From substitutions to tiling spaces and C^*-algebras

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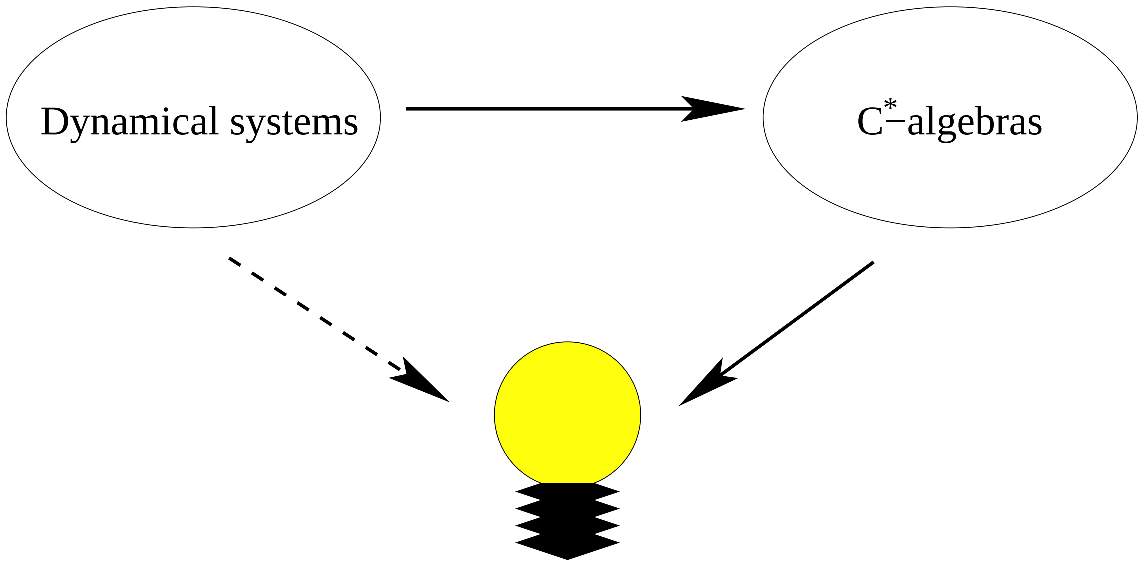


M. Barge and M. Smith: Augmented Dimension Groups and Ordered Cohomology. [Private communication.]

A substitution is a map $\tau : \mathfrak{a} \to \mathfrak{a}^* \backslash \{\epsilon\}$ mapping letters to nonempty words in the same alphabet.

The *abelianization matrix* A_τ counts the number of occurrences of each letter in each substitute.

Example $\mathfrak{a} = \{0, 1\}$, $\tau(0) = 1000$, $\tau(1) = 101$.

$$A_\tau = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Define

$$\underline{\mathsf{X}}_\tau = \{(x_i) \in \mathfrak{a}^{\mathbb{Z}} \mid \forall i < j \exists n, a : x_{[i,j]} \text{ occurs in } \tau^n(a)\}.$$

and equip with

$$\sigma(x_n) = (x_{n+1})$$

$(\underline{\mathsf{X}}_\tau, \sigma)$ is an example of a **shift space**. A construction by Matsumoto allows the association of a C^*-algebra $\mathcal{O}_{\underline{\mathsf{X}}}$ to any such shift space. We abbreviate

$$\mathcal{O}_\tau = \mathcal{O}_{\underline{\mathsf{X}}_\tau}$$

Associated to any shift space there is a **flow space** given by product topology on

$$S\underline{\mathsf{X}} = \frac{\underline{\mathsf{X}} \times \mathbb{R}}{(x,t) \sim (\sigma(x), t+1)}$$

Definition $\underline{\mathsf{X}}$ and $\underline{\mathsf{Y}}$ are *flow equivalent* (written $\underline{\mathsf{X}} \simeq_{FE} \underline{\mathsf{Y}}$) when $S\underline{\mathsf{X}}$ and $S\underline{\mathsf{Y}}$ are homeomorphic in a way preserving direction in $\mathbb{R}$.

Since the (stabilized) Matsumoto algebra is an invariant of the shift space up to flow equivalence the process

$$\tau \curvearrowright \mathcal{O}_\tau \curvearrowright K_0(\mathcal{O}_\tau)$$

leads to a flow invariant. It is computable:

Theorem [Carlsen-E 02]

$$K_0(\mathcal{O}_\tau) = \varinjlim \left(\mathbb{Z}^{|\mathfrak{a}|+\mathsf{n}_\tau}/\mathbb{Z}\widetilde{\mathsf{p}_\tau} \xrightarrow{\widetilde{\mathsf{A}}} \mathbb{Z}^{|\mathfrak{a}|+\mathsf{n}_\tau}/\mathbb{Z}\widetilde{\mathsf{p}_\tau} \xrightarrow{\widetilde{\mathsf{A}}} \cdots \right)$$

where

$$\widetilde{\mathsf{A}} = \begin{bmatrix} A_\tau & 0 \\ E_\tau & I \end{bmatrix} \qquad \widetilde{\mathsf{p}_\tau} = \begin{bmatrix} 0 & \mathsf{p}_\tau \end{bmatrix}$$

The data $\mathsf{n}_\tau, \mathsf{p}_\tau, E_\tau$ is computable as exemplified below.

$$
\begin{aligned}
\tau(\mathsf{a}) &= \mathsf{abbdadcc} \\
\tau(\mathsf{b}) &= \mathsf{aabdbdcc} \\
\tau(\mathsf{c}) &= \mathsf{abdbcadc} \\
\tau(\mathsf{d}) &= \mathsf{abbcdadc}
\end{aligned}
$$

$$
\mathsf{n}_\tau = 2, \mathsf{p}_\tau = \begin{bmatrix} 1 & 1 \end{bmatrix}, E\tau = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}
$$

$$\begin{aligned}\tau(\mathsf{a}) &= \mathsf{abbd\,a\,dcc}\\ \tau(\mathsf{b}) &= \mathsf{aabd\,b\,dcc}\\ \tau(\mathsf{c}) &= \mathsf{abdb\,c\,adc}\\ \tau(\mathsf{d}) &= \mathsf{abbc\,d\,adc}\end{aligned}$$

$$\mathsf{n}_\mathcal{T} = 2, \mathsf{p}_\mathcal{T} = \begin{bmatrix} 1 & 1 \end{bmatrix}, E_\mathcal{T} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

	2-1=1
	2-1=1
	$\mathsf{n}_\mathcal{T} = 2$

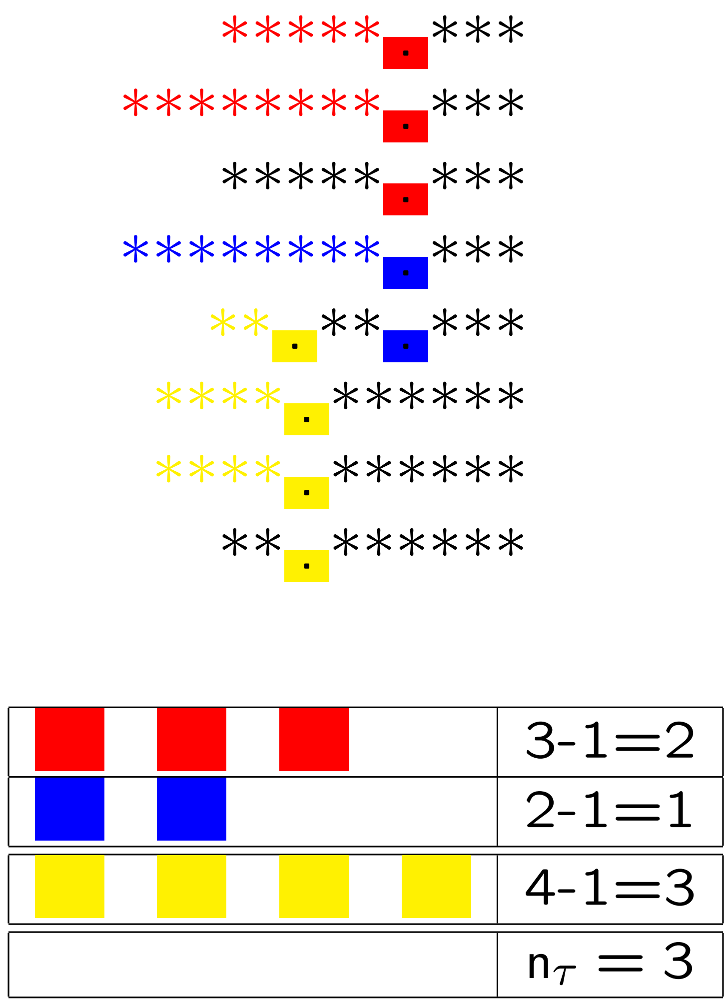
Crossmap: $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} : \mathbb{Z} \to \mathbb{Z}^4$.

$$
\begin{aligned}
\upsilon(\mathrm{a}) &= \mathrm{abbcc\,a\,dd} \\
\upsilon(\mathrm{b}) &= \mathrm{aabcc\,b\,dd} \\
\upsilon(\mathrm{c}) &= \mathrm{abbd\,c\,acd} \\
\upsilon(\mathrm{d}) &= \mathrm{abcb\,d\,acd}
\end{aligned}
$$

$$
\mathsf{n}_\upsilon = 2, \mathsf{p}_\upsilon = \begin{bmatrix} 1 & 1 \end{bmatrix}, E_\upsilon = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}
$$

	2-1=1
	2-1=1
	$\mathsf{n}_\upsilon = 2$

Crossmap: $\begin{bmatrix} 0 & 0 & 2 & -1 \end{bmatrix} : \mathbb{Z} \to \mathbb{Z}^4.$


3-1=2
2-1=1
4-1=3
$n_{\mathcal{T}} = 3$

(i) $\underline{\mathsf{X}}_\tau \simeq_{FE} \underline{\mathsf{X}}_\upsilon$

(ii) $\mathcal{O}_\tau \otimes \mathbb{K} \simeq \mathcal{O}_\upsilon \otimes \mathbb{K}$

(iii) $[\,\mathbb{Z} \xrightarrow{\mathsf{p}_\tau} \mathbb{Z}^{\mathsf{n}_\tau} \longrightarrow K_0(\mathcal{O}_\tau)\,] \simeq [\,\mathbb{Z} \xrightarrow{\mathsf{p}_\upsilon} \mathbb{Z}^{\mathsf{n}_\upsilon} \longrightarrow K_0(\mathcal{O}_\upsilon)\,]$

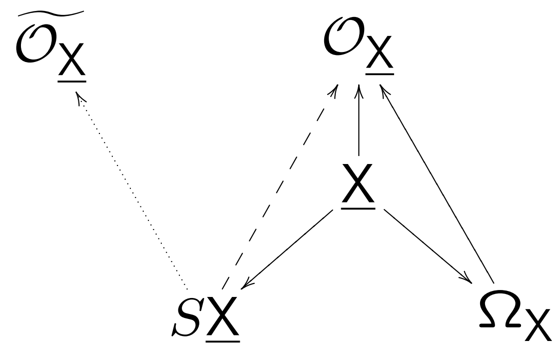
Theorem [Carlsen,E,Matsumoto,Restorff,Ruiz]

$$\text{(i)} \Longrightarrow \text{(ii)} \Longleftrightarrow \text{(iii)}$$

(i) $\Rightarrow$ (ii): Matsumoto, Carlsen
(ii) $\Rightarrow$ (iii): Carlsen, E
(iii) $\Rightarrow$ (ii): E, Restorff, Ruiz
(ii) $\not\Rightarrow$ (i): E, Restorff, Ruiz

Theorem [E-Restorff-Ruiz]
Let E_1 and E_2 be C^*-algebras each having an ideal B_i such that B_i are stable AF algebras and $A_i = E_i/B_i$ are simple AT algebras of real rank zero. Then $E_1 \otimes \mathbb{K} \simeq E_2 \otimes \mathbb{K}$ precisely when

$$\begin{array}{ccccccccc}
K_1(E_1) & \longrightarrow & K_1(A_1) & \longrightarrow & K_0(B_1) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A_1) \\
\simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
K_1(E_2) & \longrightarrow & K_1(A_2) & \longrightarrow & K_0(B_2) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A_2)
\end{array}$$



Goals:

• Explain how the data $\mathtt{n}_{\mathcal{T}}, \mathtt{p}_{\mathcal{T}}, E_{\mathcal{T}}$ can be flow invariants
• Understand the "Matsumoto relation" induced by stable isomorphism of the $\mathcal{O}_{\underline{\mathsf{X}}}$
• Give a better explanation of flow invariance of $\mathcal{O}_{\underline{\mathsf{X}}}$; or produce another C^*-algebra $\widetilde{\mathcal{O}_{\underline{\mathsf{X}}}}$ based directly on $S\underline{\mathsf{X}}$.

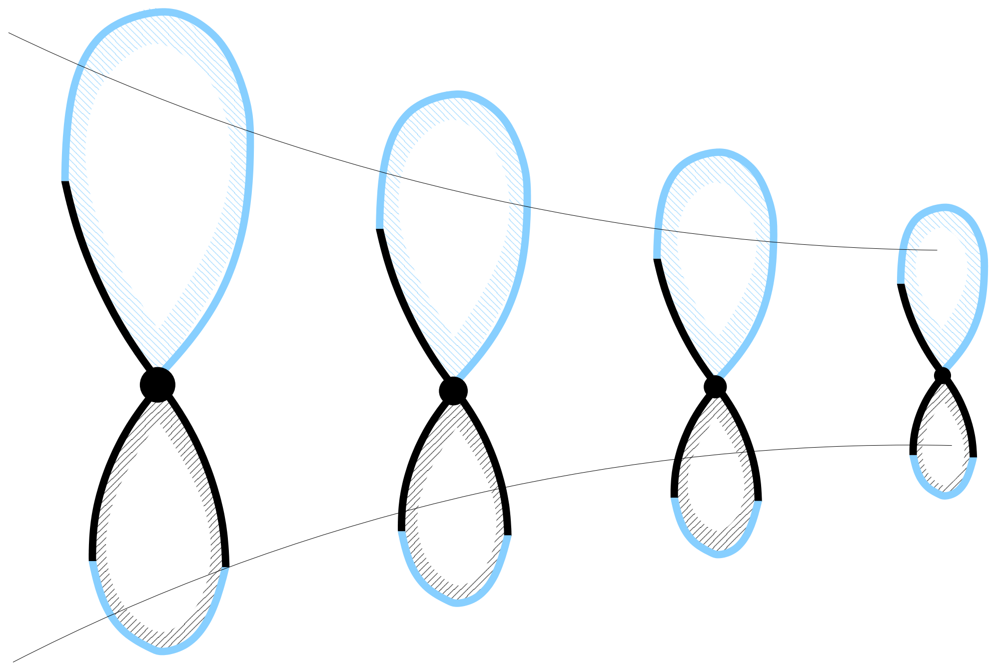
Using Perron-Frobenius theory one can describe $S\underline{\mathsf{X}}_\tau$ as a tiling space in $\mathbb{R}$ with tile lengths given by the P-F eigenvector for A_τ. For instance we have

Example $\mathfrak{a} = \{0, 1\}$, $\tau(0) = 1000$, $\tau(1) = 101$.

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ (\sqrt{5} - 1)/2 \end{bmatrix} = \frac{5 + \sqrt{5}}{2} \begin{bmatrix} 1 \\ (\sqrt{5} - 1)/2 \end{bmatrix}$$

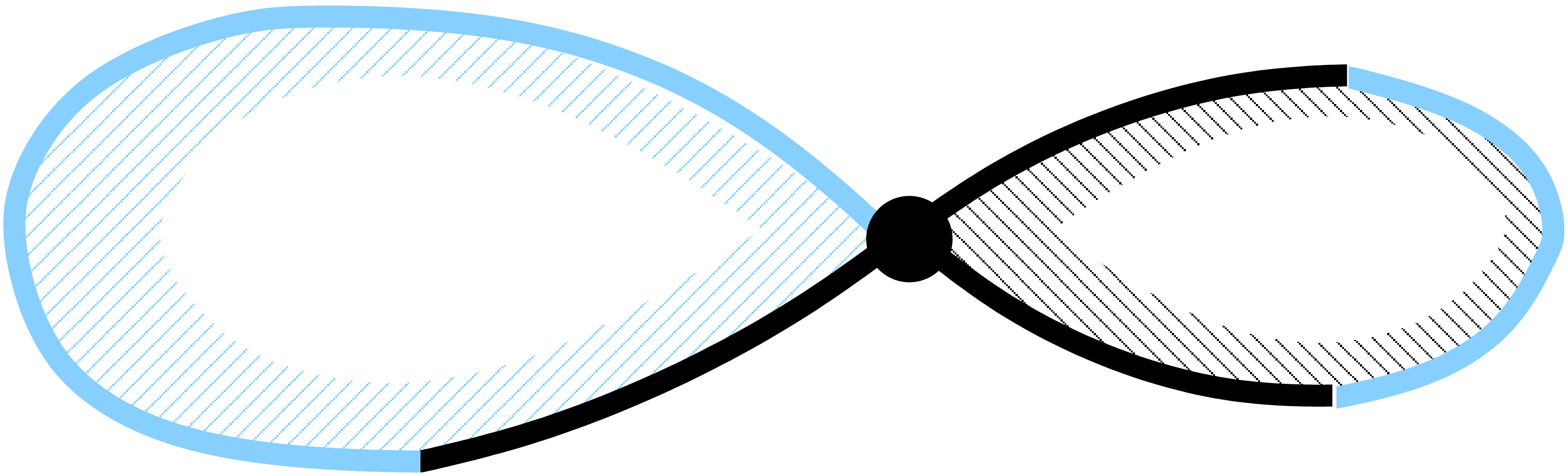
The elements in $S\underline{\mathsf{X}}_\tau$ can then be visualized as

The tiling space can in turn be described as an inverse limit



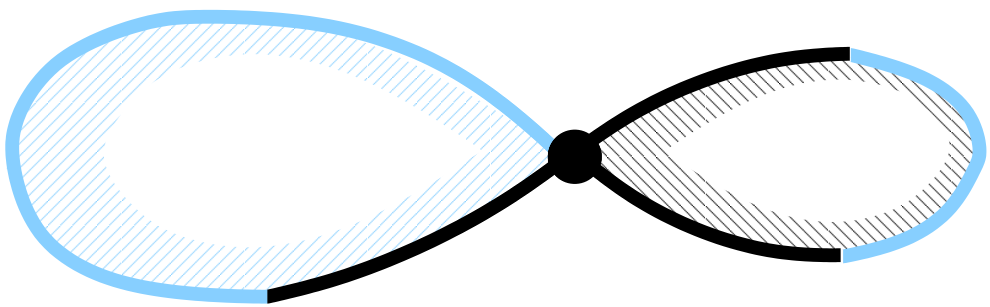
where the substitution is coded geometrically as a “map on the rose” the petals of which are precisely the tiles.

Example $\mathfrak{a} = \{0, 1\}$, $\tau(0) = 1000$, $\tau(1) = 101$.



Lemma $\varinjlim \left(\mathbb{Z}^{|\mathfrak{a}|} \xrightarrow{A_\tau} \mathbb{Z}^{|\mathfrak{a}|} \xrightarrow{A_\tau} \cdots \right)$ is a flow invariant

Proof It is the first Čech cohomology group of $S\underline{\mathsf{X}}_\tau$.

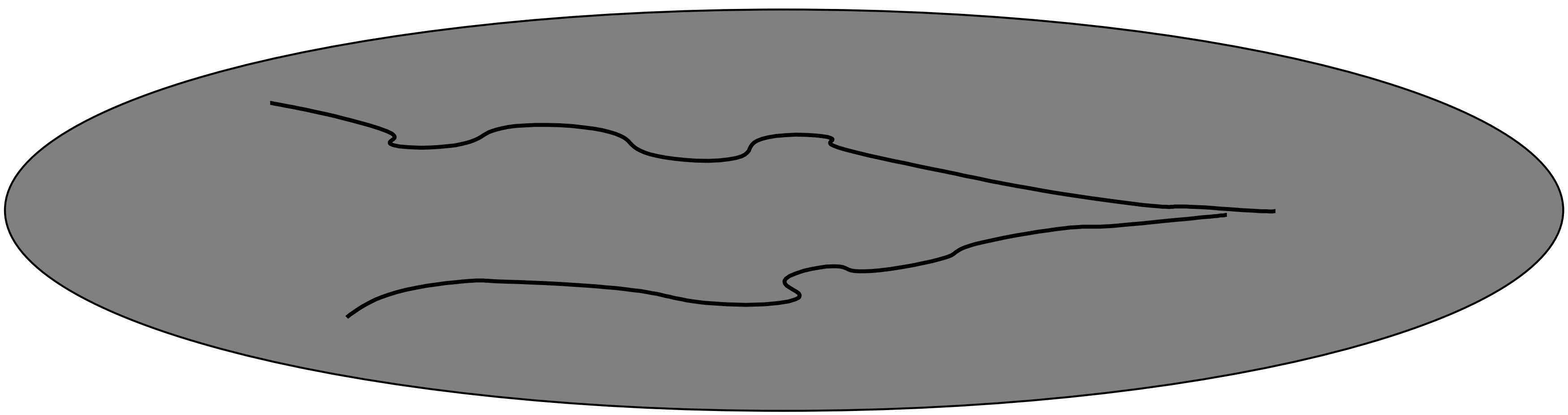


It is possible to also prove this way that the ordered group is a flow invariant.

Forward confluency in $S\underline{\mathsf{X}}_{\mathcal{T}}$ in the sense

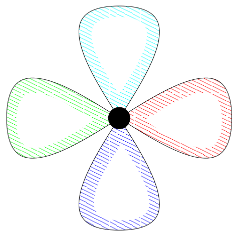
$$d\left([(x, a+t)], [(y, b+t)]\right) \to 0, \qquad t \to \infty$$

is extremely rare – it only happens for finitely many x up to orbit equivalence. [The same applies to backward confluency!]



Such x are hence called **special** words.

Example



$$\begin{aligned}\tau(\mathrm{a}) &= \mathrm{abbd\,a\,dcc}\\ \tau(\mathrm{b}) &= \mathrm{aabd\,b\,dcc}\\ \tau(\mathrm{c}) &= \mathrm{abdb\,c\,adc}\\ \tau(\mathrm{d}) &= \mathrm{abbc\,d\,adc}\end{aligned}$$

$$\cdots\tau^2(\mathrm{abbd})\tau(\mathrm{abbd})\mathrm{abbd\,a\,dcc}\tau(\mathrm{dcc})\tau^2(\mathrm{dcc})\cdots$$
$$\cdots\tau^2(\mathrm{aabd})\tau(\mathrm{aabd})\mathrm{aabd\,b\,dcc}\tau(\mathrm{dcc})\tau^2(\mathrm{dcc})\cdots$$

The **forward augmented space** contains $S\underline{\mathrm{X}}_\mathcal{T}$ as well a rays approximating the forward confluencies.