Ss $\longrightarrow$ C^* $\longrightarrow$ Ab

Classification of C^*-algebras associated to minimal dynamics

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Ss

Definition A *shift space* is a subset of $\mathfrak{a}^{\mathbb{Z}}$ which is closed in the product topology and under the shift map

$$\sigma((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}.$$

The *alphabet* $\mathfrak{a}$ is a finite set.

A shift space is *of finite type* when it can be specified by a finite list of forbidden words.

Example $\underline{\mathsf{X}}_{\{11,21\}} \ni \cdots 0010012220001020101 2\cdots$

Ss

Example With primitive substitutions such as

$$\tau(0) = 01 \qquad \tau(1) = 0$$

one gets a shift space

$$\underline{\mathsf{X}}_\tau = \overline{\{\sigma^m(u) \mid m \in \mathbb{Z}\}}$$

where u is any periodic point of τ, i.e. $\tau^n(u) = u$.

In this case we could take $n = 2$ and

$$u = \cdots 01001001.0100101001001010010100 1\cdots$$

Some substitutions

$$\tau_1(\text{א}) = \text{אבא} \qquad \tau_1(\text{ב}) = \text{באאב}$$

$$\tau_2(\alpha) = \alpha\beta \qquad \tau_2(\beta) = \alpha\beta\gamma\delta\epsilon \qquad \tau_2(\gamma) = \alpha\beta$$

$$\tau_2(\delta) = \gamma\delta\epsilon \qquad \tau_2(\epsilon) = \alpha\beta\gamma\delta\epsilon$$

$$\tau_3(1) = 1212345$$

$$\tau_3(2) = 12123451234512345$$

$$\tau_3(3) = 1212345 \qquad \tau_3(4) = 1234512345$$

$$\tau_3(5) = 12123451234512345$$

$$\tau_4(a) = ababacb \qquad \tau_4(b) = ababacbabacbabacb$$

$$\tau_4(c) = abacbabacb$$

Ss $\longrightarrow$ C^*

Let X^+ denote the projection of $\underline{\mathsf{X}}$ down on $\mathfrak{a}^{\mathbb{N}}$. We set for finite words u, v written with letters from $\mathfrak{a}$

$$C(u|v) = \{ux \in \mathsf{X}^+ \mid vx \in \mathsf{X}^+\}$$

and define a C^*-algebra $\mathcal{O}_{\underline{\mathsf{X}}}$ by generators $(S_a)_{a\in\mathfrak{a}}$ and relations

$$\begin{aligned}
a \in \mathfrak{a} \quad &: \quad S_a S_a^* S_a = S_a \\
C(u|v) = \bigcap_{i=1}^{n} C(u_i|v_i) \quad &: \quad S_u S_v^* S_v S_u^* = \prod_{i=1}^{n} S_{u_i} S_{v_i}^* S_{v_i} S_{u_i}^* \\
C(u|v) = \bigsqcup_{i=1}^{n} C(u_i|v_i) \quad &: \quad S_u S_v^* S_v S_u^* = \sum_{i=1}^{n} S_{u_i} S_{v_i}^* S_{v_i} S_{u_i}^*
\end{aligned}$$

with "$\bigsqcup$" denoting disjoint unions. *[Matsumoto]*

Ss $\longrightarrow$ C^*

Key features:

- Matsumoto algebras associated to shifts of finite type are Cuntz-Krieger algebras.

- *[Matsumoto, Carlsen]*

$$\underline{\mathsf{X}} \simeq_{flow} \underline{\mathsf{Y}} \Longrightarrow \mathcal{O}_{\underline{\mathsf{X}}} \otimes \mathbb{K} \simeq \mathcal{O}_{\underline{\mathsf{Y}}} \otimes \mathbb{K}$$

Complication:

- Not always simple, even for minimal shift spaces like $\underline{\mathsf{X}}_{\mathcal{T}}$.

$$\text{Ss} \longrightarrow C^* \xrightarrow{K_*} \text{Ab}$$

flow stable $*$-iso order iso

When are these maps injective?

This question has been successfully worked out in the case of irreducible shifts of finite type. *[Cuntz, Krieger, Bowen, Franks, Rørdam]*

The case for

$$\mathcal{O}_\tau := \mathcal{O}_{\underline{\mathsf{X}}_\tau}$$

is stucturally very different but shares the property of the shift space being finitely presented. Work by Carlsen-E shows that not both maps are injective.

Good and bad news

:-($\mathcal{O}_\tau$ is never simple.

:-) *[Carlsen]* $0 \longrightarrow \mathbb{K}^{\mathsf{n}_\tau} \longrightarrow \mathcal{O}_\tau \longrightarrow C(\underline{\mathsf{X}}_\tau) \rtimes_\sigma \mathbb{Z} \longrightarrow 0$

:-) *[Carlsen-E]* Computable K-theory:

$$K_1(\mathcal{O}_\tau) \longrightarrow K_1(C(\underline{\mathsf{X}}_\tau) \rtimes_\sigma \mathbb{Z}) \longrightarrow K_0(\mathbb{K}^{\mathsf{n}_\tau}) \longrightarrow K_0(\mathcal{O}_\tau) \longrightarrow K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_\sigma \mathbb{Z})$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{n_\tau} \longrightarrow \mathsf{DG}(\widetilde{\mathbf{A}}_\tau) \longrightarrow \mathsf{DG}(\mathbf{A}_\tau)$$

:-) *[Putnam]* $C(\underline{\mathsf{X}}_\tau) \rtimes_\sigma \mathbb{Z}$ is simple, AT and of real rank zero. $\mathcal{O}_\tau$ has real rank zero.

:-($\mathcal{O}_\tau$ is not stably finite, nor purely infinite.

Bootstrap classification

Theorem [Rørdam 1994]
Let A and B be C^*-algebras each having one essential ideal I and J such that $I, J, A/I$ and B/J are stable Kirchberg algebras satisfying the UCT. Then $A \simeq B$ precisely when

$$\begin{array}{ccccccccccccc}
\longrightarrow & K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) & \longrightarrow & K_1(I) & \longrightarrow & K_1(A) & \longrightarrow & K_1(A/I) & \longrightarrow \\
 & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & \\
\longrightarrow & K_0(J) & \longrightarrow & K_0(B) & \longrightarrow & K_0(B/J) & \longrightarrow & K_1(J) & \longrightarrow & K_1(B) & \longrightarrow & K_1(B/J) & \longrightarrow
\end{array}$$

Unfortunately, this has proven very difficult to generalize to more ideals. Restorff is able to do the next case $0 \triangleleft I_0 \triangleleft I_1 \triangleleft A$, but by completely different methods.

Another bootstrap classification

Theorem [E-Restorff-Ruiz]
Let E_1 and E_2 be C^*-algebras each having an ideal B_i such that B_i are stable AF algebras and $A_i = E_i/B_i$ are simple AT algebras of real rank zero. Then $E_1 \otimes \mathbb{K} \simeq E_2 \otimes \mathbb{K}$ precisely when

$$\begin{array}{ccccccccc} K_1(E_1) & \longrightarrow & K_1(A_1) & \longrightarrow & K_0(B_1) & \longrightarrow & K_0(E_1) & \longrightarrow & K_0(A_1) \\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\ K_1(E_2) & \longrightarrow & K_1(A_2) & \longrightarrow & K_0(B_2) & \longrightarrow & K_0(E_2) & \longrightarrow & K_0(A_2) \end{array}$$

Ss $\longrightarrow$ C^* $\longrightarrow$ Ab

Combine with results by Carlsen-E to get

Corollary $\mathcal{O}_\tau \otimes \mathbb{K} \simeq \mathcal{O}_\upsilon \otimes \mathbb{K}$ precisely when

$$\begin{array}{ccccccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z}^{n_\tau} & \longrightarrow & K_0(\mathcal{O}_\tau) & \longrightarrow & K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_\sigma \mathbb{Z}) \\
\| & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}^{n_\upsilon} & \longrightarrow & K_0(\mathcal{O}_\upsilon) & \longrightarrow & K_0(C(\underline{\mathsf{X}}_\upsilon) \rtimes_\sigma \mathbb{Z})
\end{array}$$

Corollary $\mathcal{O}_{\underline{\mathsf{X}}_\tau} \otimes \mathbb{K} \simeq \mathcal{O}_{\underline{\mathsf{X}}_\upsilon} \otimes \mathbb{K} \not\Longrightarrow \underline{\mathsf{X}}_\tau \simeq_{flow} \underline{\mathsf{X}}_\upsilon$

Corollary $K_0(\mathcal{O}_{\underline{\mathsf{X}}_\tau}) \simeq K_0(\mathcal{O}_{\underline{\mathsf{X}}_\upsilon}) \not\Longrightarrow \mathcal{O}_\tau \otimes \mathbb{K} \simeq \mathcal{O}_\upsilon \otimes \mathbb{K}$

Sketch of proof

Assume

$$\begin{array}{ccccc} K_*(B_1) & \longrightarrow & K_*(E_1) & \longrightarrow & K_*(A_1) \\ \beta_* \downarrow & & \eta_* \downarrow & & \alpha_* \downarrow \\ K_*(B_2) & \longrightarrow & K_*(E_2) & \longrightarrow & K_*(A_2) \end{array} \qquad \begin{array}{l} \tau_1 : A_1 \to M(B_1)/B_1 \\ \\ \tau_2 : A_2 \to M(B_2)/B_2 \end{array}$$

① WLOG we may assume $A_1 = A_2 = A$, $B_1 = B_2 = B$, $\alpha_* = \mathrm{id}$, $\beta_* = \mathrm{id}$. *[Strong classification/Elliott[2]]*

Sketch of proof, continued

② Then with $x_i \in KK^1(A, B)$ representing the extensions we have

$$x_1 b = a x_2$$

for some $a \in KK(A, A)^{-1}$, $b \in KK(B, B)^{-1}$. *[Generalization of method by Rørdam]*

③ $b = KK(\mathrm{id}_B)$. And WLOG we may assume that also $a = KK(\mathrm{id}_A)$ *[Kishimoto-Kumjian]*

④ Now $x_1 = x_2$. Thus $\tau_1 \oplus \tau_0 \sim \tau_2 \oplus \tau_0$. WLOG we may assume $\tau_1 \sim \tau_2$ *[Kucerovsky-Ng]*

⑤ Done.

Good and bad news

:-(A weak classification result

:-(Range of the invariant currently unknown.

:-) Similar methods apply to

$$0 \longrightarrow B \otimes \mathbb{K} \longrightarrow E \longrightarrow A \longrightarrow 0$$

if A and B both are simple unital AT algebras of real rank zero.

:-) Similar methods apply to

$$0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$$

if A and B both are simple unital nuclear algebras of tracial rank zero with finitely generated K-theory [Dadarlat].