Combinatorial aspects of pyramids of one-dimensional pieces of fixed integer length

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Content

Outline

LEGO Company profile 2004

LEGO facts and figures

- It would take 40,000,000,000 LEGO bricks stacked on top of each other to reach from the Earth to the Moon.

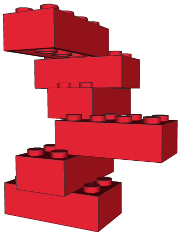
- A LEGO set is sold across the counter somewhere in the world every 7 seconds.

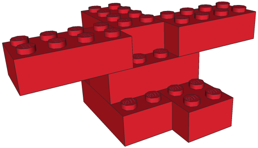
- The eight robots in the LEGO Warehouse in Billund can move 660 crates of LEGO bricks an hour.

- Children all over the world spend 5 billion hours a year playing with LEGO bricks.

- There are 102,981,500 different ways of combining six eight-stud bricks of the same colour.

- On average each person on earth owns 52 LEGO bricks.





Selected LEGO statistics

- More than 400,000,000 children and adults will play with LEGO bricks this year.
- LEGO products are on sale in more than 130 countries.
- If you built a column of about 40,000,000,000 LEGO bricks, it would reach the moon.
- Approx. four LEGO sets are sold each second.
- There are 915,103,765 different ways of combining six eight-stud bricks of the same colour.
- On average every person on earth has 52 LEGO bricks.
- With a production of about 306 million tyres a year, the LEGO Group is the world's largest tyre manufacturer.
- If all the LEGO sets sold over the past 10 years were placed end to end, they would reach from London, England, to Perth, Australia.

Theorem (Abrahamsen/Durhuus-E)

The number of LEGO buildings constructable by n blocks of size $b \times w$ grows asymptotically as $h_{b\times w}^n$ with

$$w^2 + b^2 + 6bw - 4b - 4w + 2 \le h_{b\times w} \le 24w^2 + 36bw - 48w$$

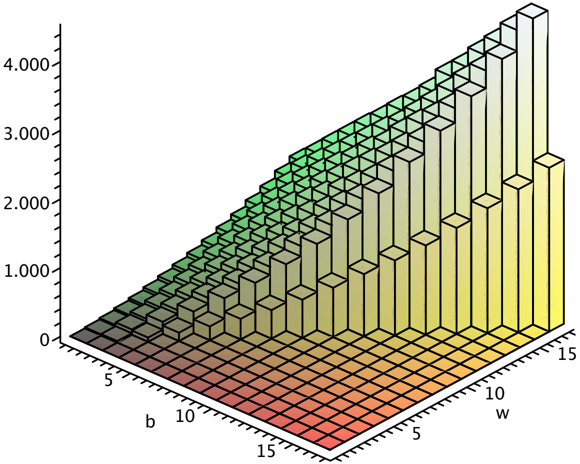
if $b \neq w$, and

$$4b^2 - 4b + 1 \le h_{b\times b} \le 18b^2$$

otherwise. We have

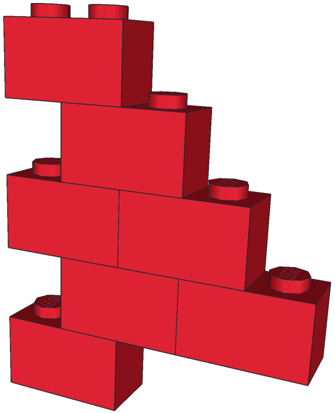
$$78 \le h_{2\times 4} \le 192$$

Quadratic dependence (empirical evidence)



Outline

Flat buildings



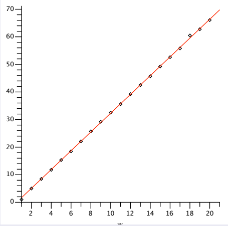
Theorem (Abrahamsen-E)

The number of flat LEGO buildings constructable by n blocks of size $1 \times w$ grows asymptotically as $\widehat{h}_w^n$ with

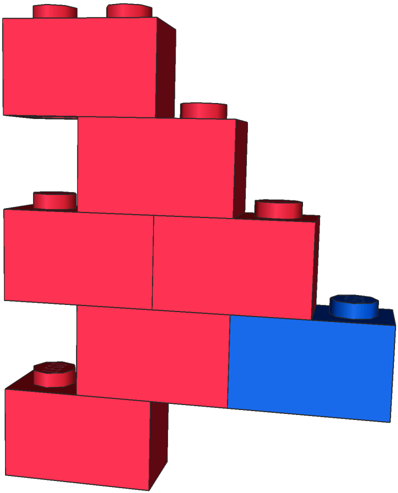
$$2w - 1 \leq \widehat{h}_w \leq 7w$$

Conjecture and wild guess

$\widehat{h}_w$ grows linearly in w



$\widehat{h}_2 = 5$



Theorem [Bousquet-Mélou & Rechnitzer]

The number of pyramids constructable by m dimers equals

$$\binom{2m-1}{m-1}$$

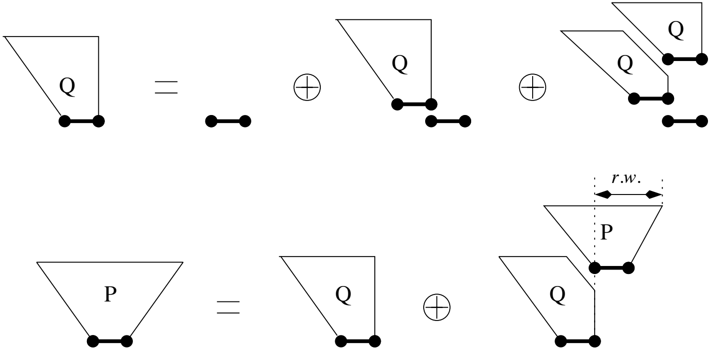
and hence grows like

$$\frac{1}{\sqrt{4\pi m}} 4^m$$

The average width of such a pyramid is (caveat!) asymptotic to

$$16\sqrt{\pi m}$$

Bousquet-Mélou & Rechnitzer



Worth pondering

$\binom{2m-1}{m-1}$ is the number of strings of $2m$ symbols drawn from $\{0, 1\}$ with exactly m ones and starting with a one, like

$$10010011$$

visualizable as

Outline

Theorem

The number of pyramids constructable by m polymers/LEGOs of width a equals

$$\binom{am-1}{m-1}$$

and hence grows like

$$\frac{1}{\sqrt{2\pi a(a-1)m}}\left(\frac{a^a}{(a-1)^{a-1}}\right)^m$$

The average width of such a pyramid is asymptotic to

$$\sqrt{\frac{\pi}{2}a(a-1)m}$$

Outline

Positive strings

Definition

A string

$$x_1 \cdots x_n$$

with n symbols in $\{0,1\}$ is a-**positive** when

$$\forall j \in \{1,\dots,n\} : \sum_{i=1}^{j} (ax_i - 1) \geq 0$$

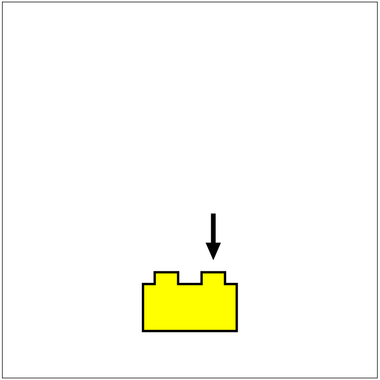
We say that $x_n \cdots x_1$ is a-**negative** in this case.

Examples

110100 is 2-positive. 100011 is not.

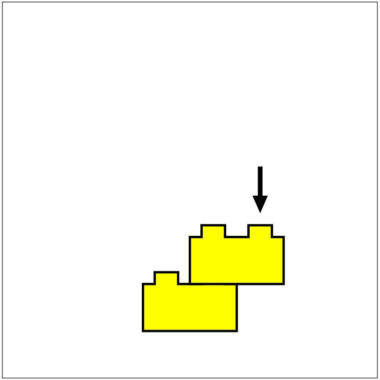
P case ($a = 2$)

1	1	0	1	0	0



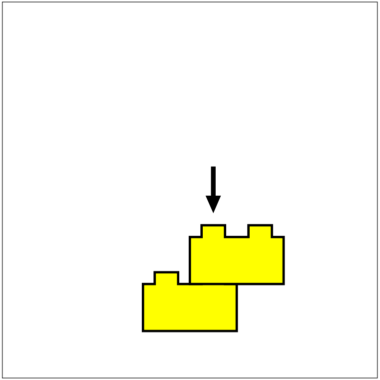
P case $(a = 2)$

1	1	0	1	0	0



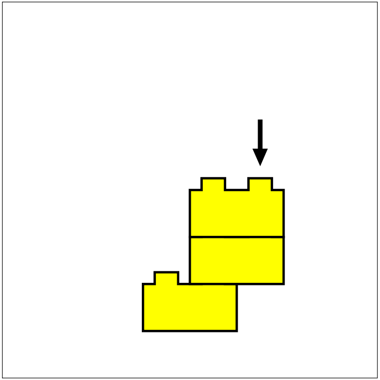
P case ($a = 2$)

1 1 0 1 0 0



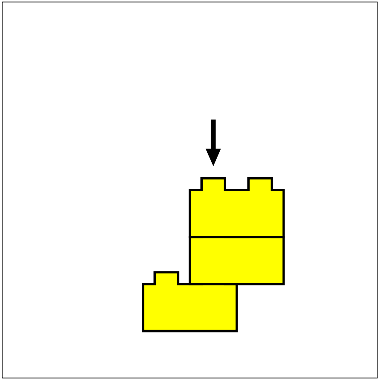
P case $(a = 2)$

1	1	0	1	0	0



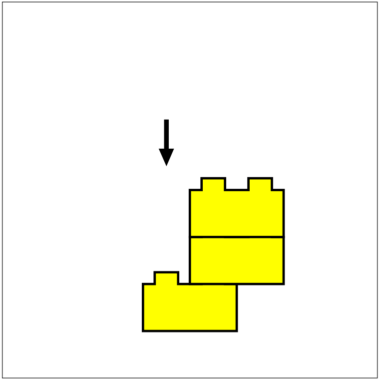
P case $(a = 2)$

1 1 0 1 0 0



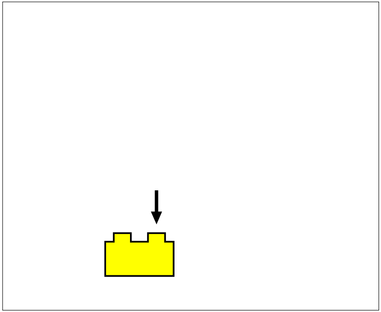
P case $(a = 2)$

1	1	0	1	0	0



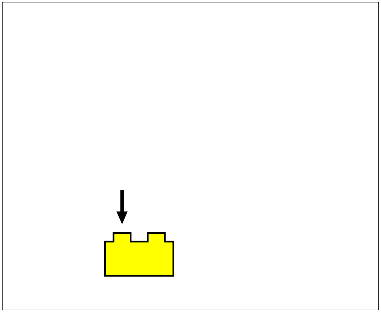
PN case ($a = 2$)

1 0 0 0 1 1



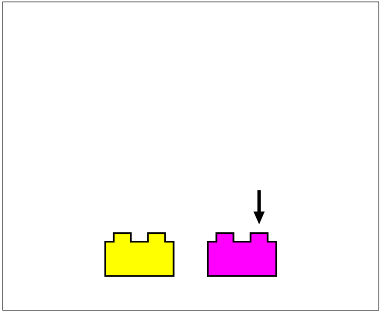
PN case $(a = 2)$

1 0 0 0 1 1



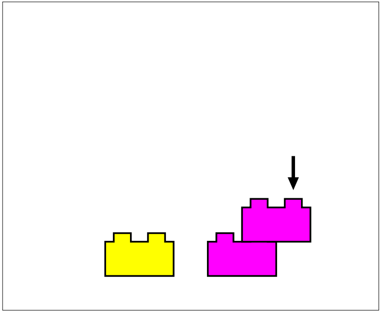
PN case ($a = 2$)

1 0 0 0 1 1



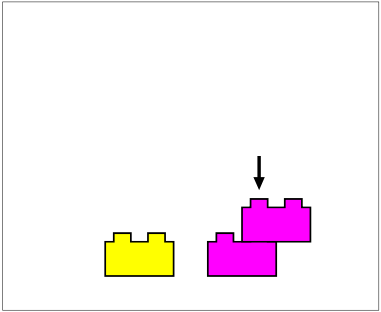
PN case $(a = 2)$

1 0 0 0 1 1



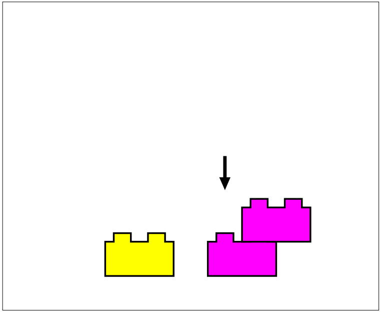
P case $(a = 2)$

1 0 0 0 1 1



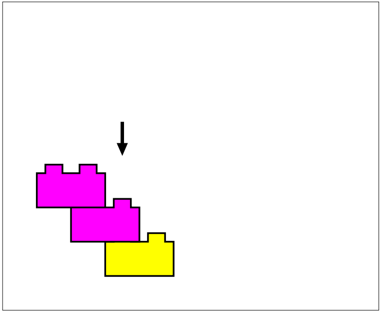
PN case ($a = 2$)

1 0 0 0 1 1



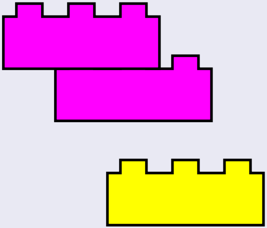
PN case ($a = 2$)

1	0	0	0	1	1



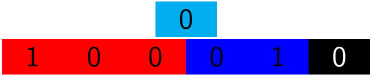
Generalizing to $a > 2$

Ambiguity ($a = 3$)

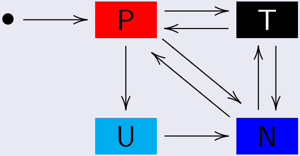


Indecomposability ($a = 3$)

100010

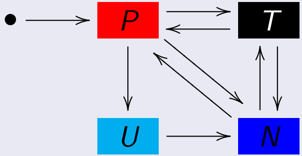


Coding automaton



Lemma

Any $\{0,1\}$-string of length am which starts with one and has exactly m ones may be uniquely decomposed into a sequence of strings P, N, T, U *satisfying the constraints of*



Lemma

Fix $a \geq 2$. The number A_n of one-sided pyramids coincides with the number of sequences in P, or N, by the coding procedure outlined earlier. Thus the number of pyramids is

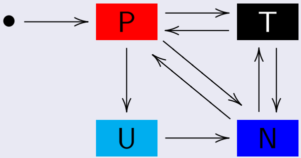
$$\sum_{r \geq 1} \sum_{m_1+\cdots+m_r=m} (a-1)^{r-1} A_{m_1} \ldots A_{m_r},$$

Observation

The number of $\{0,1\}$-strings of length am which starts with one and has exactly m ones can be written in the form

$$\sum_{r\geq 1}\ \sum_{m_1+\cdots+m_r=m} a_r\, A_{m_1}\ldots A_{m_r},$$

where r denotes the total number of substrings P or N, with sizes $m_1,\ldots,m_r \geq 1$, in a composition and the factor a_r counts the number of admissible compositions subject to the boundary conditions specified by



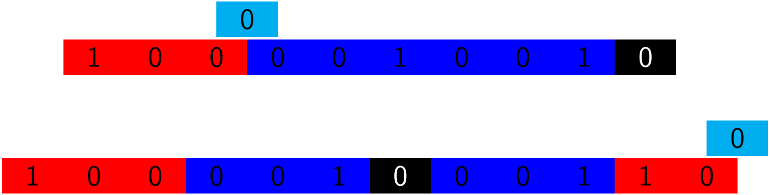
Theorem

$a_r = (a-1)^{r-1}$

Corollary

The exponential rate of growth is

$$\frac{a^a}{(a-1)^{a-1}} \sim e(a-1)$$


0
1 0 0 0 0 1 0 0 1 0
0
1 0 0 0 0 1 0 0 0 1 1 0