C^*-algebras associated to dynamical systems

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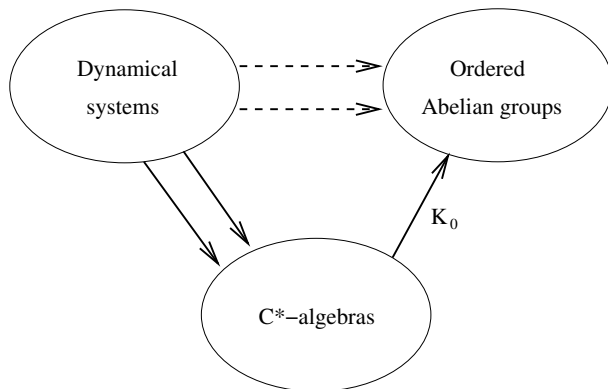
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1 Introduction

There is a long history of interaction between operator algebras and dynamical systems. At the core of this interaction are constructions of operator algebras which have that in common that they replace dynamical behavior by something *static* at the prize of non-commutativity.

This interaction is asymmetric, but mutually beneficial. There is much left to learn about the universe of so-called C^*-algebras, and examples with an origin in dynamics have proven to be important and amenable test cases, as they often come with extra structure provided by our understanding of the underlying dynamical system.

In the other direction, methods and strategies from C^*-algebras have been successfully translated to dynamical systems. One of the most prominent examples of this transport of ideas is the Bratteli-Vershik model for Cantor minimal systems. I shall try to give an overview here of an equally important area of contact, that of *classification* of dynamical systems up to various coarse equivalences.

1.1 Overview

My main ambition is to make sense of the diagram

```
  ┌─────────────┐  - - - - →  ┌──────────────┐
  │  Dynamical  │             │   Ordered    │
  │   systems   │  - - - - →  │ Abelian groups│
  └─────────────┘             └──────────────┘
         ↘ ↗                       ↗
                                K_0
        ┌──────────────┐
        │  C*–algebras │
        └──────────────┘
```

I will rather carefully explain two important constructions which in an invariant fashion associates C^*-algebras to certain dynamical systems. Of course, since operator algebras come with more structure than dynamical systems, the direct gain in using such objects as invariants is limited. However, doing so offers access to the quite developed structure and classification theory for C^*-algebras. We shall focus on the so-called *K-theory* which associates an ordered Abelian group to any C^*-algebra.

Since all diagonal maps are invariant, so are the horizontal maps, and it is often possible to *a posteriori* prove directly that the pairing of dynamical systems and ordered Abelian groups thus obtained is invariant up to various coarse equivalence relations. In some cases, the invariants thus obtained have been well known. But in most cases, what is naturally obtained by taking this detour over operator algebras has been quite different in nature from the known invariants.

1.2   Acknowledgments

I wish to thank the organizers of the *Ecole pluri-thématique de théorie ergodique*, in particular Yves Lacroix, for giving me the opportunity to attend this very inspiring meeting and to give the two lectures on which these notes are based.

The exposition of the construction of the C^*-algebras by Matsumoto draws heavily on ideas by my former PhD student Toke M. Carlsen, see [Car04a] and the associated thesis defense slides. I am very grateful to Dr. Carlsen for letting me borrow his point of view, and for supplying constructive criticism of a first version of these notes. I also wish to thank the referee for his carefully prepared comments which have eradicated a embarrasingly

high number of minor errors which would have been very disturbing for the reader.

2 External and universal origins of C^*-algebras

A C^*-algebra is ususally defined as a normed $*$-algebra subject to a (rather long) list of axioms, see for example [Ped79, 1.1.1], but as a consequence of the representation theorem of Gelfand, Naimark and Segal (see [Ped79, 3.3]), we may – and will – choose an external definition as follows:

Definition 2.1 A C^*-algebra is a closed $*$-subalgebra of the set $\mathbb{B}(\mathfrak{H})$ of bounded operators on a Hilbert space $\mathfrak{H}$.

The operations inherited from $\mathbb{B}(\mathfrak{H})$ to a C^*-algebra are the pointwise linear operations $S + T$ and λS, the composition ST, adjunction S^*, and the operator norm $\|S\|$. A $*$-*homomorphism* between C^*-algebras A and B is a map $\phi : A \longrightarrow B$ which is linear, multiplicative and $*$-preserving.

We shall not venture into the basic properties of C^*-algebras and their morphisms, but to illustrate the inherent rigidity of these objects, let us note

Proposition 2.2 *[cf. [Ped79, 1.5.7]] For any $*$-homomorphism $\phi : A \longrightarrow B$ we have*

$$\|\phi(a)\| \leq \|a\| \qquad a \in A$$

and if ϕ is injective we further get

$$\|\phi(a)\| = \|a\| \qquad a \in A$$

The first part of the result explains why we do not have to require explicitly that morphisms are continuous, but in a sense the second part is more surprising, stating as it does that any $*$-monomorphism is an isometry. This observation is a basis for a collection of rigidity results about C^*-algebras, saying for instance that there is at most one way to equip a given $*$-algebra as a C^*-algebra.

This is an important technical prerequisite for coding mathematical structure in terms of C^*-algebras. Our philosophical basis for wishing to do so it that it is often much easier to work with whole algebras than individual operators. The most basic examples of this phenomenon will be well known:

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Example 2.3 *[Commutative C^*-algebras] Let Ω be a (second countable) compact Hausdorff space. There exists a Radon measure μ such that*

$$C(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \mid f \text{ is continuous}\}$$

can be realized as a C^*-algebra by multiplication operators $M_f \in \mathbb{B}(L_2(\Omega, \mu))$ acting by

$$M_f g = fg, \qquad g \in L_2(\Omega, \mu)$$

Example 2.4 *[Spectral theory] Let T denote a normal (i.e. $TT^* = T^*T$) operator on $\mathbb{B}(\mathfrak{H})$. The smallest C^*-algebra containing T is $*$-isomorphic to $C(\mathrm{sp}(T))$, the C^*-algebra of continuous functions on*

$$\mathrm{sp}(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not invertible }\}$$

2.1 Reduced crossed products

Let (Ω, ϕ) denote an invertible topological system; Ω compact Hausdorff and ϕ a homeomorphism. We pick an invariant probability measure μ on Ω and work in the Hilbert space $L^2(\Omega, \mu)$. In $\mathbb{B}(L^2(\Omega, \mu))$ we define operators U and M_f, for each $f \in C(\Omega)$, by

$$
\begin{aligned}
Ug &= g \circ \phi^{-1} & g \in L_2(\Omega, \mu) \\
M_f g &= fg & g \in L_2(\Omega, \mu)
\end{aligned}
$$

It is worthwhile to note

$$UU^* - 1 = 0 \qquad U^*U - 1 = 0 \tag{1}$$

$$M_f M_g - M_{fg} = 0 \qquad M_f M_g - M_g M_f = 0 \tag{2}$$

$$M_f^* - M_{\overline{f}} = 0 \tag{3}$$

$$U M_f U^* - M_{f \circ \phi^{-1}} = 0 \tag{4}$$

which is easily seen by evaluating on the dense set of continuous functions in $L^2(\Omega, \mu)$; for instance

$$U M_f g = (fg) \circ \phi^{-1} = (f \circ \phi^{-1})(g \circ \phi^{-1}) = M_{f \circ \phi^{-1}} Ug$$

The smallest C^*-algebra containing U and M_f for all continuous f is denoted the *reduced crossed product* of (Ω, ϕ, μ) or sometimes

$$C(\Omega) \rtimes_{\phi, \mu, \mathrm{red}} \mathbb{Z}$$

However, this object in general depends on the choice of μ and is hence not a good invariant for the system itself. We shall subsequently refine the construction, to achieve something which is, in Section 2.3.

2.2 Reduced Matsumoto algebras

Let us now specialize to the case of a symbolic dynamical system given as a closed, shift invariant subspace $\underline{\mathsf{X}}$ of $\mathfrak{a}^{\mathbb{Z}}$, where the alphabet $\mathfrak{a}$ is a finite set with discrete topology, and $\mathfrak{a}^{\mathbb{Z}}$ is equipped with the product topology and the shift map $\sigma((x_n)) = (x_{n+1})$. We will need to work with one-sided shift spaces as well, and will do so through the restriction map

$$\pi_+ : \mathfrak{a}^{\mathbb{Z}} \longrightarrow \mathfrak{a}^{\mathbb{N}_0}$$

and set $\mathsf{X}^+ = \pi_+(\underline{\mathsf{X}})$. We let $\ell^2(\mathsf{X}^+)$ be a (not necessarily separable) Hilbert space with orthonormal basis $\{e_x \mid x \in \mathsf{X}^+\}$ and define, for each $a \in \mathfrak{a}$, the operator $S_a \in \mathbb{B}(\ell^2(\mathsf{X}^+))$ by

$$S_a e_x = \begin{cases} e_{ax} & ax \in \mathsf{X}^+ \\ 0 & \text{otherwise} \end{cases}$$

This is possible because one easily sees that S_a defines a norm-decreasing map on $\operatorname{span}\{e_x \mid x \in \mathsf{X}^+\}$.

We have

$$S_a^* e_x = \begin{cases} e_y & x = ay \\ 0 & \text{otherwise} \end{cases}$$

so

$$S_a S_a^* e_x = \begin{cases} e_x & x = ay \\ 0 & \text{otherwise} \end{cases} \qquad S_a^* S_a e_x = \begin{cases} e_x & ax \in \mathsf{X}^+ \\ 0 & \text{otherwise} \end{cases}$$

If we let

$$S_{a_1 \ldots a_n} = S_{a_1} \ldots S_{a_n}$$

with $S_\epsilon = 1$, where ϵ denotes the empty word, and let

$$C(u|v) = \{ux \in \mathsf{X}^+ \mid vx \in \mathsf{X}^+\},$$

then as proved above for $\{u, v\} = \{\epsilon, a\}$ we get

$$S_u S_v^* S_v S_u^* = \operatorname{Proj}_{\overline{\operatorname{span}\{e_x \mid x \in C(u|v)\}}}$$

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We get

$$a \in \mathfrak{a} \implies S_a S_a^* S_a - S_a = 0 \tag{5}$$

$$C(u|v) - \bigcap_{i=1}^{n} C(u_i|v_i) = 0 \implies S_u S_v^* S_v S_u^* - \prod_{i=1}^{n} S_{u_i} S_{v_i}^* S_{v_i} S_{u_i}^* = 0 \tag{6}$$

$$C(u|v) - \bigsqcup_{i=1}^{n} C(u_i|v_i) = 0 \implies S_u S_v^* S_v S_u^* - \sum_{i=1}^{n} S_{u_i} S_{v_i}^* S_{v_i} S_{u_i}^* = 0 \tag{7}$$

with "$\bigsqcup$" denoting disjoint unions.

Definition 2.5 The smallest C^*-algebra containing all the operators S_a, $a \in \mathfrak{a}$ is denoted

$$\mathcal{O}_{\underline{X},\mathrm{red}}$$

This C^*-algebra was introduced by Carlsen and Matsumoto in [CM04] as a *reduced* variant of the C^*-algebras considered earlier by Matsumoto.

Example 2.6 *Consider the* golden mean shift $\underline{X}_{gm}$ *consisting of all sequences* $(x_n)_{n \in \mathbb{Z}}$ *where for all n, $x_n x_{n+1} \neq 11$. We have $C(\epsilon|0) = X_{gm}^+$, $C(\epsilon|1) = C(0|\epsilon)$ and $C(0|\epsilon) \sqcup C(1|\epsilon) = X_{gm}^+$, so we get from (5)–(7) that*

$$S_0^* S_0 = 1 \tag{8}$$
$$S_1^* S_1 = S_0 S_0^* \tag{9}$$
$$S_0 S_0^* + S_1 S_1^* = 1 \tag{10}$$

2.3 Universal C^*-algebras

It follows by the work of Cuntz and Krieger ([CK80]) that all unital C^*-algebras generated by elements S_0 and S_1 satisfying (8)–(10) are mutually $*$-isomorphic. We thus, in this way, arrive at a universal C^*-model for the golden mean shift. However, such *canonically universal* relations are too rare for us to restrict our attention to this case. We shall instead explain how to define an explicitly universal C^*-algebra from such relations. A good general source for this is [Lor97].

A *set of relations* in some (not necessarily finite) set of generators $\mathcal{G} = \{g_i\}_{i \in I}$ is a (not necessary finite) collection of equations

$$\mathcal{R} = \{p_j(g_{i_{j1}}, \ldots, g_{i_{jn_j}}, g_{i_{j1}}^*, \ldots, g_{i_{jn_j}}^*) = 0\}_{j \in J}$$

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where the p_j's are all polynomials in $2n_j$ free variables (this restriction is not necessary, as noted in [EL99]). A *representation* of these generators in a C^*-algebra A is a map $G : I \longrightarrow A$ such that

$$p_j(G(i_{j1}), \ldots, G(i_{jn_j}), G(i_{j1})^*, \ldots, G(i_{jn_j})^*) = 0$$

for all $j \in J$. The set of all representations of $\langle \mathcal{G} | \mathcal{R} \rangle$ we will denote by $\text{Rep}\langle \mathcal{G} | \mathcal{R} \rangle$.

Whenever there is a generator named "1" we shall tacitly impose the relations

$$1x - x = 0 \qquad x1 - x = 0$$

for all generators x.

Example 2.7 *We are going to consider the following examples of generators and relations:*

$$\begin{aligned}
\mathcal{G}_{00} &= \{x\}, & \mathcal{R}_{00} &= \{x - x^* = 0\} \\
\mathcal{G}_0 &= \{y\}, & \mathcal{R}_0 &= \{yy^* = 0\} \\
\mathcal{G}_{\text{Proj}} &= \{p\}, & \mathcal{R}_{\text{Proj}} &= \{p^* - p = 0, p^2 - p = 0\} \\
\mathcal{G}_\phi &= \{U\} \cup \{T_f \mid f \in C(\Omega)\} \cup \{0\}, & \mathcal{R}_\phi &= \{(1)\text{--}(4)\} \\
\mathcal{G}_{\underline{\mathsf{X}}} &= \{S_a \mid a \in \mathfrak{a}\} \cup \{1\}, & \mathcal{R}_{\underline{\mathsf{X}}} &= \{(5)\text{--}(7)\}
\end{aligned}$$

To each set of generators and relations we may try to associate a C^*-algebra $C^*\langle \mathcal{G} \mid \mathcal{R} \rangle$ with the universal property indicated by

$$\begin{array}{ccc}
I & & \\
\Big\downarrow{\scriptstyle G_0} & \searrow^{G} & \\
C^*\langle \mathcal{G} \mid \mathcal{R} \rangle & \dashrightarrow_{\exists! \psi} & A
\end{array} \tag{11}$$

i.e., that every representation of $\langle \mathcal{G} | \mathcal{R} \rangle$ factors uniquely through the given canonical representation G_0 through the $*$-homomorphism ψ.

The theory of universal objects in C^*-algebras theory differs significantly from general universal algebra by the lack of free objects. Indeed, suppose that $I = \{i_0\}$ and $\mathcal{R} = \emptyset$ in the diagram above. By Proposition 2.2, $\|G_0(i_0)\| \geq \|G(i_0)\|$ for each $G \in \text{Rep}\langle \mathcal{G} \mid \emptyset \rangle$, and since any x in any C^*-algebra A induces such a representation, such a $G_0(i_0)$ can not exist.

We hence have to restrict our attention to relations which are *bounded* in the sense that any representation of them has a uniformly bounded norm. For such relations, this object exists and is unique. However, because of the inherent rigidity of C^*-algebras, A will be trivial in many cases. We have:

Proposition 2.8 *There is a nonzero C^*-algebra $C^*\langle \mathcal{G} \mid \mathcal{R} \rangle$ with the universal property (11) precisely when*

$$\forall i \in I : \sup_{G \in Rep\langle \mathcal{G}, \mathcal{R} \rangle} \|G(i)\| < \infty$$
$$\exists i \in I : \sup_{G \in Rep\langle \mathcal{G}, \mathcal{R} \rangle} \|G(i)\| > 0$$

Example 2.9 *If x satisfies $x = x^*$, so does λx for any real λ. Thus there is no bound of the norm of $G(x)$ in a representation of $\langle \mathcal{G}_{00}|\mathcal{R}_{00} \rangle$, and as above, $C^*\langle \mathcal{G}_{00}|\mathcal{R}_{00} \rangle$ does not exist. If y satisfies $yy^* = 0$ then by the C^*-identity [Ped79, 1.1.1]*

$$\|yy^*\| = \|y\|^2$$

we have $y = 0$. Thus as indicated by Proposition 2.8, $C^*\langle \mathcal{G}_0|\mathcal{R}_0 \rangle = 0$.

For any representation G of $\langle \mathcal{G}_{\mathrm{Proj}} \mid \mathcal{R}_{\mathrm{Proj}} \rangle$ we have, again by the C^*-identity, that $\|G(p)\| \in \{0,1\}$. The element $1 \in \mathbb{B}(\mathbb{C})$ is a non-trivial representation of the relations. Thus $C^*\langle \mathcal{G}_{\mathrm{Proj}}|\mathcal{R}_{\mathrm{Proj}} \rangle \neq 0$. In fact, one may identify the universal C^*-algebra as $\mathbb{C}$.

Our work in Sections 2.1 and 2.2 showed that there are nontrivial representations of the relations (1)–(4) and (5)–(7). It is further possible to prove by spectral considerations that any representation of the generators 1, U and S_a has norm at most one, and that any representation of the generators M_f has norm at most $\|f\|_\infty$. Thus the following definitions are meaningful and nontrivial:

Definition 2.10 [Crossed product, cf. [Dav96]] $C(\Omega) \rtimes_\phi \mathbb{Z} = C^*\langle \mathcal{G}_\phi|\mathcal{R}_\phi \rangle$

Definition 2.11 [Matsumoto algebra, cf. [Mat97], [CM04]] $\mathcal{O}_{\underline{\mathsf{x}}} = C^*\langle \mathcal{G}_{\underline{\mathsf{x}}}|\mathcal{R}_{\underline{\mathsf{x}}} \rangle$

Remark 2.12 It is possible to associate crossed products to actions on any Abelian group. For instance, an action of $\mathbb{Z}/n\mathbb{Z}$ can be successfully modelled by adding the relation $U^n = 1$ to $\mathcal{R}_\phi$.

We clearly have, by the universal properties,

$$C(\Omega) \rtimes_\phi \mathbb{Z} \longrightarrow C(\Omega) \rtimes_{\phi,\mu,\mathrm{red}} \mathbb{Z}$$
$$\mathcal{O}_{\underline{X}} \longrightarrow \mathcal{O}_{\underline{X},\mathrm{red}}$$

The first type of map is almost never a $*$-isomorphism, but it is possible to refine our construction of a concrete representation of the relations (1)–(4) to a so-called *regular* representation which identifies $C(\Omega) \rtimes_\phi \mathbb{Z}$ canonically as an algebra of operators.

Our claims concerning the golden mean shift at the beginning of Section 2.3 show that the second type of map is a $*$-isomorphism in this case. It is possible (cf. [CM04]) to give general criteria for when this happens.

We have reached our goal of associating a C^*-algebra to a dynamical system in an invariant way:

Proposition 2.13 *If (Ω, ϕ) is conjugate to $(\widetilde{\Omega}, \widetilde{\phi})$, then*

$$C(\Omega) \rtimes_\phi \mathbb{Z} \simeq C(\widetilde{\Omega}) \rtimes_{\widetilde{\phi}} \mathbb{Z}$$

Proof: Assume that the conjugacy is induced by $\chi : \Omega \longrightarrow \widetilde{\Omega}$. If then $T_f, U \in \mathbb{B}(\mathfrak{H})$ is a representation of $\langle \mathcal{G}_\phi | \mathcal{R}_\phi \rangle$, then with

$$\widetilde{T}_g = T_{g\circ\chi} \qquad \widetilde{U} = U$$

we get a representation of $\langle \mathcal{G}_{\widetilde{\phi}} | \mathcal{R}_{\widetilde{\phi}} \rangle$. For instance,

$$\widetilde{U}\widetilde{T}_g\widetilde{U}^* = UT_{g\circ\chi}U^* = T_{g\circ\chi\circ\phi^{-1}} = T_{g\circ\widetilde{\phi}^{-1}\circ\chi} = \widetilde{T}_{g\circ\widetilde{\phi}^{-1}}$$

$$\square$$

We will postpone a discussion of invariance of $\mathcal{O}_{\underline{X}}$ to Section 3.1 below.

3 K-theory and internal characterization of C^*-algebras

Instead of trying to describe C^*-algebras by the external or universal means of the previous section, it can be very useful to think of C^*-algebras, when possible, as constructed from a short list of basic examples of C^*-algebras using a number of basic operations.

Two of the most important constructions yield from a C^*-algebra A the new *matrix C^*-algebra*

$$\mathbf{M}_n(A)$$

consisting of $n \times n$-matrices with entries in A and operations equalling standard matrix multiplication and $*$-transposition in A. This is an algebra of operators on $\mathfrak{H}^n$ when A is an algebra of operators on $\mathfrak{H}$. Similarly, for Ω some compact Hausdorff space, we have the C^*-algebra

$$C(\Omega, A)$$

consisting of continuous A-valued functions with pointwise operations. This is an algebra of operators on the Hilbert space

$$\left\{ f : \Omega \longrightarrow H \,\middle|\, \int_\Omega \|f\|^2 d\mu < \infty \right\}$$

where A is an algebra of operators on $\mathfrak{H}$ and μ is an appropriately chosen Radon measure on Ω.

The *direct sum* of two C^*-algebras A and B – acting on the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ – is also a C^*-algebra $A \oplus B$ acting on the Hilbert space $\mathfrak{H} \oplus \mathfrak{K}$. If we have an increasing sequence $(A_n)_{n=1}^\infty$ of C^*-algebras, all acting on the same Hilbert space $\mathfrak{H}$, then we can consider $\bigcup_{n=1}^\infty A_i \subset \mathbb{B}(\mathfrak{H})$, but since this will in general fail to be closed, we need to consider

$$\overline{\bigcup_{n=1}^\infty A_i} \subset \mathbb{B}(\mathfrak{H}),$$

to get a C^*-algebra. The latter construction can in fact be extended to the case of a general system

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots$$

by the construction of *inductive limits* (see e.g. [RLL00, 6.2]), but we shall not go into this here.

Combining two of these ideas we arrive at

$$A_\infty = \bigcup_{n \geq 1} \mathbf{M}_n(A)$$

with $\mathbf{M}_n(A)$ embedded into $\mathbf{M}_{n+1}(A)$ at the top left corner. Again, this is not a C^*-algebra since it is not complete, but taking its closure, we get a C^*-algebra called

$$A \otimes \mathbb{K}$$

which is. It is possible ([Was94]) to make sense out of the tensor product notation in a much more general setting, but we shall not attempt to do so here. Suffice it to say that with $A = \mathbb{C}$ we get the C^*-algebra of compact operators on a separable Hilbert space and that one may think of $A \otimes \mathbb{K}$ as infinite A-valued matrices with decaying entries, just like one could do for compact operators.

3.1 K-theory

There is a functor K_0 mapping the category of C^*-algebras and $*$-homomorphisms to the category of Abelian groups and group homomorphisms. The construction is somewhat complicated – we recommend sources [Bla86] and [RLL00] – but shall attempt here to give a rough sketch. The orthogonal projections (cf. $\mathcal{R}_{\mathrm{Proj}}$ from Example 2.7) form a semigroup

$$V(A) = \left[\{ p \in A_\infty \mid p = p^* = p^2 \}, \oplus \right] / \sim$$

where

$$p \oplus q = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

and $p \sim q \in A_\infty$ when there exists $v \in A_\infty$ such that

$$vv^* = p \qquad v^*v = q.$$

By the standard (Grothendieck) construction, $V(A)$ is made into an Abelian group, which we denote $K_0(A)$, by forming formal differences

$$[p] - [q]$$

of elements from $V(A)$. However, we will keep track of $V(A)$ by denoting by $K_0(A)_+$ the set of elements which may be represented as $[p] = [p] - [0]$ for some $p \in \mathrm{Proj}(A_\infty)$. This is a cone inside $K_0(A)$, so that one may consider $K_0(A)$ as an ordered group. Since a $*$-homomorphism $\phi : A \longrightarrow B$ preserves projections (it preserves the relations $\mathcal{R}_{\mathrm{Proj}}$), it induces a positive

group homomorphism $\phi_* : K_0(A) \longrightarrow K_0(B)$. Thus $K_0(A)$, considered as an ordered group, is an invariant for C^*-algebras.

It is essential to note, however, that the positive elements do not necessarily generate the group, and that it is possible that elements x, other than 0, have the property that $\pm x \in K_0(A)_+$. We shall use the space-efficient, but somewhat nonstandard, notation "$G \supseteq H$" to specify an ordered group G with $G_+ = H$. For a unital C^*-algebra A one often wants to keep track of the distinguished element $[1] \in K_0(A)$ defined as the class of the unit.

Example 3.1

(1) $[K_0(\mathbf{M}_n(\mathbb{C})), [1]] = [\mathbb{Z} \supseteq \mathbb{N}_0, n]$

(2)
$$[K_0(C(S^n)), [1]] = \begin{cases} [\mathbb{Z} \supseteq \mathbb{N}_0, 1] & n \text{ odd} \\ [\mathbb{Z}^2 \supseteq A_n, (1,0)] & n \text{ even} \end{cases}$$

where A_n can only be partially specified by

$$(\{0, \ldots, n-1\} \times \{0\}) \cup (\{n, \ldots\} \times \mathbb{Z}) \subseteq A_n \subseteq \mathbb{N}_0 \times \mathbb{Z}$$

(3) If $\dim(\Omega) = 0$

$$[K_0(C(\Omega)), [1]] = [C(\Omega, \mathbb{Z}) \supseteq C(\Omega, \mathbb{N}_0), 1]$$

It is rather humbling to note that (unless there has been a recent development in algebraic topology that I am not aware of) we do not know exacly what the order on something as fundamental as $K_0(C(S^n))$ is for all n. Somewhat counterintuitively, this is only a minor problem for the theory, as the partial information given in (2) above is often sufficient to analyze the *simple* or *real rank zero* C^*-algebras that one is often lead to consider.

One may define another functor $K_1(-)$ with the property that

$$K_0(C(S^1, A)) = K_0(A) \oplus K_1(A)$$

We shall not go into this construction here; suffice it to say that because of the phenomenon *Bott periodicity* ([RLL00, 11]) there is no $K_2(-)$, or rather, it is the same as $K_0(-)$. Thus the long exact sequences in this theory become periodic of order six. Of main importance in this context is:

Theorem 3.2 *[Pimser-Voiculescu ([PV80])] There is an exact sequence*

$$K_0(C(\Omega)) \xrightarrow{\text{id} - \phi_*} K_0(C(\Omega)) \longrightarrow K_0(C(\Omega) \rtimes_\phi \mathbb{Z})$$

$$K_1(C(\Omega) \rtimes_\phi \mathbb{Z}) \longleftarrow K_1(C(\Omega)) \xleftarrow[\text{id} - \phi_*]{} K_1(C(\Omega))$$

In the case of minimal actions on zero-dimensional spaces we get an even finer result:

Theorem 3.3 *[[Put90], [BH96]] If ϕ acts minimally on Ω with $\dim \Omega = 0$ then*

$$K_0(C(\Omega) \rtimes_\phi \mathbb{Z}) = \left[\frac{C(\Omega, \mathbb{Z})}{\text{Im}(\text{id} - \phi_*)} \supseteq \frac{C(\Omega, \mathbb{N}_0)}{\text{Im}(\text{id} - \phi_*)} \right]$$

Theorem 3.4 *[[Mat98], [Mat01]]*

$$K_0(\mathcal{O}_{\underline{X}}) = \frac{\text{span}_{\mathbb{Z}} \{ 1_{C(u|v)} \}}{\text{Im}(\text{Id} - \lambda)}$$

where

$$\lambda(f)(x) = \sum_{\sigma_+(y) = x} f(y),$$

$\sigma_+ : \mathsf{X}^+ \longrightarrow \mathsf{X}^+$ *being the shift map.*

Note that this last result does not specify $K_0(\mathcal{O}_{\underline{X}})$ as an ordered group; only as a group.

In the special case where $\underline{\mathsf{X}}$ above is a shift of finite type, $K_0(\mathcal{O}_{\underline{X}})$ becomes a well-known invariant for such shifts, namely the Bowen-Franks group denoted $BF(\cdot)$ ([BF77],[LM95]) defined by considering such shift spaces as egde shift spaces such as

and

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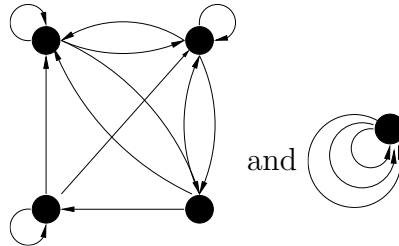
with adjacency matrices

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } B = [3].$$

Indeed,

$$\begin{aligned} K_0(\mathcal{O}_{\underline{X}_A}) &= BF(\underline{X}_A) = \mathbb{Z}^4/(\mathrm{Id} - A^t)\mathbb{Z}^4 &\simeq& \quad \mathbb{Z}/2 \\ K_0(\mathcal{O}_{\underline{X}_B}) &= BF(\underline{X}_B) = \mathbb{Z}^2/(\mathrm{Id} - B^t)\mathbb{Z}^2 &\simeq& \quad \mathbb{Z}/2 \end{aligned}$$

One may prove that the order structure is trivial in the sense that every element in $K_0(\mathcal{O}_{\underline{X}_A})$ and $K_0(\mathcal{O}_{\underline{X}_B})$ is positive.

This computation sheds light on the invariance problem for Matsumoto algebras mentioned above. Since the class of the unit is represented by $(1, 1, 1, 1) + (\mathrm{Id} - A^t)\mathbb{Z}^4$ and $(1) + (\mathrm{Id} - B^t)\mathbb{Z}$, respectively, we get that

$$(K_0(\mathcal{O}_{\underline{X}_A}), [1_A]) \simeq (\mathbb{Z}/2, 1) \not\simeq (\mathbb{Z}/2, 0) \simeq (K_0(\mathcal{O}_{\underline{X}}B), [1_B])$$

so $\mathcal{O}_{\underline{X}_A}$ and $\mathcal{O}_{\underline{X}_B}$ are <u>not</u> isomorphic, and we have seen that the map $\underline{X} \mapsto \mathcal{O}_{\underline{X}}$ fails to be conjucacy invariant. This is in fact not so surprising given the focus on one-sided shift spaces in the definition of Matsumoto algebras, and the example given above is just the first textbook example (from [Kit98]) of a pair of graphs yielding two-sided shift spaces which are conjugate, for which the one-sided shift spaces are not conjugate.

There is a general principle in classification theory for C^*-algebras, based on

Proposition 3.5 *For any C^*-algebra A, $K_0(A \otimes \mathbb{K}) = K_0(A)$*

and the observation that $A \otimes \mathbb{K}$ has no unit, which indicates that when two (ordered) groups $K_0(A)$ and $K_0(B)$ are the same in a way not preserving the unit, so that $A \not\simeq B$, one may try instead to establish so-called *stable* isomorphism: $A \otimes \mathbb{K} \simeq B \otimes \mathbb{K}$. This does the trick:

Theorem 3.6 *[Carlsen] If $\underline{X}$ and $\underline{Y}$ are conjugate shift spaces, then*

$$\mathcal{O}_{\underline{X}} \otimes \mathbb{K} \simeq \mathcal{O}_{\underline{Y}} \otimes \mathbb{K}$$

The result above is not easy to prove in full generality. The proof which is going to appear in [Car] draws on deep facts from both symbolic dynamics (Nasu bipartite codes, [Nas86]) and operator algebras (Morita equivalence, [Bro77]).

3.2 Classification

As noted in Example 3.1, $[K_0(C(S^1)), [1]] = [K_0(C(S^3)), [1]]$, and in general, one cannot hope that K-theory is a complete invariant for C^*-algebras up to $*$-isomorphism. The *classification theory* for C^*-algebras is concerned with finding large classes of C^*-algebras $\mathcal{C}$ for which this invariant, or variations of it, is complete. Such results have historically been centered around classes of C^*-algebras given as unions or inductive limits of C^*algebras of the form

$$\mathbf{M}_{n_1}(A_1) \oplus \cdots \oplus \mathbf{M}_{n_m}(A_m)$$

where the C^*-algebras A_i come from a short list. The defining examples, both classified by Elliott ([Ell76], [Ell93]) are the AF (approximately finite) algebras obtained by requiring that $A_i = \mathbb{C}$ for all i, and the real rank zero AT (approximately toral) algebras obtained by requiring that $A_i = C(S^1)$. The blueprint for such classification results is

$$\left. \begin{array}{c} [K_0(A), [1_A], \dots] \simeq [K_0(B), [1_B], \dots] \\ A, B \in \mathcal{C} \end{array} \right\} \Rightarrow A \simeq B$$

or

$$\left. \begin{array}{c} [K_0(A), \dots] \simeq [K_0(B), \dots] \\ A, B \in \mathcal{C} \end{array} \right\} \Rightarrow A \otimes \mathbb{K} \simeq B \otimes \mathbb{K}$$

where one must adjust the elliptical parts of the invariant to the considered classes $\mathcal{C}$. Note how the second version corresponds to the principle explained around Proposition 3.5. For AF algebras, the ordered group $K_0(A)$ is itself a complete invariant. For AT algebras, one needs to add $K_1(-)$ and define an ordered group structure on $K_0(A) \oplus K_1(A)$. An overview of the status of classification of C^*-algebras appears in [Rør02] and [Lin01].

For this to bear relevance for operator algebras associated to dynamical systems we hence need to establish that such algebras fall in classifiable classes, so that the diagram above becomes
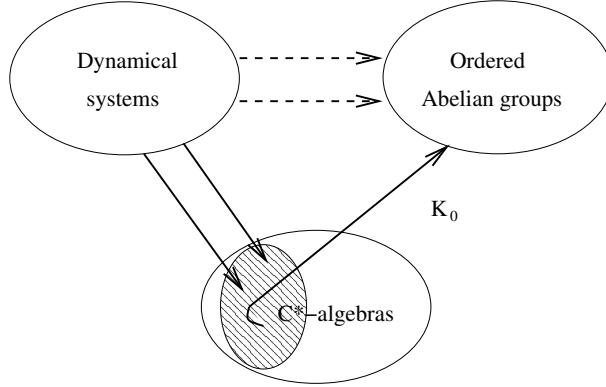
The defining example of this strategy is that of the *irrational rotation*. We consider $\Omega = S^1$ acted upon by ϕ_θ; the rotation by an irrational angle θ. We have by an easy application of Theorem 3.2 that $K_0(C(\Omega) \rtimes_{\phi_\theta} \mathbb{Z}) = \mathbb{Z}^2$ as a group, but to identify the ordered group requires a deeper analysis of the C^*-algebra, yielding (cf. [Rie81])

$$K_0(C(\Omega) \rtimes_{\phi_\theta} \mathbb{Z}) = [\mathbb{Z} + \theta\mathbb{Z} \supseteq (\mathbb{Z} + \theta\mathbb{Z}) \cap \mathbb{R}_+]$$

It turns out ([EE93]) that for suitably chosen p_i, q_i we get

$$C(\Omega) \rtimes_{\phi_\theta} \mathbb{Z} = \overline{\bigcup_{i=1}^{\infty} \mathbf{M}_{p_i}(C(S^1)) \oplus \mathbf{M}_{q_i}(C(S^1))}$$

which is manifestly in the class of *AT* algebras of real rank zero, classified in [Ell93].

A result by Lin and Phillips ([LP]) is a recent breakthrough on a generalization of this result to a more general setting. It takes as a starting point the strategy of proof in the following theorem, which is of great relevance to the theme in the remaining part of these notes.

Theorem 3.7 *[Putnam [Put90]] If (Ω, ϕ) is a Cantor minimal system then $C(\Omega) \rtimes_\phi \mathbb{Z}$ is an AT algebra of real rank zero.*

Because of this result, the crossed products end up in the classifiable class in [Ell93], leading to the following crucial result. Two shift spaces $\underline{X}$ and $\underline{Y}$ are *orbit equivalent* if there is a homeomorphism $F : \underline{X} \longrightarrow \underline{Y}$ with

the property that $F(\mathrm{orbit}(x)) = \mathrm{orbit}(F(x))$ for all x. In this case, there are maps $n : \underline{X} \longrightarrow \mathbb{Z}$ and $m : \underline{Y} \longrightarrow \mathbb{Z}$ such that

$$F(\sigma^m(x)(x)) = \sigma(F(x)) \qquad F(\sigma(x)) = \sigma^{n(x)}(F(x))$$

for all $x \in \underline{X}$. We say that $\underline{X}$ and $\underline{Y}$ are *strong orbit equivalent* when m, n can be chosen discontinuous only at one point each.

Theorem 3.8 *[[GPS95]] The following are equivalent*

 (i) (Ω, ϕ) *is strong orbit equivalent to* $(\widetilde{\Omega}, \widetilde{\phi})$

 (ii) $[K_0(C(\Omega) \rtimes_\phi \mathbb{Z}), [1]] \simeq [K_0(C(\widetilde{\Omega}) \rtimes_{\widetilde{\phi}} \mathbb{Z}), [1]]$

 (iii) $C(\Omega) \rtimes_\phi \mathbb{Z} \simeq C(\widetilde{\Omega}) \rtimes_{\widetilde{\phi}} \mathbb{Z}$

As decribed in [GPS95], it is possible to adjust the invariant $K_0(C(\Omega) \rtimes_\phi \mathbb{Z})$ to achieve a complete invariant for the perhaps more natural notion of orbit equivalence as well.

4   A comparison

4.1   Substitutional dynamical systems

We have thus far worked with two different ways of associating C^*-algebras to dynamical systems. One way, the crossed product, is available for any action, and is particularly tractable when the action is minimal on a Cantor set. The other way, the Matsumoto algebra, is only available for shift spaces.

We are going to compare the constructions and the information they carry on the dynamics in the case of substitutional dynamical systems [Que87], [Fog02]. Starting with a map

$$\tau : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp.$$

we associate a shift space $\underline{X}_\tau$ consisting of all biinfinite words for which any finite subword is a subword of $\tau^n(a)$ for some $n \in \mathbb{N}$ and $a \in \mathfrak{a}$. One associates (cf. [DHS99]) to any substitution τ the *abelianization matrix* which is the $|\mathfrak{a}| \times |\mathfrak{a}|$-matrix $\mathbf{A}_\tau$ given by

$$(\mathbf{A}_\tau)_{a,b} = \#[b, \tau(a)].$$

As usual, one focuses on the *primitive* (i.e., $\mathbf{A}_\tau$ is primitive) and *aperiodic* (i.e., $\underline{X}_\tau$ is infinite) substitutions. A main accomplishment in [DHS99] is the description of $K_0(C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z})$ as a stationary inductive limit with matrices for the connecting maps read off directly from the substitution:

Theorem 4.1 *[[DHS99], Theorem 22(i)] Let τ be a primitive, aperiodic and proper substitution. There is an order isomorphism*

$$K_0(C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z}) \simeq \varinjlim (\mathbb{Z}^{|\mathfrak{a}|}, \mathbf{A}_\tau)$$

where each $\mathbb{Z}^{|\mathfrak{a}|}$ is ordered by

$$(x_a) \geq 0 \iff \forall a \in \mathfrak{a} : x_a \geq 0.$$

A substitution τ is *proper* if for some $\tau' : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp \cup \{\epsilon\}$,

$$\exists n \in \mathbb{N} \exists l, r \in \mathfrak{a} \forall a \in \mathfrak{a} : \tau^n(a) = l\tau'(a)r.$$

In [DHS99, Proposition 20, Lemma 21] an algorithmic way is given for passing from a primitive and aperiodic substitution τ' to a primitive, aperiodic and proper substitution τ such that $\underline{X}_{\tau'} \simeq \underline{X}_\tau$, so asking for this property in the theorem is not a restriction.

With this result in hand, it can be shown that strong orbit equivalence is decidable among primitive and aperiodic substitutions, cf. [BJKR01].

In general, as one would expect from the difference between their defining relations (1)–(4) and (5)–(7), the C^*-algebras associated to a shift space as a crossed product and associated to a shift space as a Matsumoto algebra are completely unrelated objects. For instance, the Matsumoto algebra associated to an irreducible shift of finite type is simple, whereas the crossed product has infinitely many ideals.

However, as explained in the work by Carlsen ([Car04b], [Car04a]) it is possible to understand Matsumoto algebras as a kind of crossed products over non-invertible systems. Thus the following observation, which is the starting point of our comparison in the case considered, is a consequence of the fact that the difference between one- and twosided substitutional systems is limited:

Theorem 4.2 *[Carlsen, [Car04b]] Let τ be a primitive and aperiodic substitution. There is a surjective $*$-homomorphism*

$$\rho : \mathcal{O}_{\underline{X}_\tau} \longrightarrow C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z},$$

To understand the interrelation between $\mathcal{O}_{\underline{X}_\tau}$ and $C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z}$, we need to invoke a well-known concept from substitutional dynamics, that of *special words*.

We say (cf. [HZ01]) that $y \in \underline{X}_\tau$ is *left special* if there exists $y' \in \underline{X}_\tau$ such that
$$y_{-1} \neq y'_{-1} \qquad \pi_+(y) = \pi_+(y').$$
If there exist an n and an M such that $x_m = y_{n+m}$ for all $m > M$ then we say that x and y are *right shift tail equivalent* and write $x \sim_r y$.

By [Que87, p. 107] and [BL97, Theorem 3.9], there is a finite, but nonzero, number of left special words in $\underline{X}_\tau$. Thus the number n_τ of right shift tail equivalence classes of left special words in $\underline{X}_\tau$, and the n_τ-vector p_τ with each entry one less than the number of representative in each such equivalence class are defined. Further, by a method described in [CE] and [CE04a] these numbers are computable by a procedure which also generates an $\mathsf{n}_\tau \times |\mathfrak{a}|$-matrix $\mathbf{B}_\tau$ for a class of so-called *basic* substitutions. We get:

Theorem 4.3 *Let τ be a basic substitution. There is a group isomorphism*
$$K_0(\underline{X}\tau) \simeq \varinjlim \left(\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau \mathbb{Z}, \begin{bmatrix} \mathbf{A}_\tau & 0 \\ \mathbf{B}_\tau & \mathsf{Id} \end{bmatrix} \right).$$
where each $\mathbb{Z}^{|\mathfrak{a}|} \oplus \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau \mathbb{Z}$ is ordered by
$$(x_a, y_i + \mathsf{p}\mathbb{Z}) \geq 0 \iff \forall a \in \mathfrak{a} : x_a \geq 0.$$

Again, there is an algorithmic way of passing from any aperiodic and primitive substitution to one which is basic and flow equivalent to it, so that the result above can be used to compute the ordered Matsumoto K_0-group of any aperiodic and primitive substitution.

The notion of *flow equivalence* among two-sided shift spaces is of importance here. This notion is defined using the *suspension flow space* of $(\underline{X}, \sigma)$ defined as $S\underline{X} = (\underline{X} \times \mathbb{R})/\sim$ where the equivalence relation $\sim$ is generated by requiring that $(x, t+1) \sim (\sigma(x), t)$. Equipped with the quotient topology, we get a compact space with a *continuous flow* consisting of a family of maps (ϕ_t) defined by $\phi_t([x, s]) = [x, s + t]$. We say that two shift spaces $\underline{X}$ and $\underline{X}'$ are *flow equivalent* and write $\underline{X} \cong_f \underline{X}'$ if a homeomorphism $F : S\underline{X} \longrightarrow S\underline{X}'$ exists with the property that for every $x \in S\underline{X}$ there is a monotonically increasing map $f_x : \mathbb{R} \longrightarrow \mathbb{R}$ such that
$$F(\phi_t(x)) = \phi'_{f_x(t)}(F(x)).$$

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In words, F takes flow orbits to flow orbits in an orientation-preserving way.

One may prove that both C^*-algebras $\mathcal{O}_{\underline{X}_\tau} \otimes \mathbb{K}$ and $C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z} \otimes \mathbb{K}$ are invariants of $\underline{X}_\tau$ up to flow equivalence. Hence also the ordered groups

$$K_0(\mathcal{O}_{\underline{X}_\tau}) \qquad K_0(C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z})$$

are flow invariants, and one can prove using Proposition 4.2 that $K_0(\mathcal{O}_{\underline{X}_\tau})$ contains $K_0(C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z})$. The following example shows that $K_0(\mathcal{O}_{\underline{X}_\tau})$ is a strictly finer invariant among the two:

Remark 4.4 We consider

$$\tau(a) = dbdbaaaaadddddddbbbbbbcccabd$$
$$\tau(b) = d^{11}b^{10}cccca^{10}bdbbbdddddbbbbccaadd$$
$$\tau(c) = dbdbaaaacdddddddbbbbbbcaacbd$$
$$\tau(d) = d^{11}b^{10}cccca^{10}dddddbd^{31}b^{39}c^{12}a^{24}dddddbbbbccaadd$$

where "$\bullet^i$" is just an abbreviation of the concatenation of i instances of "$\bullet$".

Computations using our program [CE02], cf. [CE], show that this substitution is aperiodic, elementary and basic with $\mathsf{n}_\tau = 2, \mathsf{p}_\tau = (1,1)$ and

$$\mathbf{B}_\tau = \begin{bmatrix} 10 & 13 & 4 & 12 \\ 6 & 8 & 2 & 8 \end{bmatrix}.$$

Now consider the opposite substitution τ^{-1} with $\tau^{-1}(\bullet)$ equalling $\tau(\bullet)$ read from right to left. By definition, $\mathbf{A}_\tau = \mathbf{A}_{\tau^{-1}}$ so that by Theorem 4.1,

$$K_0(C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z}) \simeq K_0(C(\underline{X}_{\tau'}) \rtimes_\sigma \mathbb{Z})$$

as ordered groups. But for τ^{-1} we get $\mathsf{n}_{\tau^{-1}} = 2$, $\mathsf{p}_{\tau^{-1}} = (1,1)$, and

$$\mathbf{B}_{\tau^{-1}} = \begin{bmatrix} 2 & 7 & 2 & 7 \\ 2 & 7 & 2 & 7 \end{bmatrix}$$

It is elementary (but cumbersome, cf. [CE04b]) to show that

$$K_0(\mathcal{O}_{\underline{X}_\tau}) \not\simeq K_0(\mathcal{O}_{\underline{X}_{\tau^{-1}}})$$

We have chosen the example so that no other flow invariant known to us can tell the flow equivalence classes of $\underline{X}_\tau$ and $\underline{X}_{\tau^{-1}}$ apart. Surely shorter examples could be found – the repeated letters are only used to get computationally convenient invariants.

Comparing to Theorem 3.8, one is now lead to the following open questions, with which I will conclude:

(i) Are the C^*-algebras $\mathcal{O}_{\underline{\mathsf{X}}_\tau}$ classifiable by ordered K-theory?

(ii) Which relation on the spaces $\underline{\mathsf{X}}_\tau$ is induced by isomorphism of the associated ordered K-groups?

(iii) Which relation on the spaces $\underline{\mathsf{X}}_\tau$ is induced by stable $*$-isomorphism of the associated C^*-algebras?

It is possible to prove that there exist non-flow equivalent substitutions whose ordered K_0-groups are isomorphic. But I can not say at present whether this indicates that the C^*-algebras are not classifiable, or that the relation induced by stable $*$-isomorphism on the substitutional systems is much coarser than flow equivalence. Work is needed to realize in dynamical terms the meaning of stable isomorphism of Matsumoto algebras.

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