The C^*-algebras of right-angled Artin–Tits monoids

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Outline

1. Isometries
2. Encoding by C^*-algebras and monoids
3. The co-irreducible case
4. The general case
5. $n = 5$ case by case

Isometric operators on Hilbert space

Let H be a separable Hilbert space.

Lemma

An operator $S \in \mathbb{B}(H)$ is an isometry precisely when $S^*S = I$.

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Proof

We have

$$\|Sx\| = \|x\| \iff \langle (S^*S - I)x, x\rangle = 0$$

so the result follows by the polarization identity.

Example

On $\ell^2(\mathbb{N})$, we have

$$S_{\text{even}} e_n = e_{2n} \qquad S_{\text{odd}} e_n = e_{2n-1}$$

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On $\ell^2(\mathbb{N} \times \mathbb{N})$, we have

$$S_{\rightarrow} e_{n,m} = e_{n+1,m} \qquad S_{\uparrow} e_{n,m} = e_{n,m+1}$$

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Observation

$$S_{\text{even}}^* S_{\text{odd}} = 0 \qquad S_{\rightarrow} S_{\uparrow} = S_{\uparrow} S_{\rightarrow} \qquad S_{\rightarrow}^* S_{\uparrow} = S_{\uparrow} S_{\rightarrow}^*$$

Stable relations

An almost isometry is close to an isometry, *i.e.*

Lemma

For any $\epsilon > 0$*, there exists* $\delta > 0$ *such that whenever* $T \in \mathbb{B}(H)$ *satisfies*

$$\|T^*T - I\| < \delta$$

there exists $S \in \mathbb{B}(H)$ *satisfying*

$$S^*S = I \qquad \|S - T\| < \epsilon$$

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Proof

We can let

$$S = T(T^*T)^{-1/2}$$

when $\delta < 1$.

A pair of almost orthogonal almost isometries is close to a pair of orthogonal isometries:

Lemma

For any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $T_1, T_2 \in \mathbb{B}(H)$ satisfy

$$\|T_i^* T_i - I\| < \delta \qquad \|T_1^* T_2\| < \delta$$

there exist $S_1, S_2 \in \mathbb{B}(H)$ satisfying

$$S_i^* S_i = I \qquad S_1^* S_2 = 0 \qquad \|S_i - T_i\| < \epsilon$$

A pair of almost commuting almost isometries is not close to a pair of commuting orthogonal isometries:

Theorem

Irrespective of $\delta > 0$, *there is* $T_1, T_2 \in \mathbb{B}(H)$ *satisfying*

$$\|T_i^* T_i - I\| < \delta \qquad \|T_1 T_2 - T_2 T_1\| < \delta \qquad \|T_1^* T_2 - T_2 T_1^*\| < \delta$$

where no $S_1, S_2 \in \mathbb{B}(H)$ *can satisfy*

$$S_i^* S_i = I \qquad S_1 S_2 = S_2 S_1 \qquad S_1^* S_2 = S_2^* S_1 \qquad \|S_i - T_i\| < 1/\sqrt{2}$$

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Proof idea

Voiculescu matrices or BDF-theory and the Bott element of $K^0(S^2)$.

Question

For families $T_1, \ldots, T_n$ of almost isometries, where each pair is given to either be orthogonal or to commute, are the relations stable?

Encoding by graphs

Graphs

We work with finite, simple, undirected graphs with no loops and call them

$$\Gamma = (V, E), \quad \Gamma' = (V', E').$$

Definition

For $\Gamma = (V, E)$ we let $\Gamma^{\text{op}} = (V, E^{\text{op}})$ with

$$E^{\text{op}} = (V \times V) \setminus (E \cup \{(v, v) \mid v \in V\}).$$

We call Γ co-irreducible when Γ^{op} is irreducible.

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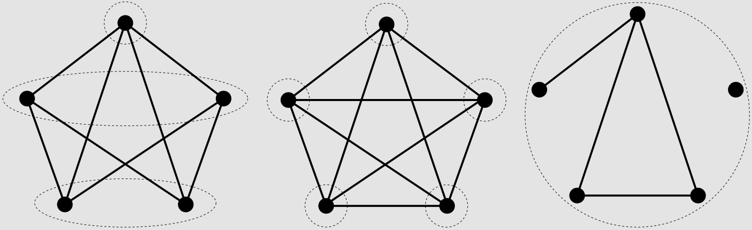
$$E^{\mathrm{op}} = (V \times V) \backslash (E \cup \{(v, v) \mid v \in V\}).$$

We call Γ co-irreducible when Γ^{op} is irreducible, and for non-co-irreducible graphs consider co-irreducible components:

$$\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n$$

Graphs (cont'd)

Examples



Graphs (cont'd)

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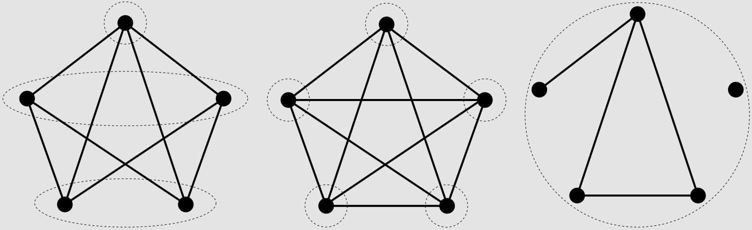
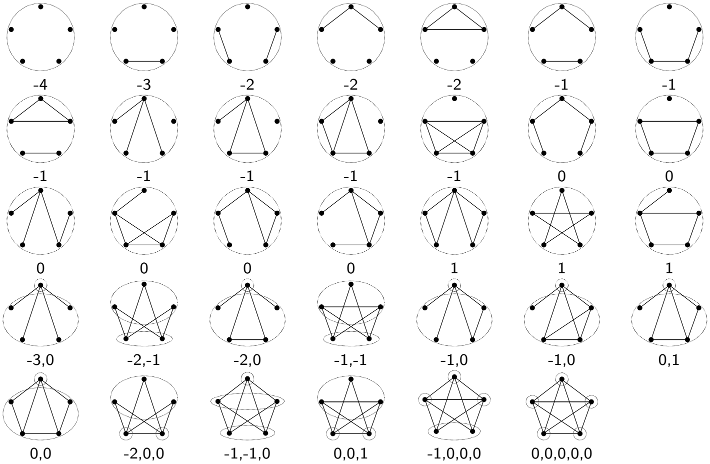


Definition (Euler characteristic)

$$\chi(\Gamma) = \sum_{K\ \Gamma\text{-simplex}} (-1)^{|K|}$$

χ is multiplicative over co-irreducible components.

$n = 5$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Examples of C^*-algebras

- $C(X)$, X a compact Hausdorff space
- $M_n(\mathbb{C})$
- $\mathbb{K} = \mathbb{K}(H)$
- $\mathcal{T} = C^* \langle S \mid S^*S = 1 \rangle$
- $\mathcal{E}_2 = C^* \langle S_1, S_2 \mid S_i^*S_i = 1, S_1^*S_2 = 0 \rangle$
- $\mathcal{O}_2 = C^* \langle S_1, S_2 \mid S_i^*S_i = 1, S_1S_1^* + S_2S_2^* = I \rangle$
- $A \otimes B$ when A and B are C^*-algebras

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Key notions

Simplicity, nuclearity, pure infiniteness.

The Elliott Program

Goal

Classify nuclear C^*-algebras by K-theoretical invariants.

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Classify nuclear C^*-algebras by K-theoretical invariants.

Progress bars

Simple, purely infinite C^*-algebras	98%
Simple C^*-algebras	91%
Purely infinite C^*-algebras with finitely many ideals	37%
C^*-algebras with finitely many ideals	8%

Artin-Tits constructions

Let Γ be a graph.

Right-angled Artin-Tits group

$$A_\Gamma = \langle \{\sigma_v\}_{v\in V} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ if } (v,w)\in E\rangle$$

Right-angled Artin-Tits monoid

$$A_\Gamma^+ = \langle \{\sigma_v\}_{v\in V} \mid \sigma_v\sigma_w = \sigma_w\sigma_v \text{ if } (v,w)\in E\rangle^+$$

Definition (Crisp–Laca '02)

The C^*-algebra associated to the Artin-Tits monoid of Γ is

$$C^*(A_\Gamma^+) = C^*\left\langle \{s_v\}_{v\in V} \;\middle|\; \begin{array}{ll} s_v s_w = s_w s_v & (v,w)\in E \\ s_v s_w^* = s_w^* s_v & (v,w)\in E \\ s_v^* s_w = \delta_{v,w}\cdot 1 & (v,w)\notin E \end{array} \right\rangle.$$

Observation

$$C^*(A_\Gamma^+) = C^*(A_{\Gamma_1}^+) \otimes C^*(A_{\Gamma_2}^+) \otimes \cdots \otimes C^*(A_{\Gamma_n}^+)$$

when

$$\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n.$$

Example

- $C^*(A^+_{\bullet}) = \mathcal{T}$
- $C^*(A^+_{\bullet - \bullet}) = \mathcal{T} \otimes \mathcal{T}$
- $C^*(A^+_{\bullet \quad \bullet}) = \mathcal{E}_2$

Semiprojectivity

Stability of relations $\mathcal{R}$ on generators $x_1, \dots, x_n$ is ensured by *semiprojectivity* of the universal C^*-algebra $C^*\langle x_1, \dots, x_n \mid \mathcal{R} \rangle$:

Definition (Semiprojectivity)

$$\begin{array}{ccccccccc} & & & & & & & & A \\ & & & & & \swarrow & & & \downarrow \\ B/I_1 & \longrightarrow & B/I_2 & \longrightarrow \dots \longrightarrow & B/I_{n_0} & \longrightarrow & \dots & \longrightarrow & B/\overline{\bigcup I_n} \end{array}$$

Theorem (Thiel-Sørensen)

$C(X)$ is semiprojective precisely when X is a 1-dimensional ANR.

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Example

- $C^*(A^+_{\bullet}) = \mathcal{T}$ is semiprojective
- $C^*(A^+_{\bullet - \bullet}) = \mathcal{T} \otimes \mathcal{T}$ is not semiprojective
- $C^*(A^+_{\bullet \quad \bullet}) = \mathcal{E}_2$ is semiprojective

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Ideal structure

Theorem (Coburn)

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

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Observation (Li)

$\mathbb{K}$ is always an ideal of $C^*(A_\Gamma^+)$. When Γ is co-irreducible with $|\Gamma| > 1$,

$$C^*(A_\Gamma^+)/\mathbb{K}$$

is simple and purely infinite.

Classifying C^*-algebras with 1 ideal

Goal

Classify C^*-algebras E with a unique ideal I by their six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(E) & \longrightarrow & K_0(E/I) \\ \uparrow & & & & \downarrow \\ K_1(E/I) & \longleftarrow & K_1(E) & \longleftarrow & K_1(I) \end{array}$$

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Progress bars

Stable, purely infinite	98%
Unital, purely infinite	98%
Stable, mixed	41%
Unital, mixed	7%

Theorem (E–Restorff–Ruiz)

Unital C^*-algebras E of the form

$$0 \longrightarrow \mathbb{K} \longrightarrow E \longrightarrow Q \longrightarrow 0$$

with Q a purely infinite and simple C^*-algebra (nuclear, UCT) are classified by their six-term exact sequence when moreover

- $K_*(Q)$ *finitely generated*
- $K_1(Q)$ *free*
- $\operatorname{rank} K_1(Q) \leq \operatorname{rank} K_0(Q)$

C^*-algebras of Artin-Tits monoids (cont'd)

Theorem (Cuntz–Echterhoff–Li)

For any Γ,

$$K_*(C^*(A_\Gamma^+)) = \mathbb{Z} \oplus 0$$

with $[1] = 1$.

Proof

The Baum–Connes conjecture holds for the group A_Γ since it has the Haagerup property.

Case I

When Γ is co-irreducible with $|\Gamma| > 1$ and $\chi(\Gamma) \neq 0$, we have

$$0 \longrightarrow \mathbb{K} \longrightarrow C^*(A_\Gamma^+) \longrightarrow \mathcal{O}_{|\chi(\Gamma)|+1} \longrightarrow 0$$

with K-theory

$$\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{\chi(\Gamma)} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/\chi(\Gamma)\mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & 0 & \longleftarrow & 0
\end{array}$$

Case II

When Γ is co-irreducible with $|\Gamma| > 1$ and $\chi(\Gamma) = 0$, we have

$$0 \longrightarrow \mathbb{K} \longrightarrow C^*(A_\Gamma^+) \longrightarrow \mathcal{O}_1 \longrightarrow 0$$

with K-theory

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

Here $\mathcal{O}_1$ is the unique unital Kirchberg algebra with the indicated K-theory and $[1] = 1$.

Theorem (E–Li–Ruiz)

When Γ, Γ' *are co-irreducible with* $|\Gamma|, |\Gamma'| > 1$ *we have*

$$C^*(A_\Gamma^+) \simeq C^*(A_{\Gamma'}^+) \Longleftrightarrow \chi(\Gamma) = \chi(\Gamma')$$

Theorem (E–Li–Ruiz)

When Γ, Γ' are co-irreducible with $|\Gamma|, |\Gamma'| > 1$ we have

$$C^*(A^+_\Gamma) \simeq C^*(A^+_{\Gamma'}) \iff \chi(\Gamma) = \chi(\Gamma')$$

Note that, *e.g.*, $C^*(A^+_{\bullet - \bullet \quad \bullet}) \simeq C^*(A^+_{\bullet \quad \bullet})$. Since the latter is semiprojective, so is the former.

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When Γ, Γ' are co-irreducible with $|\Gamma|, |\Gamma'| > 1$ we have

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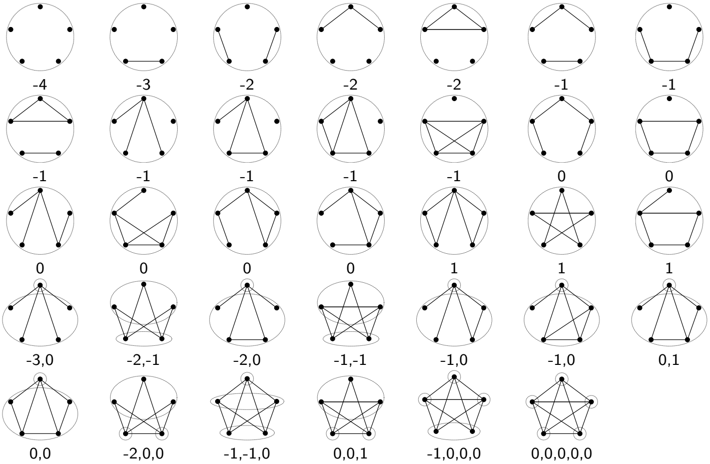
Corollary

When Γ is co-irreducible with $\chi(\Gamma) < 0$, $C^*(A^+_\Gamma)$ is semiprojective.

$n = 5$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Outline

Classifying C^*-algebras with finitely many ideals

Progress bars

Stable, purely infinite	32%
Unital, purely infinite	15%
Stable, mixed	3%
Unital, mixed	1%

The general case

Definition

When $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n$, define

$$t(\Gamma) = \#\{i \mid |\Gamma_i| = 1\}$$
$$N_k(\Gamma) = \#\{i \mid \chi(\Gamma_i) = k\}$$

The general case

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When $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_n$, define

$$t(\Gamma) = \#\{i \mid |\Gamma_i| = 1\}$$
$$N_k(\Gamma) = \#\{i \mid \chi(\Gamma_i) = k\}$$

Theorem (E–Li–Ruiz)

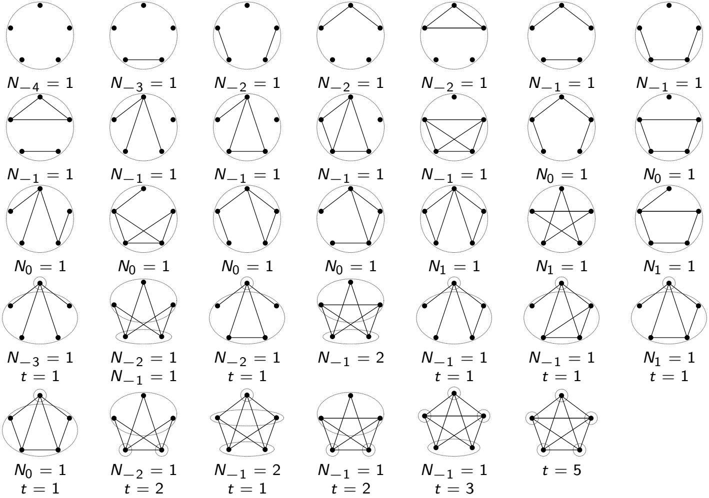
For general graphs Γ, Γ' we have

$$C^*(A^+_\Gamma) \simeq C^*(A^+_{\Gamma'})$$

precisely when

1. $t(\Gamma) = t(\Gamma')$
2. $N_k(\Gamma) + N_{-k}(\Gamma) = N_k(\Gamma') + N_{-k}(\Gamma')$ *for all* k
3. $N_0(\Gamma) > 0$ *or* $\sum_{k>0} N_k(\Gamma) \equiv \sum_{k>0} N_k(\Gamma') \mod 2$

$n = 5$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Subtler isomorphisms

Semiprojectivity

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The graph C^*-algebra $C^*(E)$ given by a finite directed graph E is semiprojective.

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Observation (Szymański)

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Theorem (E-Li-Ruiz)

When $t(\Gamma) = 0$, $C^*(A_\Gamma^+)$ is a graph C^*-algebra precisely when

$$\sum_{|k| \neq 1} N_k \leq 1$$

Corollary

- *When* $t(\Gamma) > 1$, $C^*(A_\Gamma^+)$ *is not semiprojective.*
- *When* $t(\Gamma) = 1$, $C^*(A_\Gamma^+)$ *is semiprojective precisely when*

$$\sum_k N_k = 0$$

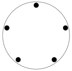
- *When* $t(\Gamma) = 0, C^*(A_\Gamma^+)$ *is semiprojective when*

$$\sum_{|k|\neq 1} N_k \leq 1$$

First open case

Outline

$N_{-4} = 1$: Semiprojective



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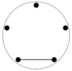
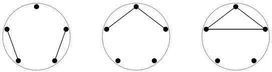
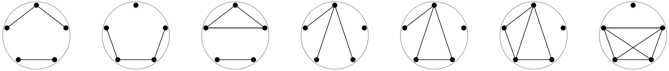


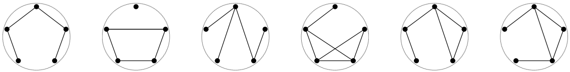
$N_{-2} = 1$: Semiprojective



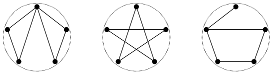
$N_{-1} = 1$: Semiprojective



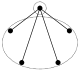
$N_0 = 1$: Semiprojective



$N_1 = 1$: Semiprojective



$N_{-1} = 1, t = 1$: Not semiprojective



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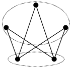
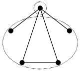
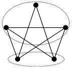


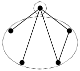
$N_{-2} = 1, t = 1$: Not semiprojective

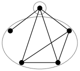


$N_{-1} = 2$: Semiprojective

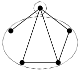


$N_{-1} = 1, t = 1$: Not semiprojective

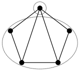




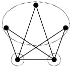
$N_1 = 1, t = 1$: Not semiprojective



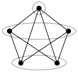
$N_0 = 1, t = 1$: Semiprojective



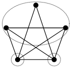
$N_{-2} = 1, t = 2$: Not semiprojective



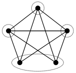
$N_{-1} = 2, t = 1$: Not semiprojective



$N_{-1} = 1, t = 2$: Not semiprojective



$N_{-1} = 1, t = 3$: Not semiprojective



$t = 5$: Not semiprojective