-\Delta u = f \quad \text{in } \Omega, \qquad \gamma_0 u = 0 \quad \text{on } \Sigma_-, \qquad \gamma_1 u = 0 \quad \text{on } \Sigma_+, \tag{4.27}$$

with $f \in L_2(\Omega)$, let v be the solution of the Dirichlet problem

$$-\Delta v = f \quad \text{in } \Omega, \qquad \gamma_0 v = 0 \quad \text{on } \Sigma; \tag{4.28}$$

then $z = u - v$ should be a solution of (4.26) with $\varphi = 0$, $\psi = -\gamma_1 v|_{\Sigma_+}$. Since $v \in H^2(\Omega)$ by [18], Th. 10.4, $\gamma_1 v|_{\Sigma_+} \in H^{\frac{1}{2}}(\Sigma_+^\circ) \subset L_2(\Sigma_+)$, so the result of Brown [11] implies that $\gamma_1 z \in H^1(\Sigma)$. Then the regularity theorem for the Neumann problem [18], Th. 10.8 implies that $z \in H^{\frac{3}{2}}(\Omega)$, hence $u = v + z \in H^{\frac{3}{2}}(\Omega)$. (Since A_{ν,Σ_+} is bijective, the solutions we consider are consistent with those considered in [11].) We conclude that $D(A_{\nu,\Sigma_+}) \subset H^{\frac{3}{2}}(\Omega)$. □

Note the contrast with the informations obtained in Proposition 4.1ff. where Σ is smooth and $D(A_{\nu,\Sigma_+})$ is in general only in $H^{\frac{3}{2}-\varepsilon}(\Omega)$. But that is a case where Σ_+ and Σ_- meet at the angle π, which is explicitly excluded in [11].

The mixed problem in these various forms can, for piecewise smooth domains, be regarded as a special case of crack problems and edge problems, as studied e.g. by Duduchava, Dauge, Costabel, Mazya, Solonnikov and their collaborators, see also Schulze et al. [49,33]. The results are often described in terms of norms weighted by powers of the distance to the edge; this gives a clarification of the singularities, but can lead outside the Sobolev spaces considered here.

Let us finally mention that for quasi-convex domains there is in [18], Th. 10.4 established a homeomorphism $\hat{\gamma}_D : Z \xrightarrow{\sim} (N^{\frac12}(\Sigma))^*$ (generalizing γ_0), which allows translating formula (2.10) in Section 2 above to a formula like (2.16) with (2.17). Here $N^{\frac12}(\Sigma)$ is a certain Hilbert space related to $H^{\frac12}(\Sigma)$ explained in [18], and $(N^{\frac12}(\Sigma))^*$ is its dual space with respect to a sesquilinear duality consistent with the $L_2(\Sigma)$-scalar product, such that $N^{\frac12}(\Sigma) \subset L_2(\Sigma) \subset (N^{\frac12}(\Sigma))^*$, with dense, continuous injections. For a general closed realization $\widetilde{A}$ of $-\Delta$, let X and Y be the closures of $\hat{\gamma}_D(D(\widetilde{A}))$ resp. $\hat{\gamma}_D(D(\widetilde{A}^*))$ in $(N^{\frac12}(\Sigma))^*$, and let V resp. W be their inverse images in Z (by $\hat{\gamma}_D^{-1}$); in the selfadjoint case, $Y = X$. We can then define γ_V to be the restriction of $\hat{\gamma}_D$ mapping V homeomorphically to X; similarly, γ_W is the restriction of $\hat{\gamma}_D$ mapping W homeomorphically to Y. With this, the considerations in (2.15)–(2.18) are valid, leading to:

Theorem 4.9. *When Ω is quasi-convex and $\widetilde{A}$ is a general closed realization of $-\Delta$ with $0 \in \varrho(\widetilde{A})$, it satisfies the Kreĭn resolvent formula* (2.16) *with* (2.17)*, L defined as after* (2.15)*. In particular, for the realization A_{ν,Σ_+} of the mixed problem, one has with $X =$ the closure of $\hat{\gamma}_D(D(A_{\nu,\Sigma_+}))$,*

$$A_{\nu,\Sigma_+}^{-1} - A_\gamma^{-1} = K_{\gamma,X} L^{-1} (K_{\gamma,X})^*, \qquad K_{\gamma,X} = \mathrm{i}_V \gamma_V^{-1} : X \to V \subset H. \tag{4.29}$$

The rest of Section 2 likewise carries over to the quasi-convex setting, but it must be noted that the Dirichlet-to-Neumann operator is then an abstractly defined operator whose local structure is not so well known. Also λ-dependent variants of (2.16) for realizations of $-\Delta - \lambda$ are valid when $\lambda \in \varrho(A_\gamma) \cap \varrho(\widetilde{A})$. The interpretation of the general theory of [22] for quasi-convex domains is worked out in great detail in [18].

We remark however that the Kreĭn formula in [18], Th. 16.3 differs from our formula (2.16), particularly when $X \neq (N^{\frac12}(\Sigma))^*$ (which holds for genuine mixed problems).

Upper eigenvalue estimates (1.6) for the resolvent difference (4.29) follow from [6], cf. Remark 3.3. Asymptotic estimates would demand an effort that to our knowledge has not yet been taken up.

5. Spectral asymptotics for the mixed problem

5.1. Notation

In this section we go back to smooth domains and restrict the attention to the case $a_{jk} = \delta_{jk}$, i.e., we take A principally equal to $-\Delta$, in order to use some detailed formulas in Eskin [14].

We want to show a spectral asymptotic formula for the operator

$$(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1} = K^\lambda_{\gamma,X}\big(L^\lambda\big)^{-1}\big(K^{\bar\lambda}_{\gamma,X}\big)^* = -K^\lambda_{\gamma,X}\big(P^\lambda_{\gamma,\chi,+}\big)^{-1}\big(K^{\bar\lambda}_{\gamma,X}\big)^*$$

from Theorem 4.2. As done also earlier, we begin by taking λ as a sufficiently low fixed real number such that the considered realizations of $A-\lambda$ are positive, and then omit λ from the notation. General λ are included in the proof of the final Theorem 5.17.

In view of the formula (2.17) for $K_{\gamma,X}$, we are considering the operator

$$A_{\chi,\Sigma_+}^{-1} - A_\gamma^{-1} = \mathrm{i}_V \gamma_V^{-1} L^{-1} \big(\gamma_V^{-1}\big)^* \operatorname{pr}_V; \tag{5.1}$$

it is compact selfadjoint nonnegative.

Let us first recall some facts on spaces describing the spectral behavior of compact operators. For $p > 0$ we denote by $\mathcal{C}_p$ the Schatten class of compact linear operators B (in a Hilbert space H) with singular value sequences $(s_j(B))_{j\in\mathbb{N}}$ belonging to ℓ_p, and by $\mathfrak{S}_p$ the quasi-normed space of compact operators B with $s_j(B) = O(j^{-1/p})$ (sometimes called a weak Schatten class); here $\mathfrak{S}_p \subset \mathcal{C}_{p'}$ for $p' > p$. Moreover, we denote by $\mathfrak{S}_{p,0}$ the subset of operators $B \in \mathfrak{S}_p$ for which $s_j(B) = o(j^{-1/p})$, i.e., $s_j(B) j^{1/p} \to 0$ for $j \to \infty$. Clearly, $\mathfrak{S}_p \subset \mathfrak{S}_{p',0}$ for $p' > p$.

The rules shown by Ky Fan [15]

$$s_{j+k-1}\big(B + B'\big) \leqslant s_j(B) + s_k\big(B'\big), \qquad s_{j+k-1}\big(BB'\big) \leqslant s_j(B) s_k\big(B'\big),$$

imply that $\mathcal{C}_p$, $\mathfrak{S}_p$ and $\mathfrak{S}_{p,0}$ are vector spaces, and that there are the following product rules:

$$\mathcal{C}_p \cdot \mathcal{C}_q \subset \mathcal{C}_{1/(p^{-1}+q^{-1})}, \qquad \mathfrak{S}_p \cdot \mathfrak{S}_q \subset \mathfrak{S}_{1/(p^{-1}+q^{-1})}, \qquad \mathfrak{S}_p \cdot \mathfrak{S}_{q,0} \subset \mathfrak{S}_{1/(p^{-1}+q^{-1}),0}. \tag{5.2}$$

Moreover, the rule for $F_1, F_2 \in \mathcal{L}(H)$,

$$s_j(F_1 B F_2) \leqslant \|F_1\| s_j(B) \|F_2\| \tag{5.3}$$

implies that $\mathcal{C}_p$, $\mathfrak{S}_p$ and $\mathfrak{S}_{p,0}$ are preserved under compositions with bounded operators. They are also preserved under taking adjoints. We recall two perturbation results:

Lemma 5.1.

1° *If $s_j(B)j^{1/p} \to C_0$ and $s_j(B')j^{1/p} \to 0$ for $j \to \infty$, then $s_j(B+B')j^{1/p} \to C_0$ for $j \to \infty$.*
2° *If $B = B_M + B'_M$ for each $M \in \mathbb{N}$, where $s_j(B_M)j^{1/p} \to C_M$ for $j \to \infty$ and $s_j(B'_M)j^{1/p} \leqslant c_M$ for $j \in \mathbb{N}$, with $C_M \to C_0$ and $c_M \to 0$ for $M \to \infty$, then $s_j(B)j^{1/p} \to C_0$ for $j \to \infty$.*

The statement in 1° is the Weyl–Ky Fan theorem (cf. e.g. [19], Th. II 2.3), and 2° is a refinement shown in [26], Lemma 4.2.2°.

We also recall that when Ξ and Ξ_1 are m-dimensional manifolds (possibly with a boundary, sufficiently smooth), $\overline{\Xi}_1$ being compact, and B is a bounded linear operator from $L_2(\Xi)$ to $H^t(\Xi_1)$ for some $t > 0$, then $B \in \mathfrak{S}_{m/t}$ as an operator from $L_2(\Xi)$ to $L_2(\Xi_1)$, with

$$s_j(B)j^{t/m} \leqslant C\|B\|_{\mathcal{L}(L_2(\Xi),H^t(\Xi_1))}, \tag{5.4}$$

with a constant C depending on t and the manifolds (references e.g. in [26]).

5.2. Constant coefficients

One ingredient in the analysis of the spectrum of (5.1) is an application of the constant-coefficient situation, so we begin by working that out, in the case $b = 0$. Here Ω, Σ and $\Sigma_\pm$ are replaced by $\mathbb{R}^n_+$, $\mathbb{R}^{n-1}$ and $\overline{\mathbb{R}}^{n-1}_\pm$, and we take $A = -\Delta + \alpha^2$ for some $\alpha > 0$; marking the operators with a subscript 0. The Poisson operator $K_{0,\gamma}$ solving the Dirichlet problem is the operator $\varphi(x') \mapsto \mathcal{F}^{-1}_{\xi'\to x'}[e^{-x_n(|\xi'|^2+\alpha^2)^{\frac12}}\hat\varphi(\xi')]$, so the Dirichlet-to-Neumann operator $P_{0,\gamma,\nu}$ is the ψdo with symbol $-(|\xi'|^2+\alpha^2)^{\frac12}$, i.e.,

$$P_{0,\gamma,\nu} = -\operatorname{Op}\bigl(\bigl(|\xi'|^2+\alpha^2\bigr)^{\frac12}\bigr) = -\bigl(-\Delta_{x'}+\alpha^2\bigr)^{\frac12}, \quad \text{with inverse } P_{0,\nu,\gamma} = -\operatorname{Op}\bigl(\bigl(|\xi'|^2+\alpha^2\bigr)^{-\frac12}\bigr).$$

($\mathcal{F}$ denotes the Fourier transform, and $\operatorname{Op}(a(x',\xi'))v = \mathcal{F}^{-1}_{\xi'\to x'}(a(x',\xi')\mathcal{F}_{x'\to\xi'}v)$.) Then with $\xi'' = (\xi_1,\dots,\xi_{n-2})$,

$$L_0 = -r^+P_{0,\gamma,\nu}e^+ = r^+\operatorname{Op}\bigl(\bigl(|\xi''|^2+\xi_{n-1}^2+\alpha^2\bigr)^{\frac12}\bigr)e^+ : H^s_0\bigl(\mathbb{R}^{n-1}_+\bigr) \to H^{s-1}\bigl(\mathbb{R}^{n-1}_+\bigr); \tag{5.5}$$

it will be used with $s = 1-\varepsilon$, cf. Proposition 4.1. According to Eskin [14], Ch. 7, one has in view of the factorization

$$\bigl(|\xi''|^2+\xi_{n-1}^2+\alpha^2\bigr)^{\frac12} = \bigl(\bigl(|\xi''|^2+\alpha^2\bigr)^{\frac12} - i\xi_{n-1}\bigr)^{\frac12}\bigl(\bigl(|\xi''|^2+\alpha^2\bigr)^{\frac12} + i\xi_{n-1}\bigr)^{\frac12},$$

that L_0 has the inverse

$$L_0^{-1} = r^+\Lambda_+e^+r^+\Lambda_-\ell^+_s : H^{s-1}\bigl(\mathbb{R}^{n-1}_+\bigr) \to H^s_0\bigl(\mathbb{R}^{n-1}_+\bigr), \quad 0 < s < 1, \tag{5.6}$$

where

$$\Lambda_\pm = \operatorname{Op}\bigl(\lambda_\pm(\xi')\bigr), \quad \lambda_\pm(\xi') = \bigl(\bigl(|\xi''|^2+\alpha^2\bigr)^{\frac12} \pm i\xi_{n-1}\bigr)^{-\frac12}, \tag{5.7}$$

and ℓ^+_s denotes a smooth extension operator, continuous from $H^t(\mathbb{R}^{n-1}_+)$ to $H^t(\mathbb{R}^{n-1})$ for all t. The operators $\Lambda_\pm$ are a "plus-operator" resp. a "minus-operator" in the terminology of [14]; plus-operators preserve support in $\overline{\mathbb{R}}^{n-1}_+$, and minus-operators are adjoints of plus-operators and preserve support in $\overline{\mathbb{R}}^{n-1}_-$.

When the formula is used for $s = 1-\varepsilon > \frac12$, we can replace ℓ^+_s by e^+, so

$$L_0^{-1} = r^+\Lambda_+e^+r^+\Lambda_-e^+ = \Lambda_{+,+}\Lambda_{-,+} : H^{-\varepsilon}\bigl(\mathbb{R}^{n-1}_+\bigr) \to H^{1-\varepsilon}_0\bigl(\mathbb{R}^{n-1}_+\bigr) \tag{5.8}$$

(recall the notation $Q_+ = r^+Qe^+$). L_0^{-1} is of course different from $(\Lambda_+\Lambda_-)_+ = -P_{0,\nu,\gamma,+}$, that we shall compare it with further below. We note that $\Lambda_{-,+}$ maps $H^{-\varepsilon}(\mathbb{R}^{n-1}_+)$ to $H^{\frac12-\varepsilon}(\mathbb{R}^{n-1}_+) = H_0^{\frac12-\varepsilon}(\mathbb{R}^{n-1}_+)$. Then the fact that $\Lambda_{+,+}$ preserves support in $\overline{\mathbb{R}}^{n-1}_+$, confirms that the range of L^{-1} is in the subspace $H_0^{1-\varepsilon}(\mathbb{R}^{n-1}_+)$ of $H^{1-\varepsilon}(\mathbb{R}^{n-1}_+)$.

We shall treat our general problem by reducing to cases in local coordinates with ingredients principally of this form. Then L_0^{-1} is multiplied on both sides with cutoff functions, so we shall now also consider $\psi L_0^{-1}\psi_1$, where $\psi,\psi_1 \in C_0^\infty(B_R)$ for some ball $B_R = \{|x'| < R\} \subset \mathbb{R}^{n-1}$. It is continuous

$$\psi L_0^{-1}\psi_1 : L_2\bigl(B_R \cap \mathbb{R}^{n-1}_+\bigr) \to H^{1-\varepsilon}\bigl(B_R \cap \mathbb{R}^{n-1}_+\bigr), \quad \text{any } \varepsilon > 0; \tag{5.9}$$

hence in view of (5.4),

$$\psi L_0^{-1}\psi_1 \in \mathfrak{S}_{n-1+\delta}, \quad \text{any } \delta > 0. \tag{5.10}$$

(Better estimates will be obtained below.) We shall compare it with $-\psi r^+ P^{-1}_{0,\gamma,\nu} e^+ \psi_1 = -\psi P_{0,\nu,\gamma,+}\psi_1$, and for this purpose we observe that

$$\begin{aligned} -r^+ P_{0,\nu,\gamma} e^+ - L^{-1} &= r^+ \Lambda_+ \Lambda_- e^+ - r^+ \Lambda_+ e^+ r^+ \Lambda_- e^+ \\ &= r^+ \Lambda_+ e^- J J r^- \Lambda_- e^+ \\ &= G^+(\Lambda_+) G^-(\Lambda_-), \end{aligned}$$

where r^- is the restriction operator from $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-1}_-$, e^- is the corresponding extension-by-zero operator, and J is the reflection operator $J: u(x'', x_{n-1}) \mapsto u(x'', -x_{n-1})$. We have used that $I - e^+ r^+ = e^- r^-$, and denoted

$$G^+(Q) = r^+ Q e^- J, \qquad G^-(Q) = J r^- Q e^+, \tag{5.11}$$

as in [26] and subsequent papers and books of the author. Note that the distribution kernel of $G^+(\Lambda_+)$ is obtained from that of Λ_+ by restriction to the second quadrant in (y_{n-1}, x_{n-1})-space, so that the singularity at the diagonal $\{x_{n-1} = y_{n-1}\}$ is only felt at 0.

On the manifold $\Sigma = \Sigma_+ \cup \Sigma_-$, $G^\pm(Q)$ make sense only in local coordinates, but

$$L(Q_1, Q_2) = (Q_1 Q_2)_+ - Q_{1,+} Q_{2,+} \tag{5.12}$$

is well defined when Q_1 and Q_2 are of order $\leqslant 0$, and locally has the structure $G^+(Q_1)G^-(Q_2)$.

For later purposes we recall the result of Laptev [37] (also shown for ψdo's having the transmission property in [26]):

Theorem 5.2. *(See [37].) Let $n-1 \geqslant 2$. When Q is a ψdo on $\mathbb{R}^{n-1}$ of order $-r < 0$, and $\psi \in C_0^\infty(\mathbb{R}^{n-1})$, then $\psi G^\pm(Q)$ and $G^\pm(Q)\psi$ are in $\mathfrak{S}_{(n-2)/r}$, with $s_j j^{r/(n-2)}$ converging to a limit determined from the principal symbol.*

When Q_1 and Q_2 are ψdo's on $\Sigma = \Sigma_+ \cup \Sigma_-$ of orders $-r_1, -r_2 < 0$, then $L(Q_1, Q_2)$ is in $\mathfrak{S}_{(n-2)/(r_1+r_2)}$.

The operators $\Lambda_\pm$ are of order $-\frac{1}{2}$, but are not standard ψdo's, since the symbols $\lambda_\pm$ are not in Hörmander's symbol space $S^{-\frac{1}{2}}_{1,0}$ as functions of ξ' (high derivatives in ξ'' do not satisfy the required estimates in terms of powers of $1 + |\xi'|$). Then Laptev's theorem is not applicable to $G^\pm(\Lambda_+)$ and $G^\pm(\Lambda_-)$. In fact, one can check that the associated integral operator kernels, calculated explicitly, do not satisfy all the estimates required for Th. 3 in [37]. We expect that it should be possible to show a spectral estimate as in Theorem 5.2 for these operators, but leave out further investigations here, settling for some weaker estimates that still serve our purpose.

In the following, we denote $x_{n-1} = t$, $y_{n-1} = s$, with dual variables τ, σ, to simplify the notation. Let $\zeta(t) \in C^\infty(\mathbb{R})$, taking values in $[0, 1]$ and equal to 1 for $t \geqslant 1$, equal to 0 for $t \leqslant \frac{2}{3}$. For $\varepsilon > 0$, denote $\zeta(t/\varepsilon) = \zeta_\varepsilon(t)$.

Lemma 5.3. *Let $\varepsilon > 0$. The operators $\zeta_\varepsilon G^+(\Lambda_\pm)$ are of order $-\frac{3}{2}$, and $\psi\zeta_\varepsilon G^+(\Lambda_\pm)$ as well as $\zeta_\varepsilon G^+(\Lambda_\pm)\psi$ belong to $\mathfrak{S}_{2(n-1)/3} \cup \mathcal{C}_1$. Similarly, $G^-(\Lambda_\pm)\zeta_\varepsilon$ are of order $-\frac{3}{2}$, and $G^-(\Lambda_\pm)\zeta_\varepsilon\psi$, $\psi G^-(\Lambda_\pm)\zeta_\varepsilon$ belong to $\mathfrak{S}_{2(n-1)/3} \cup \mathcal{C}_1$.*

Proof. It suffices to give the details for $\varepsilon = 1$. Consider $G^+(\Lambda_+)$. First we note that

$$\zeta G^+(\Lambda_+) = \zeta r^+ \Lambda_+ e^- J = r^+ \Lambda_+ \zeta e^- J + r^+[\zeta, \Lambda_+] e^- J = r^+[\zeta, \Lambda_+] e^- J,$$

since $\zeta e^- = 0$; here $[\zeta, \Lambda_+]$ is the commutator $\zeta\Lambda_+ - \Lambda_+\zeta$. As for ordinary ψdo's, the commutator is of lower order; since Λ_+ is nonstandard, we work out proof details:

For $t, s \in \mathbb{R}$, ζ has the Taylor-expansion

$$\zeta(t) = \sum_{0 \leqslant j < J} \frac{1}{j!}\zeta^{(j)}(s)(t-s)^j + (t-s)^J \varrho_J(s,t), \quad \text{where}$$

$$\varrho_J(s,t) = \frac{1}{(J-1)!}\int_0^1 (1-h)^{J-1} \partial^J \zeta\big(s + h(t-s)\big)\, dh.$$

Then, using that $(t-s)^j e^{i(t-s)\tau} = D_\tau^j e^{i(t-s)\tau}$ and integrating by parts (as allowed in oscillatory integrals), we find, denoting $(2\pi)^{1-n}\, d\xi' = đ\xi'$:

$$[\zeta, \Lambda_+]u = \int e^{i(x'-y')\cdot\xi'}\big(\zeta(t)-\zeta(s)\big)\lambda_+\big(\xi'\big)u\big(y'\big)\,đ\xi'\,dy'$$
$$= \int e^{i(x'-y')\cdot\xi'}\Big(\sum_{1\leqslant j<J}\frac{1}{j!}\zeta^{(j)}(s)(t-s)^j + (t-s)^J\varrho_J(s,t)\Big)\lambda_+\big(\xi'\big)u\big(y'\big)\,đ\xi'\,dy'$$
$$= \int e^{i(x'-y')\cdot\xi'}\Big(\sum_{1\leqslant j<J}\frac{1}{j!}\zeta^{(j)}(s)\overline{D}_\tau^j + \varrho_J(s,t)\overline{D}_\tau^J\Big)\lambda_+\big(\xi'\big)u\big(y'\big)\,đ\xi'\,dy'$$
$$= \sum_{1\leqslant j<J}\frac{1}{j!}\operatorname{Op}\big(\overline{D}_\tau^j\lambda_+\big(\xi'\big)\big)\zeta^{(j)}u + \operatorname{Op}\big(\varrho_J(s,t)\overline{D}_\tau^J\lambda_+\big(\xi'\big)\big)u. \tag{5.13}$$

Here

$$\overline{D}_\tau^j\lambda_+\big(\xi'\big) = \overline{D}_\tau^j\big(\big|\big(\xi'',\alpha\big)\big| + i\tau\big)^{-\frac{1}{2}} = c_j\big(\big|\big(\xi'',\alpha\big)\big| + i\tau\big)^{-\frac{1}{2}-j};$$

they are of order $-\frac{1}{2}-j$. Take J so large that the last symbol is integrable in ξ', e.g. $J=n$. Then the terms in the sum over j map $H^r(\mathbb{R}^{n-1})$ into $H^{r+\frac{1}{2}+j}(\mathbb{R}^{n-1})$ ($r\in\mathbb{R}$) by elementary considerations, and the last term has a continuous kernel, supported for $s,t\in[\frac{1}{3},\frac{4}{3}]$. Similar considerations hold for $\zeta G^+(\Lambda_-)$. When we cut down with multiplication by ψ, and functions $1_{\mathbb{R}^{n-1}_\pm}$, we can use the spectral estimates (5.4) and the trace-class property of operators with continuous kernel, to see that $\psi\zeta G^+(\Lambda_\pm)$ are in $\mathfrak{S}_{2(n-1)/3}\cup\mathcal{C}_1$.

The statements for $G^-(\Lambda_\pm)\zeta$ are shown similarly, and for the operators with ψ to the right one can use that $G^+(\Lambda_\pm)$ and $G^-(\Lambda_\mp)$ are adjoints. □

One could argue in a more refined way (e.g. with sequences of nested cutoff functions), to show that since ζ is supported away from 0, *all* the terms in (5.13) give spectrally negligible contributions (in $\bigcap_p\mathfrak{S}_p$) when we take G^+ of them (as for ordinary singular Green operators), but that extra information will not be needed in the following.

Next, we shall show spectral estimates of the contributions to $G^\pm(\Lambda_\pm)$ supported near $t=0$. Here we shall profit from the fact that Birman and Solomyak in [8] showed far-reaching spectral results for nonstandard ψdo's, taking L_p-norms (not just L_∞-norms) of cutoff functions into account. Anisotropic symbols are allowed there, but we just need the case of isotropic symbols with low smoothness.

Theorem 5.4. *(See [8].) Let $A=\operatorname{Op}(b(x)a(x,\xi)c(y))$ on $\mathbb{R}^m$, with $a(x,\xi)$ homogeneous in ξ of degree $-\mu\in]-m,0[$. Denote $m/\mu=\nu$. Then $A\in\mathfrak{S}_\nu$ with*

$$\sup_{j\in\mathbb{N}} s_j(A)j^{1/\nu} \leqslant C\|b\|_{L_{q_1}}\|c\|_{L_{q_2}}[a_{|\xi|=1}]_\beta,$$

if

$$q_1,q_2\in]2,\infty],\qquad \frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{\nu},\qquad \beta=q_1.$$

(Here $[\Phi(x,\xi)]_\beta$ denotes the norm of a certain linear operator on $\mathfrak{S}_\beta$ defined from Φ.) A sufficient condition for the boundedness of $[a_{|\xi|=1}]_\beta$ is that

$$a(x,\xi)|_{|\xi|=1}\in L_\infty\big(S_\xi^{m-1}, W_p^l\big(\mathbb{R}_x^m\big)\big),\quad \textit{with } \frac{1}{2}-\frac{1}{q_1}<\frac{1}{p}\leqslant\frac{1}{2},\ pl>m. \tag{5.14}$$

The paper [8] also covers cases where $\nu\leqslant 1$, and gives spectral asymptotics formulas under additional mild regularity hypotheses (in (5.14), L_∞ is then replaced by C^0).

In order to apply the result we must estimate the effect of replacing the $\Lambda_\pm(\alpha)$ by the operators $\Lambda_\pm(0)$ with strictly homogeneous symbols $\lambda_\pm(\xi',0)$.

Lemma 5.5. *The symbol $\lambda_\pm(\xi',\alpha)-\lambda_\pm(\xi',0)$ of the difference $\Lambda_\pm(\alpha)-\Lambda_\pm(0)$ satisfies*

$$\lambda_\pm\big(\xi',\alpha\big)-\lambda_\pm\big(\xi',0\big) = O\big(\alpha^2\big|\big(\xi',\alpha\big)\big|^{-\frac{3}{2}}\big|\xi'\big|^{-1}\big|\big(\xi'',\alpha\big)\big|^{-1}\big).$$

Hence it defines an operator mapping $H^r(\mathbb{R}^{n-1})$ into $H^{r+\frac{5}{2}}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ for $r\in\mathbb{R}$.

It follows that $\psi(\Lambda_+(\alpha)-\Lambda_+(0))$, $\psi(G^+(\Lambda_+(\alpha))-G^+(\Lambda_+(0)))$, $(\Lambda_-(\alpha)-\Lambda_-(0))\psi_1$ and $(G^-(\Lambda_-(\alpha))-G^-(\Lambda_-(0)))\psi_1$ are in $\mathfrak{S}_{2(n-1)/5}$.

Proof. We give the details for $\Lambda_+(\alpha) - \Lambda_+(0)$. Here

$$\begin{aligned}
\lambda_+\big(\xi',\alpha\big) - \lambda_+\big(\xi',0\big) &= \frac{\lambda_+(\xi',\alpha)^2 - \lambda_+(\xi',0)^2}{\lambda_+(\xi',\alpha) + \lambda_+(\xi',0)} \\
&= \frac{1}{\lambda_+\big(\xi',\alpha\big) + \lambda_+\big(\xi',0\big)}\left(\frac{1}{\big|(\xi'',\alpha)\big| + i\tau} - \frac{1}{\big|\xi''\big| + i\tau}\right) \\
&= \frac{|\xi''| - |(\xi'',\alpha)|}{(\lambda_+(\xi',\alpha) + \lambda_+(\xi',0))(|(\xi'',\alpha)| + i\tau)(|\xi''| + i\tau)} \\
&= \frac{-\alpha^2}{(\lambda_+(\xi',\alpha) + \lambda_+(\xi',0))(|(\xi'',\alpha)| + i\tau)(|\xi''| + i\tau)(|\xi''| + |(\xi'',\alpha)|)} \\
&= O\big(\alpha^2\big|(\xi',\alpha)\big|^{-\frac{3}{2}}\big|\xi'\big|^{-1}\big|(\xi'',\alpha)\big|^{-1}\big).
\end{aligned}$$

The operator with symbol $\zeta(|\xi'|)(\lambda_+(\xi',\alpha) - \lambda_+(\xi',0))$ maps $H^r(\mathbb{R}^{n-1})$ into $H^{r+\frac{5}{2}}(\mathbb{R}^{n-1})$ for $r \in \mathbb{R}$, and the remainder supported near $|\xi'| = 0$ gives an operator mapping into $C^\infty(\mathbb{R}^{n-1})$. When cutoffs by compactly supported functions are applied, this gives operators in $\mathfrak{S}_{2(n-1)/5}$.

The result for Λ_- follows by similar calculations or by duality. □

In the following, $\varphi(t)$ denotes a function in $C^\infty(\mathbb{R})$ that takes values in $[0,1]$ and equals 1 for $|t| \leqslant \frac{1}{3}$, equals 0 for $|t| \geqslant \frac{2}{3}$; we denote $\varphi(t/\varepsilon) = \varphi_\varepsilon(t)$. We can assume that $1_{\mathbb{R}_+}(1-\zeta) = 1_{\mathbb{R}_+}\varphi$.

Lemma 5.6. *There are the following spectral estimates*:

$$\begin{aligned}
&\sup_j s_j\big(\varphi_\varepsilon(t)\psi\Lambda_+(0)\big)j^{1/(2n-2)} \leqslant C_\varepsilon, \\
&\sup_j s_j\big(\varphi_\varepsilon(t)\psi G^+\big(\Lambda_+(0)\big)\big)j^{1/(2n-2)} \leqslant C_\varepsilon, \\
&\sup_j s_j\big(\Lambda_-(0)\psi_1\varphi_\varepsilon(t)\big)j^{1/(2n-2)} \leqslant C_\varepsilon, \\
&\sup_j s_j\big(G^-\big(\Lambda_-(0)\big)\psi_1\varphi_\varepsilon(t)\big)j^{1/(2n-2)} \leqslant C_\varepsilon,
\end{aligned} \tag{5.15}$$

where $C_\varepsilon \to 0$ for $\varepsilon \to 0$.

Proof. For the first line in (5.15), we apply Theorem 5.4 with

$$b\big(x'\big) = \varphi_\varepsilon(t)\psi_2\big(x'\big), \qquad a\big(x',\xi'\big) = \psi\big(x'\big)\lambda_+\big(\xi',0\big), \qquad c\big(x'\big) = 1,$$

where $\psi_2 \in C_0^\infty(\mathbb{R}^{n-1})$, equal to 1 on $\operatorname{supp}\psi$. Here $m = n-1$, $\mu = \frac{1}{2}$ so that $\nu = m/\mu = 2(n-1)$, and we take $q_1 = \beta = \nu = 2(n-1)$ and $q_2 = \infty$. Moreover, since $\frac{1}{2} - \frac{1}{q_1} = \frac{1}{2} - \frac{1}{2(n-1)} = \frac{n-2}{2(n-1)}$, p is taken in $]2, \frac{2(n-1)}{n-2}]$ and l is taken $> (n-1)/p$. (5.14) is satisfied since $\psi \in C_0^\infty(\mathbb{R}^{n-1}) \subset W_p^l(\mathbb{R}^{n-1})$. Then

$$\sup_j s_j\big(\varphi_\varepsilon(t)\psi\Lambda_+\big)j^{1/(2n-2)} \leqslant C\|\varphi_\varepsilon\psi_2\|_{L_\nu} \leqslant C'\operatorname{vol}\big(\operatorname{supp}(\varphi_\varepsilon\psi_2)\big)^{1/(2n-2)} \leqslant C''\varepsilon^{1/(2n-2)} \to 0$$

for $\varepsilon \to 0$.

For the second line in (5.15) we replace b by $1_{\mathbb{R}^{n-1}_+}\varphi_\varepsilon\psi_2$ and c by $1_{\mathbb{R}^{n-1}_-}$, and use that J is an isometric isomorphism.

The proof of the third and fourth line goes in a similar way, interchanging choices for b and c. □

We can finally conclude:

Theorem 5.7. *The operator L_0^{-1} acts like*

$$L_0^{-1} = -P_{0,\nu,\gamma,+} - G^+(\Lambda_+)G^-(\Lambda_-), \tag{5.16}$$

where

$$P_{0,\nu,\gamma,+} : H^s\big(\mathbb{R}^{n-1}_+\big) \to H^{s+1}\big(\mathbb{R}^{n-1}_+\big) \quad \text{for } -\frac{1}{2} < s < \frac{1}{2}, \tag{5.17}$$

and the operators $\psi G^\pm(\Lambda_\pm)$ and $G^\pm(\Lambda_\pm)\psi$ are in $\mathfrak{S}_{2(n-1),0}$, when $\psi \in C_0^\infty(\mathbb{R}^{n-1})$.

Proof. The decomposition (5.16) was shown above. The continuity in (5.17) follows since $P_{0,\nu,\gamma}$ is a constant-coefficient ψdo of order -1. For the next statement, we give details for $\psi G^+(\Lambda_+)$; the other cases are similar. For any $\varepsilon > 0$ we can write

$$\psi G^+\big(\Lambda_+(\alpha)\big) = \varphi_\varepsilon(t)\psi G^+\big(\Lambda_+(0)\big) + \varphi_\varepsilon(t)\psi\big[G^+\big(\Lambda_+(\alpha)\big) - G^+\big(\Lambda_+(0)\big)\big] + \zeta_\varepsilon(t)\psi G^+\big(\Lambda_+(\alpha)\big). \tag{5.18}$$

Here the first term satisfies (5.15), the second term is in $\mathfrak{S}_{2(n-1)/5}$ by Lemma 5.5, and the third term is in $\mathfrak{S}_{\max\{2(n-1)/3,1+\delta\}}$ (for any $\delta > 0$) by Lemma 5.3. Thus the sum of the second and third term satisfies $s_j j^{1/(2n-2)} \to 0$ for $j \to \infty$. We can then apply Lemma 5.1.2°, with $1/p = 1/(2n-2)$, $M = 1/\varepsilon$, B_M being the sum of the second and third terms and B'_M being the first term, $c_M = C_\varepsilon$ and $C_M = C_0 = 0$. □

Remark 5.8. In the case $n = 2$, when Q is a ψdo on $\mathbb{R}^{n-1}$ of order $-r < 0$, the operators $G^\pm(Q)$ are not covered by Theorem 5.2. But certainly the calculations leading to Theorem 5.7 work in this case, so we have at least that $\psi G^\pm(Q) \in \mathfrak{S}_{(n-1)/r,0}$. Similarly, if $n = 2$ and Q_1 and Q_2 are ψdo's on Σ of negative orders $-r_1, -r_2$, then $L(Q_1, Q_2) \in \mathfrak{S}_{(n-1)/(r_1+r_2)}$.

5.3. Variable coefficients, analysis of L^{-1}

Now consider $A = -\Delta + a_0(x)$ on the smooth bounded open subset Ω of $\mathbb{R}^n$, provided with the mixed boundary condition $\nu u = b\gamma_0 u$ on Σ_+, $\gamma_0 u = 0$ on Σ_-. The operator L acts like

$$L\varphi = r^+(b - P_{\gamma,\nu})e^+\varphi = -P_{\gamma,\chi,+}\varphi$$

for $\varphi \in D(L)$, cf. (4.3), (4.6).

In the analysis of L^{-1} on Σ_+ we want to use the insight gained in Section 5.2 for the "flat" constant-coefficient case, but since the ingredients are not standard ψdo's, we do not have the usual localization tools for ψdo's available and must reason very carefully (for example in Eskin's book, formulas for coordinate changes are only worked out for a subclass of symbols with better estimates than the present $\lambda_\pm(\xi')$). The strategy will be to reduce to a situation where the results from the "flat" case can be used directly.

Our aim is to show:

Theorem 5.9. *The operator L^{-1} acts like $-P_{\nu,\gamma,+} + R$, where $R \in \mathfrak{S}_{n-1,0}$. In particular, $L^{-1} \in \mathfrak{S}_{n-1}$.*

This will be shown in several steps. We first show a preliminary spectral estimate for L^{-1}; it will be improved later.

Lemma 5.10. *The operator $L^{-1} : X^* \to X$ extends to an operator M that maps continuously*

$$M : H^s\big(\Sigma_+^\circ\big) \to H_0^{s+\frac{1}{2}-\varepsilon}(\Sigma_+) \quad \textit{for } -1 < s \leqslant \frac{1}{2}. \tag{5.19}$$

In particular, the closure of L^{-1} in $L_2(\Sigma_+)$ is a continuous operator from $L_2(\Sigma_+)$ to $H_0^{\frac{1}{2}-\varepsilon}(\Sigma_+)$; it belongs to $\mathfrak{S}_{(n-1)/(\frac{1}{2}-\varepsilon)}$ for $\varepsilon > 0$.

Proof. It follows from Proposition 4.1 that L^{-1} is continuous from $X^* = H^{\frac{1}{2}}(\Sigma_+^\circ)$ to $H_0^{1-\varepsilon}(\Sigma_+)$. Then it has an adjoint M (with respect to dualities consistent with the $L_2(\Sigma_+)$-scalar product) that is continuous from $H^{-1+\varepsilon}(\Sigma_+^\circ)$ to $H_0^{-\frac{1}{2}}(\Sigma_+)$. But since L^{-1} is known to be selfadjoint (from X^* to X, consistently with the L_2-scalar product), M must be an extension of L^{-1}. Now (5.19) follows by interpolation. For $s = 0$ we find the last statement, where the spectral information follows from (5.4); note that $H_0^{\frac{1}{2}-\varepsilon}(\Sigma_+) = H^{\frac{1}{2}-\varepsilon}(\Sigma_+^\circ)$. □

When $\{\varrho_1, \dots, \varrho_N\}$ is any partition of unity for Σ, then $L^{-1} = \sum_{k=1}^N \varrho_k L^{-1}$, and it suffices to analyze the terms $\varrho_k L^{-1}$ individually. Here we can also introduce a cutoff function ψ_k to the right, considering terms $\varrho_k L^{-1}\psi_k$ where ψ_k is 1 on the support of ϱ_k; the effect of such a modification will be studied later.

Our next observation is that it is allowed to perform smooth diffeomorphisms of Ω, in particular of Σ. Assume that κ is a diffeomorphism of an open neighborhood U_0 of $\overline{\Omega}$ onto another open set $V_0 \subset \mathbb{R}^n$, where $\kappa(\Omega) = \Omega'$, then functions $f(x)$ on Ω are carried over to functions $\underline{f}(y) = f(\kappa^{-1}(y))$ on Ω', and operators P over $\overline{\Omega}$ are carried over to operators $\underline{P}$ over $\overline{\Omega'}$:

$$(\underline{P}\,\underline{f})(y) = (Pf)\big(\kappa^{-1}(y)\big). \tag{5.20}$$

The ψdo $P_{\gamma,\chi}$ on Σ carries over to a ψdo $\underline{P_{\gamma,\chi}}$ on $\underline{\Sigma}$ according to well-known rules; it is again elliptic of order 1 and has the same principal symbol. The operator L carries over to $\underline{L}$, equal to the truncated version of $\underline{P_{\gamma,\chi}}$, where we apply e^+

and r^+ with respect to the partition $\underline{\Sigma} = \underline{\Sigma}_+ \cup \underline{\Sigma}_-$. There is again an inverse $\underline{L}^{-1}$, with mapping properties as explained for L^{-1}, relative to the transformed sets.

To find the structure of L^{-1} in a neighborhood of a point $x_0 \in \Sigma$, let us consider $\psi L^{-1}\psi_1$, where ψ and ψ_1 are C^∞-functions supported in the neighborhood, with $\psi_1 = 1$ on $\operatorname{supp}\psi$.

It can be assumed, after a translation and rotation if necessary, that $x_0 \in \overline{\mathbb{R}}^{n-1}_+$ and the interior normal at $x_0 = \{x_{0,1}, \dots, x_{0,n-1}, 0\}$ is $(0, \dots, 0, 1)$, such that $x_{0,n-1} > 0$ if $x_0 \in \Sigma_+^\circ$ and $x_0 = 0$ if $x_0 \in \partial\Sigma_+$; in the latter case we can assume that the interior normal to $\partial\Sigma_+ \subset \Sigma$ at x_0 is $\{0, \dots, 0, 1, 0\}$. We choose a diffeomorphism that changes $\overline{\Omega}$ only near x_0. If $x_0 \in \Sigma_+^\circ$, we can assume that ψ and ψ_1 are supported away from $\partial\Sigma_+$; then we let the diffeomorphism be such that it transforms a neighborhood $U \subset \mathbb{R}^n$ of x_0 over to $V \subset \mathbb{R}^n$, carrying $U \cap \Omega$ and $U \cap \Sigma_+$ over to $V \cap \mathbb{R}^n_+$ and $V \cap \overline{\mathbb{R}}^{n-1}_+$, with ψ and ψ_1 supported in $U \cap \Sigma_+^\circ$. If $x_0 \in \partial\Sigma_+$, we choose the diffeomorphism such that $U \cap \Omega$, $U \cap \Sigma$ and $U \cap \Sigma_+$ are mapped to $V \cap \mathbb{R}^n p$, $V \cap \mathbb{R}^{n-1}$ and $V \cap \overline{\mathbb{R}}^{n-1}_+$, ψ and ψ_1 supported in $V \cap \mathbb{R}^{n-1}$. (The identifications of $\mathbb{R}^{n-1}$ and $\overline{\mathbb{R}}^{n-1}_+$ with $\mathbb{R}^{n-1} \times \{0\}$ and $\overline{\mathbb{R}}^{n-1}_+ \times \{0\}$ as subsets of $\mathbb{R}^n$ are understood here.)

This gives a transformed operator $\underline{\psi}\,\underline{L}^{-1}\underline{\psi}_1$ acting on functions supported in $V' = V \cap \mathbb{R}^{n-1}$. For simplicity of notation, we drop the underlines in the following.

We shall compare $\psi L^{-1}\psi_1$ with $\psi L_0^{-1}\psi_1$ where L_0^{-1} is the constant-coefficient operator studied in Section 5.2. Let us give the details for the most delicate case $x_0 \in \partial\Sigma_+$, where the effects of truncation have to be taken into account.

Proposition 5.11. *In the setting described in the preceding lines, we have that*

$$\psi L^{-1}\psi_1 = -\psi P_{\nu,\gamma,+}\psi_1 + R_1, \tag{5.21}$$

as operators in $L_2(V' \cap \mathbb{R}^{n-1}_+)$, where $R_1 \in \mathfrak{S}_{n-1,0}$.

Proof. It follows from Theorem 5.7 that

$$\psi L_0^{-1}\psi_1 = -\psi P_{0,\nu,\gamma,+}\psi_1 + R_2, \tag{5.22}$$

where $R_2 = \psi G^+(\Lambda_+)G^-(\Lambda_-)\psi_1$ is in $\mathfrak{S}_{n-1,0}$; cf. (5.2). We shall now compare $\psi L^{-1}\psi_1$ and $\psi L_0^{-1}\psi_1$. There is the difficulty that the operators L_1^{-1} and L_0^{-1} do not act over the same manifold, but this will be dealt with by introduction of more cutoff functions. Let $\psi_2 \in C_0^\infty(V')$, satisfying $\psi_2 = 1$ on $\operatorname{supp}\psi_1$. We calculate:

$$\psi L^{-1}\psi_1 - \psi L_0^{-1}\psi_1 = \psi L^{-1}\psi_2\psi_1 - \psi\psi_2 L_0^{-1}\psi_1 = \psi L^{-1}\psi_2 L_0 L_0^{-1}\psi_1 - \psi L^{-1} L\psi_2 L_0^{-1}\psi_1. \tag{5.23}$$

We want to insert the factor ψ_2 in the middle of $L_0 L_0^{-1}$ as well as $L^{-1}L$; this is justified as follows: Write e.g.

$$\psi L^{-1}\psi_2 L_0 L_0^{-1}\psi_1 = \psi L^{-1}\psi_2 L_0\psi_2 L_0^{-1}\psi_1 + \psi L^{-1}\psi_2 L_0(1-\psi_2)L_0^{-1}\psi_1.$$

For the last term, we note that (since $(1-\psi_2)\psi_1 = 0$)

$$(1-\psi_2)L_0^{-1}\psi_1 = \big[(1-\psi_2), L_0^{-1}\big]\psi_1 = \big[L_0^{-1}, \psi_2\big]\psi_1 = L_0^{-1}[\psi_2, L_0]L_0^{-1}\psi_1,$$

where $[L_0, \psi_2] = [-P_{0,\gamma,\nu,+}, \psi_2]$ is L_2-bounded (since $P_{0,\gamma,\nu}$ is a first-order ψdo). Then

$$\psi L^{-1}\psi_2 L_0(1-\psi_2)L_0^{-1}\psi_1 = \psi L^{-1}\psi_2 L_0 L_0^{-1}[\psi_2, L_0]L_0^{-1}\psi_1 = \psi L^{-1}\psi_2[\psi_2, L_0]L_0^{-1}\psi_1,$$

which is the composition of $\psi L^{-1}\psi_2 \in \mathfrak{S}_{(n-1)/(\frac12-\varepsilon)}$ (cf. Lemma 5.10), the bounded operator $[\psi_2, L_0]$, and $L_0^{-1}\psi_1 \in \mathfrak{S}_{n-1}$ (cf. Theorem 5.7; its adjoint is $\overline{\psi}_1 L_0^{-1}$). Then the whole term is in $\mathfrak{S}_{(n-1)/(\frac32-\varepsilon)}$, and

$$\psi L^{-1}\psi_2 L_0 L_0^{-1}\psi_1 = \psi L^{-1}\psi_2 L_0\psi_2 L_0^{-1}\psi_1 + R_3, \quad \text{where } R_3 \in \mathfrak{S}_{(n-1)/(\frac32-\varepsilon)}.$$

Similarly, we can insert a factor ψ_2 between L^{-1} and L in the last term of (5.23), making an error that is in $\mathfrak{S}_{(n-1)/(\frac32-\varepsilon)}$.

It remains to consider

$$\psi L^{-1}\psi_2 L_0\psi_2 L_0^{-1}\psi_1 - \psi L^{-1}\psi_2 L\psi_2 L_0^{-1}\psi_1 = \big(\psi L^{-1}\psi_3\big)(\psi_2 L_0\psi_2 - \psi_2 L\psi_2)\big(\psi_3 L_0^{-1}\psi_1\big),$$

where we have replaced ψ_2 by $\psi_2\psi_3$, with $\psi_3 = 1$ on $\operatorname{supp}\psi_2$, in a few places. Here the first factor is in $\mathfrak{S}_{(n-1)/(\frac12-\varepsilon)}$ by Lemma 5.10, the last factor is in $\mathfrak{S}_{n-1}$ by Theorem 5.7, and the middle factor is a truncated ψdo of order zero, hence bounded in L_2, since $P_{\gamma,\chi}$ and $P_{0,\gamma,\nu}$ have the same principal symbol on V' (recall (4.6)). Then the whole expression is in $\mathfrak{S}_{(n-1)/(\frac32-\varepsilon)}$. Thus we have obtained that $\psi L^{-1}\psi_1 - \psi L_0^{-1}\psi_1 \in \mathfrak{S}_{(n-1)/(\frac32-\varepsilon)}$, which is contained in $\mathfrak{S}_{n-1,0}$. Together with (5.22) this shows

$$\psi L^{-1}\psi_1 - \psi(-P_{0,\nu,\gamma,+})\psi_1 \in \mathfrak{S}_{n-1,0}.$$

Finally, since $P_{0,\nu,\gamma}$ and $P_{\nu,\gamma}$ have the same principal symbol (of order -1) on V', $\psi P_{0,\nu,\gamma,+}\psi_1 - \psi P_{\nu,\gamma,+}\psi_1$ is a truncated ψdo of order -2; hence it is in $\mathfrak{S}_{(n-1)/2} \subset \mathfrak{S}_{n-1,0}$, and (5.21) follows. $\square$

Proof of Theorem 5.9. We now consider L^{-1} on Σ_+, written as $L^{-1} = \sum_{k=1}^N \varrho_k L^{-1}$ for some partition of unity $\sum_{k=1}^N \varrho_k = 1$. To analyze an individual term $\varrho_k L^{-1}$, we choose a cutoff function ψ_k that is 1 on $\operatorname{supp}\varrho_k$, and write

$$\begin{aligned}\varrho_k L^{-1} &= \varrho_k L^{-1}\psi_k + \varrho_k L^{-1}(1-\psi_k) = \varrho_k L^{-1}\psi_k + \varrho_k\big[L^{-1}, 1-\psi_k\big]\\ &= \varrho_k L^{-1}\psi_k + \varrho_k L^{-1}[L,\psi_k]L^{-1}.\end{aligned} \tag{5.24}$$

Since $[L,\psi_k]$ is a truncated zero-order ψdo, it is bounded in L_2. By Lemma 5.10, $L^{-1} \in \mathfrak{S}_{(n-1)/(\frac12-\varepsilon)}$, so the last term satisfies

$$\varrho_k L^{-1}[L,\psi_k]L^{-1} \in \mathfrak{S}_{n-1+\delta}, \quad \text{any } \delta > 0. \tag{5.25}$$

We can assume that the supports of ϱ_k and ψ_k are so small that a diffeomorphism as described before Proposition 5.11 can be applied in a neighborhood of the supports; then Proposition 5.11 gives that

$$\varrho_k L^{-1}\psi_k = -\varrho_k P_{\nu,\gamma,+}\psi_k + R_{1,k}, \quad \text{with } R_{1,k} \in \mathfrak{S}_{n-1,0}. \tag{5.26}$$

A first observation resulting from this is that $\varrho_k L^{-1}\psi_k$ is in $\mathfrak{S}_{n-1}$, since the ψdo of order -1 is there. Then in view of (5.24)–(5.25), $\varrho_k L^{-1} \in \mathfrak{S}_{n-1+\delta}$, any $\delta > 0$. Summation in k gives that $L^{-1} \in \mathfrak{S}_{n-1+\delta}$. Next, we go back to (5.24), where the new information allows us to conclude that

$$\varrho_k L^{-1}[L,\psi_k]L^{-1} \in \mathfrak{S}_{(n-1+\delta)/2} \subset \mathfrak{S}_{n-1,0}.$$

In view of (5.26), we finally get that

$$\varrho_k L^{-1} = -\varrho_k P_{\nu,\gamma,+}\psi_k + R_{2,k}, \quad \text{with } R_{2,k} \in \mathfrak{S}_{n-1,0}.$$

Summation in k gives that

$$L^{-1} = -\sum_{k=1}^N \varrho_k P_{\nu,\gamma,+}\psi_k + \sum_{k=1}^N R_{2,k} = -P_{\nu,\gamma,+} + R_3,$$

with $R_3 \in \mathfrak{S}_{n-1,0}$. $\square$

5.4. Reduction of the Poisson operators

We now consider the operator (5.1). To find the spectral behavior, we note that by the general rule for eigenvalues $\mu_j(ST) = \mu_j(TS)$, we can write

$$\mu_j\big(\mathrm{i}_V\gamma_V^{-1}L^{-1}\big(\gamma_V^{-1}\big)^*\operatorname{pr}_V\big) = \mu_j\big(L^{-1}\big(\gamma_V^{-1}\big)^*\operatorname{pr}_V\mathrm{i}_V\gamma_V^{-1}\big) = \mu_j\big(L^{-1}\big(\gamma_V^{-1}\big)^*\gamma_V^{-1}\big), \tag{5.27}$$

in view of (2.15).

Lemma 5.12. *The operator $(\gamma_V^{-1})^*\gamma_V^{-1}$ satisfies*

$$\big(\gamma_V^{-1}\big)^*\gamma_V^{-1} = P_{1,+}, \tag{5.28}$$

where $P_1 = K_\gamma^* K_\gamma$ is a selfadjoint nonnegative elliptic ψdo of order -1 on Σ with principal symbol $(2|\xi'|)^{-1}$.

Proof. We have for $\varphi, \psi \in X$:

$$\begin{aligned}\big(\big(\gamma_V^{-1}\big)^*\gamma_V^{-1}\varphi,\psi\big)_{X^*,X} &= \big(\gamma_V^{-1}\varphi,\gamma_V^{-1}\psi\big)_V = (K_\gamma\varphi, K_\gamma\psi)_H\\ &= \big(K_\gamma^*K_\gamma\varphi,\psi\big)_{\frac12,-\frac12} = (P_1\varphi,\psi)_{\frac12,-\frac12},\end{aligned} \tag{5.29}$$

where $P_1 = K_\gamma^* K_\gamma$ is a ψdo of order -1 on Σ, by the rules of calculus for pseudodifferential boundary operators; it is clearly selfadjoint nonnegative. The principal symbol is found from the calculation using the principal symbol-kernel $\tilde{k}^0 = e^{-x_n|\xi'|}$ of K_γ:

$$\int_0^\infty e^{-x_n|\xi'|}e^{-x_n|\xi'|}\,dx_n = \big(2|\xi'|\big)^{-1},$$

also equal to $\|\tilde{k}^0\|^2_{L_2(\mathbb{R}_+)}$. Since φ and ψ are supported in Σ_+, (5.29) may be rewritten further as

$$(P_1\varphi,\psi)_{\frac12,-\frac12} = \big(P_1 e^+\varphi, e^+\psi\big)_{\frac12,-\frac12} = \big(r^+ P_1 e^+\varphi,\psi\big)_{H^{\frac12}(\Sigma_+^\circ), H_0^{-\frac12}(\Sigma_+)} = (P_{1,+}\varphi,\psi)_{X^*,X}.$$

Then (5.28) follows since φ and ψ are arbitrary. $\square$

Next, we define

$$P_2 = P_1^{\frac12}, \tag{5.30}$$

a nonnegative selfadjoint ψdo on Σ of order $-\frac12$, by Seeley [51]. Moreover, set

$$G^{(1)} = P_{1,+} - (P_{2,+})^2, \qquad G^{(\frac12)} = (P_{1,+})^{\frac12} - P_{2,+}. \tag{5.31}$$

Lemma 5.13. *When $n \geqslant 3$, $G^{(1)} \in \mathfrak{S}_{n-2}$ and $G^{(\frac12)} \in \mathfrak{S}_{2(n-2)}$. When $n = 2$, $G^{(1)} \in \mathfrak{S}_{n-1,0}$ and $G^{(\frac12)} \in \mathfrak{S}_{2(n-1),0}$.*

Proof. We first note (cf. (5.12)) that

$$G^{(1)} = P_{1,+} - P_{2,+}P_{2,+} = r^+P_2P_2e^+ - r^+P_2e^+r^+P_2e^+ = L(P_2,P_2). \tag{5.32}$$

Since P_2 is a ψdo of order $-\frac12$, we have by Theorem 5.2 that $L(P_2,P_2)$ is in $\mathfrak{S}_{n-2}$ when $n \geqslant 3$. For $n = 2$, we see that $s_j(L(P_2,P_2))j^{1/(n-1)} \to 0$ for $j \to \infty$ by use of Remark 5.8.

To obtain the result for $G^{(\frac12)}$, we shall as in [25] appeal to a result of Birman, Koplienko and Solomyak [7]. It states that when M_1 and M_2 are compact selfadjoint nonnegative operators on a Hilbert space H such that $G^{(1)} = M_1 - M_2$ is in $\mathfrak{S}_\gamma$ for some $\gamma > 0$, then $G^{(\sigma)} = M_1^\sigma - M_2^\sigma$ is in $\mathfrak{S}_{\gamma/\sigma}$ for all $0 < \sigma < 1$. Applying this with $M_1 = P_{1,+}$, $M_2 = (P_{2,+})^2$ and $\sigma = \frac12$, we get the desired result when $n \geqslant 3$. The paper [7] also shows that $\limsup s_j(G^{(\frac12)})j^{1/(2n-2)}$ is dominated by $\limsup s_j(G^{(1)})j^{1/(n-1)}$, which assures the statement for $n = 2$. $\square$

Now we continue the analysis in (5.27) as follows:

Proposition 5.14.

$$\mu_j\big(\mathrm{i}_V\gamma_V^{-1}L^{-1}\big(\gamma_V^{-1}\big)^*\mathrm{pr}_V\big) = \mu_j\big(P_{2,+}L^{-1}P_{2,+} + G'\big), \tag{5.33}$$

where G' is the selfadjoint operator

$$G' = G^{(\frac12)}L^{-1}P_{2,+} + P_{2,+}L^{-1}G^{(\frac12)} + G^{(\frac12)}L^{-1}G^{(\frac12)}; \tag{5.34}$$

it is in $\mathfrak{S}_{(n-1)/2-r}$ for a positive r when $n \geqslant 3$, and in $\mathfrak{S}_{(n-1)/2,0}$ when $n = 2$.

Proof. Using Lemma 5.13 and (5.27) and the definitions (5.30)–(5.31) we have:

$$\begin{aligned}
\mu_j\big(\mathrm{i}_V\gamma_V^{-1}L^{-1}\big(\gamma_V^{-1}\big)^*\mathrm{pr}_V\big) &= \mu_j\big(L^{-1}P_{1,+}\big) = \mu_j\big(L^{-1}(P_{1,+})^{\frac12}(P_{1,+})^{\frac12}\big)\\
&= \mu_j\big((P_{1,+})^{\frac12}L^{-1}(P_{1,+})^{\frac12}\big)\\
&= \mu_j\big(P_{2,+}L^{-1}P_{2,+} + G'\big),
\end{aligned}$$

where G' is as in (5.34). When $n \geqslant 3$, we use that $L^{-1} \in \mathfrak{S}_{n-1}$, $P_{2,+} \in \mathfrak{S}_{2(n-1)}$, and $G^{(\frac12)} \in \mathfrak{S}_{2(n-2)}$ (by Lemma 5.13) and the rule (5.2) to see that

$$G' \in \mathfrak{S}_p \quad \text{with } p = \left(\frac{1}{n-1} + \frac{1}{2(n-1)} + \frac{1}{2(n-2)}\right)^{-1} < \frac{n-1}{2},$$

hence $G' \in \mathfrak{S}_{(n-1)/2-r}$ for a positive r. When $n = 2$, $G^{(\frac12)} \in \mathfrak{S}_{2(n-1),0}$ leads to $G' \in \mathfrak{S}_{(n-1)/2,0}$. $\square$

We can then conclude:

Theorem 5.15. *The eigenvalues of $A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma$ satisfy*:

$$\mu_j\big(A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma\big) = \mu_j\big(\mathrm{i}_V\gamma_V^{-1}L^{-1}\big(\gamma_V^{-1}\big)^*\mathrm{pr}_V\big) = \mu_j(P_{2,+}P_{\nu,\gamma,+}P_{2,+} + G), \tag{5.35}$$

where $G \in \mathfrak{S}_{(n-1)/2,0}$.

Proof. This follows by inserting the information from Theorem 5.9 in the formula (5.33), using that $P_{2,+}RP_{2,+} \in \mathfrak{S}_{(n-1)/2,0}$ by the rules in Section 5.1. $\square$

5.5. Spectral asymptotics

To find the asymptotic behavior of the s-numbers we shall use the following theorem shown in [31] (Th. 3.3):

Theorem 5.16. *(See [31].) Let P be an operator on Σ composed of l classical pseudodifferential operators $P_1, \dots, P_l$ of negative orders $-t_1, \dots, -t_l$ and $l+1$ functions $b_1, \dots, b_{l+1}$ that are piecewise continuous on Σ with possible jumps at $\partial\Sigma_+$,*

$$P = b_1 P_1 \dots b_l P_l b_{l+1}. \tag{5.36}$$

Let $t = t_1 + \dots + t_l$. Then P has the spectral behavior:

$$s_j(P) j^{t/(n-1)} \to c(P)^{t/(n-1)} \quad \textit{for } j \to \infty, \tag{5.37}$$

where

$$c(P) = \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\Sigma} \int_{|\xi'|=1} \left|b_1 \dots b_{l+1} p_1^0 \dots p_l^0\right|^{(n-1)/t} d\omega(\xi')\, dx'. \tag{5.38}$$

Let us also recall that the principal symbol of $P_{\nu,\gamma}$ is $p^0 = -|\xi'|^{-1}$. As noted in Lemma 5.12, the principal symbol of $P_1 = K_\gamma^* K_\gamma$ is $\|\tilde{k}^0\|_{L_2}^2 = (2|\xi'|)^{-1}$; that of the squareroot P_2 is $\|\tilde{k}^0\|_{L_2} = (2|\xi'|)^{-\frac{1}{2}}$.
Then we can finally show:

Theorem 5.17. *Let $\lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma)$. The s-numbers of $(A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}$ satisfy the asymptotic formula*

$$s_j\big((A_{\chi,\Sigma_+} - \lambda)^{-1} - (A_\gamma - \lambda)^{-1}\big) j^{2/(n-1)} \to C_{0,+}^{2/(n-1)} \quad \textit{for } j \to \infty, \tag{5.39}$$

where

$$C_{0,+} = \frac{1}{(n-1)(2\pi)^{n-1}} \int_{\Sigma_+} \int_{|\xi'|=1} \left(\|\tilde{k}^0\|_{L_2(\mathbb{R}_+)} |p^0|^{1/2}\right)^{n-1} d\omega(\xi')\, dx' = c_n \int_{\Sigma_+} 1\, dx', \tag{5.40}$$

for a constant c_n depending on n (see (5.41) below).

Proof. We first treat the case without λ (or with $\lambda = 0$), where the realizations are positive. Here the s-numbers are the positive eigenvalues, and we use (5.35). We can identify $P_{2,+}P_{\nu,\gamma,+}P_{2,+}$ with the operator $1_{\Sigma_+}P_2 1_{\Sigma_+}P_{\nu,\gamma}1_{\Sigma_+}P_2 1_{\Sigma_+}$ in $L_2(\Sigma)$, acting trivially (as 0) on $L_2(\Sigma_-)$. An application of Theorem 5.16 to this operator gives that

$$\mu_j(1_{\Sigma_+}P_2 1_{\Sigma_+}P_{\nu,\gamma}1_{\Sigma_+}P_2 1_{\Sigma_+}) j^{2/(n-1)} \to c^{2/(n-1)} \quad \text{for } j \to \infty,$$

where

$$\begin{aligned} c &= \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\Sigma} \int_{|\xi'|=1} \left|1_{\Sigma_+} p_2^0 p^0 p_2^0\right|^{(n-1)/2} d\omega(\xi')\, dx' \\ &= \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{\Sigma_+} \int_{|\xi'|=1} \left(\|\tilde{k}^0\|^2 |p^0|\right)^{(n-1)/2} d\omega(\xi')\, dx' = C_{0,+}. \end{aligned}$$

Since $G \in \mathfrak{S}_{(n-1)/2,0}$, this asymptotic behavior is preserved under addition of G, by Lemma 5.1.1°, which implies the main statement in the theorem for $\lambda = 0$.
Since $\|\tilde{k}^0\|^2 = (2|\xi'|)^{-1}$, $|p^0| = |\xi'|^{-1}$, the constant c_n can be calculated as

$$\begin{aligned} c_n &= \frac{1}{(n-1)(2\pi)^{(n-1)}} \int_{|\xi'|=1} 2^{-(n-1)/2}\, d\omega(\xi') \\ &= \frac{1}{(n-1)(2\pi)^{(n-1)}} 2^{-(n-1)/2}(n-1)\pi^{(n-1)/2}\Gamma\left(1 + \frac{n-1}{2}\right)^{-1} \\ &= (2\pi)^{-(n-1)/2} 2^{1-n} \Gamma\left(1 + \frac{n-1}{2}\right)^{-1}. \end{aligned} \tag{5.41}$$

For more general $\lambda \in \varrho(A_{\chi,\Sigma_+}) \cap \varrho(A_\gamma)$, we use a resolvent identity as in [31]:

$$\begin{aligned}(B-\lambda)^{-1}-(B_1-\lambda)^{-1} &= \bigl(1+\lambda(B_1-\lambda)^{-1}\bigr)\bigl(B^{-1}-B_1^{-1}\bigr)\bigl(1+\lambda(B-\lambda)^{-1}\bigr)\\ &= B^{-1}-B_1^{-1}+\lambda(B_1-\lambda)^{-1}\bigl(B^{-1}-B_1^{-1}\bigr)+\bigl(B^{-1}-B_1^{-1}\bigr)\lambda(B-\lambda)^{-1}\\ &\quad+\lambda(B_1-\lambda)^{-1}\bigl(B^{-1}-B_1^{-1}\bigr)\lambda(B-\lambda)^{-1},\end{aligned} \tag{5.42}$$

valid for $\lambda, 0 \in \varrho(B) \cap \varrho(B_1)$. We apply it to $B = A_{\chi,\Sigma_+}$ and $B_1 = A_\gamma$ for $\lambda \in \varrho(\widetilde{A}) \cap \varrho(A_\gamma)$. Since $(A_\gamma-\lambda)^{-1}$ and $(A_{\chi,\Sigma_+}-\lambda)^{-1}$ are in $\mathfrak{S}_{n/2}$ (cf. Corollary 3.2), the three last terms are in $\mathfrak{S}_{(n-1)/2-r}$ with $r > 0$. Then we find by Lemma 5.1.1° that the main asymptotic estimate of the s-numbers is the same as for $A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma$. $\square$

For $n \geqslant 3$, a generalization of Laptev's result in Theorem 5.2 to nonstandard ψdo's like Λ_+ and Λ_- would allow an estimate of $s_j(A^{-1}_{\chi,\Sigma_+} - A^{-1}_\gamma) - C_{0,+}^{2/(n-1)} j^{-2/(n-1)}$ by a lower power of j.

The methods of [30] would be useful in an extension of the results to exterior domains.

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References

[1] H. Abels, G. Grubb, I. Wood, Extension theory and Kreĭn-type resolvent formulas for nonsmooth boundary value problems, arXiv:1008.3281.
[2] D. Alpay, J. Behrndt, Generalized Q-functions and Dirichlet-to-Neumann maps for elliptic differential operators, J. Funct. Anal. 257 (2009) 1666–1694.
[3] W.O. Amrein, D.B. Pearson, M operators: a generalisation of Weyl–Titchmarsh theory, J. Comput. Appl. Math. 171 (2004) 1–26.
[4] J. Behrndt, M. Langer, Boundary value problems for elliptic partial differential operators on bounded domains, J. Funct. Anal. 243 (2007) 536–565.
[5] M.S. Birman, On the theory of self-adjoint extensions of positive definite operators, Mat. Sb. (N.S.) 38 (80) (1956) 431–450 (in Russian).
[6] M.S. Birman, Perturbations of the continuous spectrum of a singular elliptic operator by varying the boundary and the boundary conditions, Vestnik Leningrad. Univ. 17 (1962) 22–55; English translation in: Spectral Theory of Differential Operators, in: Amer. Math. Soc. Transl. Ser. (2), vol. 225, Amer. Math. Soc., Providence, RI, 2008, pp. 19–53.
[7] M.S. Birman, L.S. Koplienko, M.Z. Solomyak, Estimates of the spectrum of a difference of fractional powers of selfadjoint operators, Izv. Vyssh. Uchebn. Zaved. Mat. 3 (154) (1975) 3–10 (in Russian).
[8] M.S. Birman, M.Z. Solomyak, Asymptotic behavior of the spectrum of pseudodifferential operators with anisotropically homogeneous symbols, Vestnik Leningrad. Univ. 13 (1977) 13–21, English translation in: Vestnik Leningrad. Univ. Math. 10 (1982) 237–247.
[9] M.S. Birman, M.Z. Solomyak, Asymptotics of the spectrum of variational problems on solutions of elliptic equations in unbounded domains, Funktsional. Anal. i Prilozhen. 14 (1980) 27–35; English translation in: Funct. Anal. Appl. 14 (1981) 267–274.
[10] L. Boutet de Monvel, Boundary problems for pseudodifferential operators, Acta Math. 126 (1971) 11–51.
[11] R. Brown, The mixed problem for Laplace's equation in a class of Lipschitz domains, Comm. Partial Differential Equations 19 (1994) 1217–1233.
[12] B.M. Brown, G. Grubb, I.G. Wood, *M*-functions for closed extensions of adjoint pairs of operators with applications to elliptic boundary problems, Math. Nachr. 282 (2009) 314–347.
[13] M. Brown, M. Marletta, S. Naboko, I. Wood, Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices, J. Lond. Math. Soc. (2) 77 (2008) 700–718.
[14] G.I. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations, Transl. from Russian by S. Smith, Transl. Math. Monogr., vol. 52, Amer. Math. Soc., Providence, RI, 1981.
[15] Ky Fan, Maximum properties and inequalities for the eigenvalues of completely continuous operators, Proc. Natl. Acad. Sci. USA 37 (1951) 760–766.
[16] F. Gesztesy, M. Mitrea, Generalized Robin boundary conditions, Robin-to-Dirichlet maps, and Krein-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains, in: Perspectives in Partial Differential Equations, Harmonic Analysis and Applications, in: Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 105–173.
[17] F. Gesztesy, M. Mitrea, Robin-to-Robin maps and Kreĭn-type resolvent formulas for Schrödinger operators on bounded Lipschitz domains, in: V. Adamyan, Y.M. Berezansky, I. Gohberg, M.L. Gorbachuk, V. Gorbachuk, A.N. Kochubei, H. Langer, G. Popov (Eds.), Modern Analysis and Applications. The Mark Kreĭn Centenary Conference, vol. 2, in: Oper. Theory Adv. Appl., vol. 191, Birkhäuser, Basel, 2009, pp. 81–113.
[18] F. Gesztesy, M. Mitrea, A description of all selfadjoint extensions of the Laplacian and Kreĭn-type resolvent formulas in nonsmooth domains, J. Anal. Math. 113 (2011) 53–172.
[19] I.C. Gohberg, M.G. Kreĭn, Introduction to the Theory of Linear Nonselfadjoint Operators, Transl. from Russian by A. Feinstein, Transl. Math. Monogr., vol. 18, Amer. Math. Soc., Providence, RI, 1969.
[20] V.I. Gorbachuk, M.L. Gorbachuk, Boundary Value Problems for Operator Differential Equations, Kluwer, Dordrecht, 1991.
[21] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Monogr. Stud. Mat., vol. 24, Pitman Advanced Publishing Program, Boston, MA, 1985.
[22] G. Grubb, A characterization of the non-local boundary value problems associated with an elliptic operator, Ann. Sc. Norm. Super. Pisa 22 (1968) 425–513.
[23] G. Grubb, Les problèmes aux limites généraux d'un opérateur elliptique, provenant de la théorie variationnelle, Bull. Sci. Math. 94 (1970) 113–157.
[24] G. Grubb, Properties of normal boundary problems for elliptic even-order systems, Ann. Sc. Norm. Super. Pisa (IV) 1 (1974) 1–61.
[25] G. Grubb, Spectral asymptotics for the "soft" self-adjoint extension of a symmetric elliptic differential operator, J. Operator Theory 10 (1983) 9–20.
[26] G. Grubb, Singular Green operators and their spectral asymptotics, Duke Math. J. 51 (1984) 477–528.
[27] G. Grubb, Functional Calculus of Pseudodifferential Boundary Problems, second edition, Progr. Math., vol. 65, Birkhäuser, Boston, 1996.
[28] G. Grubb, Kreĭn resolvent formulas for elliptic boundary problems in nonsmooth domains, Rend. Semin. Mat. Univ. Politec. Torino 66 (2008) 13–39.
[29] G. Grubb, Distributions and Operators, Grad. Texts in Math., vol. 252, Springer-Verlag, New York, 2009.
[30] G. Grubb, Perturbation of essential spectra of exterior elliptic problems, Appl. Anal. 90 (2011) 103–123.
[31] G. Grubb, Spectral asymptotics for Robin problems with a discontinuous coefficient, J. Spectr. Theory, in press, arXiv:1009.0997.
[32] G. Grubb, Extension theory for elliptic partial differential operators with pseudodifferential methods, arXiv:1008.1081.

[33] G. Harutyunyan, B.-W. Schulze, Elliptic Mixed, Transmission and Singular Crack Problems, EMS Tracts Math., vol. 4, European Mathematical Society (EMS), Zürich, 2008.
[34] L. Hörmander, The Analysis of Linear Partial Differential Operators. IV. Fourier Integral Operators, Grundlehren Math. Wiss., vol. 275, Springer-Verlag, Berlin, 1985.
[35] D. Jerison, C.E. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal. 130 (1995) 161–219.
[36] M. Krein, The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I, Mat. Sb. (N.S.) 20 (62) (1947) 431–495 (in Russian).
[37] A. Laptev, Spectral asymptotics of a class of Fourier integral operators, Tr. Mosk. Mat. Obs. 43 (1981) 92–115; English translation in: Trans. Moscow Math. Soc. (1983) 101–127.
[38] M. Levitin, L. Parnovski, I. Polterovich, Isospectral domains with mixed boundary conditions, J. Phys. A 39 (2006) 2073–2082.
[39] J.-L. Lions, E. Magenes, Problèmes aux limites non homogènes et applications, vol. 1, Éditions Dunod, Paris, 1968.
[40] M.M. Malamud, V.I. Mogilevskii, Kreĭn type formula for canonical resolvents of dual pairs of linear relations, Methods Funct. Anal. Topology 8 (2002) 72–100.
[41] M.M. Malamud, Spectral theory of elliptic operators in exterior domains, Russ. J. Math. Phys. 17 (2010) 96–125.
[42] V.G. Mazya, M. Mitrea, T. Shaposhnikova, The Dirichlet problem in Lipschitz domains for higher order elliptic operators with rough coefficients, J. Anal. Math. 110 (2010) 167–239.
[43] M. Mitrea, M. Taylor, A. Vasy, Lipschitz domains, domains with corners, and the Hodge Laplacian, Comm. Partial Differential Equations 30 (2005) 1445–1462.
[44] K. Pankrashkin, Resolvents of self-adjoint extensions with mixed boundary conditions, Rep. Math. Phys. 58 (2006) 207–221.
[45] J. Peetre, Mixed problems for higher order elliptic equations in two variables. I, Ann. Sc. Norm. Super. Pisa (3) 15 (1961) 337–353.
[46] J. Peetre, Mixed problems for higher order elliptic equations in two variables. II, Ann. Sc. Norm. Super. Pisa (3) 17 (1963) 1–12.
[47] A. Posilicano, L. Raimondi, Krein's resolvent formula for self-adjoint extensions of symmetric second-order elliptic differential operators, J. Phys. A 42 (2009) 015204, 11 pp.
[48] A.J. Pryde, Second order elliptic equations with mixed boundary conditions, J. Math. Anal. Appl. 80 (1981) 203–244.
[49] S. Rempel, B.-W. Schulze, A Theory of Pseudo-Differential Boundary Value Problems with Discontinuous Coefficients I–IV, Akademie-Verlag, Berlin, 1983, Preprints 17/83, 23/83, 24/83, 25/83.
[50] V. Ryzhov, A general boundary value problem and its Weyl function, Opuscula Math. 27 (2007) 305–331.
[51] R.T. Seeley, Complex powers of an elliptic operator, in: Proc. Sympos. Pure Math., vol. 10, Amer. Math. Soc., 1967, pp. 288–307.
[52] E. Shamir, Regularization of mixed second-order elliptic problems, Israel J. Math. 6 (1968) 150–168.
[53] S. Simanca, Mixed elliptic boundary value problems, Comm. Partial Differential Equations 12 (1987) 123–200.
[54] M.I. Vishik, On general boundary problems for elliptic differential equations, Tr. Mosk. Mat. Obs. 1 (1952) 187–246 (in Russian); English translation in: Amer. Math. Soc. Transl. Ser. 2, vol. 24, 1963, pp. 107–172.