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ON HIGHER ARITHMETIC INTERSECTION THEORY

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CERTIFICA

que la present memòria ha estat realitzada sota la seva direcció per
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Contents

Introduction

The *Mordell's conjecture* states that there are a finite number of rational points on a non-singular algebraic curve C over $\mathbb{Q}$ of genus $g > 1$. The geometric analog of this conjecture was proved by Manin in 1963 (see [44]), using the Gauss-Manin connection. This suggested that the geometric tools where more developed than the arithmetic ones. *Arakelov theory* was introduced by Arakelov in [3], in order to give analogs of the algebraic geometry results in the field of arithmetic geometry. Arakelov defined a new notion of divisor class on the non-singular model of an algebraic curve defined over an algebraic number field. He then defined an intersection theory for these divisor classes, following the intersection theory of divisors in algebraic geometry. The idea is that one can compactify a curve defined over the ring of integers of a number field by considering Green functions on the associated complex curve. This initial work on arithmetic surfaces was expanded on by Deligne [16], Szpiro [56] and Faltings [19] among others. These studies provided results on arithmetic surfaces like the adjunction formula, the Hodge index theorem and the Riemann-Roch theorem. Mordell's conjecture was first proved by Faltings in [18]. A proof of the Mordell's conjecture using the tools of Arakelov theory, was given by Vojta in [58].

These studies were generalized to higher dimensions in [24] by Gillet and Soulé, who defined an intersection theory for arithmetic varieties. That paper was the starting point of a program aiming to obtain an *arithmetic intersection theory*, following the steps of the algebraic intersection theory, but suitable for arithmetic varieties. This program included, in its initial stages, the definition of arithmetic Chow groups equipped with an intersection product, the definition of the arithmetic K_0-group and the definition of characteristic classes leading to Riemann-Roch theorems.

The program should be continued with the development of higher arithmetic intersection theory, which should include the definition of higher arithmetic Chow groups with an intersection pairing, the definition of higher arithmetic K-theory, the definition of characteristic class maps between them and higher Riemann-Roch theorems.

We will now review the Arakelov program and explain the contribution of this thesis to its fulfill. We start by studying the algebraic analogues.

Algebraic intersection theory. Let X be an equidimensional algebraic variety and let $CH^p(X)$ be the *Chow group* of codimension p algebraic cycles. Different approaches may be used to equip it with a product structure

$$CH^p(X) \otimes CH^q(X) \xrightarrow{\cdot} CH^{p+q}(X).$$

The first theory is based on the moving lemma. Given the class of two irreducible subvarieties, the method consists of finding representatives that intersect properly. This approach is valid for quasi-projective schemes over a field. Another approach due to Fulton and MacPherson is based on the deformation to the normal cone. In this case, the scheme need not be quasi-projective and is valid for schemes over the spectrum of a Dedekind domain.

Alternatively, in [23], Gillet and Soulé showed that the intersection theory can be developed by transferring the product of the *algebraic K-groups* of a regular noetherian scheme X to the Chow groups. This relies on the graded isomorphism

$$\bigoplus_{p\geq 0} K_0(X)_{\mathbb{Q}}^{(p)} \cong K_0(X)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} CH^p(X)_{\mathbb{Q}},$$

where the pieces $K_0(X)_{\mathbb{Q}}^{(p)}$ are the eigenspaces of the *Adams operations* Ψ^k on $K_0(X)_{\mathbb{Q}}$ and "ch" is the *Chern character*.

The commutation relation of the Chern character with push-forward maps is given by the *Grothendieck-Riemann-Roch theorem*. Let Td denote the Todd class of the tangent bundle over an algebraic variety. Let X, Y be regular schemes which are quasi-projective and flat over the spectrum S of a Dedekind domain and let $f : X \to Y$ be a flat and projective S-morphism. Then, the Grothendieck-Riemann-Roch theorem says that there is a commutative diagram

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{Td}(X)\cdot \mathrm{ch}(\cdot)"] \arrow[d, "f_*"'] & \bigoplus_{p\geq 0} CH^p(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{Td}(Y)\cdot \mathrm{ch}(\cdot)"] & \bigoplus_{p\geq 0} CH^p(Y)_{\mathbb{Q}}. \end{tikzcd}$$

In [7], Bloch developed a theory of *higher algebraic Chow groups* for smooth algebraic varieties over a field. If X is such a variety, these groups are denoted by $CH^p(X,n)$, for $n, p \geq 0$. He proved that there is an isomorphism

$$\bigoplus_{p\geq 0} K_n(X)_{\mathbb{Q}}^{(p)} \cong K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}.$$

Bloch also gave a product structure on $CH^*(X,*)$, which relied on the moving lemma.

This theory established itself as a candidate for motivic cohomology. Since then, there have been many other proposals for motivic cohomology, which apply to bigger classes of schemes. Under certain conditions, the new definitions agree with the higher Chow groups. For this reason, the Bloch Chow groups have remained as a basis and simple description of motivic cohomology for smooth schemes over certain fields.

Arithmetic Chow groups and arithmetic intersection theory. As mentioned above, the advent of arithmetic intersection theory in arbitrary dimensions is due to Gillet and Soulé in [24]. In loc. cit., an arithmetic variety is a regular, quasi-projective scheme flat over an arithmetic ring. Let X be an arithmetic variety. An *arithmetic cycle* on X is a pair (Z, g) where Z is an algebraic cycle and g is a Green current for Z, that is, a current on the complex manifold associated to X satisfying the relation

$$dd^c g + \delta_Z = [\omega],$$

with ω a smooth differential form and δ_Z the current associated to Z. Then, the *arithmetic Chow group* $\widehat{CH}^*(X)$ is defined as the quotient of the free abelian group generated by the arithmetic cycles by an appropriate equivalence relation.

If X is an arithmetic variety, let $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ be the complex conjugation, and let $E^{p,q}(X)$ denote the vector space of $\mathbb{C}$-value differential forms ω on $X(\mathbb{C})$ of type (p, q) that satisfy the relation $F_\infty^*\omega = (-1)^p\omega$. Denote by $\widetilde{E}^{p,p}(X)$ the quotient of $E^{p,p}(X)$ by $(\operatorname{im}\partial + \operatorname{im}\bar{\partial})$.

Gillet and Soulé proved the following properties:

(i) The groups $\widehat{CH}^p(X)$ fit into an exact sequence:

$$CH^{p-1,p}(X) \xrightarrow{\rho} \widetilde{E}^{p-1,p-1}(X) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0, \tag{1}$$

where $CH^{p-1,p}(X)$ is the term $E_2^{p-1,-p}(X)$ of the Quillen spectral sequence (see [48], §7) and ρ the Beilinson regulator (up to a constant factor).

(ii) There is a pairing

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \xrightarrow{\cdot} \widehat{CH}^{p+q}(X)_{\mathbb{Q}}$$

turning $\bigoplus_{p\geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$ into a commutative graded unitary $\mathbb{Q}$-algebra.

(iii) If X, Y are projective and $f : X \to Y$ is a morphism, there exists a pull-back morphism

$$f^* : \widehat{CH}^p(Y) \to \widehat{CH}^p(X).$$

If f is proper, X, Y are equidimensional and $f_{\mathbb{Q}} : X_Q \to Y_{\mathbb{Q}}$ is smooth, there is a push-forward morphism

$$f_* : \widehat{CH}^p(X) \to \widehat{CH}^{p-\delta}(Y),$$

where $\delta = \dim X - \dim Y$. Moreover, the projection formula holds.

Gillet and Soulé continued the project in [25] and [26] defining *characteristic classes* for a hermitian vector bundle over an arithmetic variety X. In order to define the *arithmetic Chern character* "ch", they introduced the *arithmetic K_0-group*, $\widehat{K}_0(X)$, and showed that "ch" gives an isomorphism between $\widehat{K}_0(X)_{\mathbb{Q}}$ and $\bigoplus_{p\geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$. Let us briefly review the definition of $\widehat{K}_0(X)$ and "ch".

Let X be an arithmetic variety. A *hermitian vector bundle* $\overline{E}=(E,h)$ over X is a locally free sheaf of finite rank on X together with a hermitian metric on the associated holomorphic bundle. Let $\overline{E}$ be a hermitian vector bundle over X. Then, there is a Chern character

$$\widehat{\mathrm{ch}}(\overline{E})\in\bigoplus_{p\geq 0}\widehat{CH}^p(X)_{\mathbb{Q}}$$

characterized by five properties: the functoriality, additivity, multiplicativity, compatibility with the Chern forms properties and a normalization condition. Moreover, for every exact sequence of hermitian vector bundles $\epsilon: 0\to\overline{S}\to\overline{E}\to\overline{Q}\to 0$, the Chern character satisfies

$$\widehat{\mathrm{ch}}(\overline{E})=\widehat{\mathrm{ch}}(\overline{S})+\widehat{\mathrm{ch}}(\overline{Q})-(0,\widetilde{\mathrm{ch}}(\epsilon)),$$

where $\widetilde{\mathrm{ch}}(\epsilon)$ is the secondary Bott-Chern class of ϵ. This leads to the following definition of $\widehat{K}_0(X)$. Let $\widehat{K}_0(X)$ be the group generated by pairs $(\overline{E},\alpha)$, with $\alpha\in\bigoplus_{p\geq 0}\widetilde{\mathcal{D}}^{2p-1}(X,p)$, modulo the relation

$$(\overline{S},\alpha')+(\overline{Q},\alpha'')=(\overline{E},\alpha'+\alpha''+\widetilde{\mathrm{ch}}(\epsilon)),$$

for every exact sequence ϵ as above. This group fits in an exact sequence

$$K_1(X)\xrightarrow{\rho}\bigoplus_{p\geq 0}\widetilde{\mathcal{D}}^{2p-1}(X,p)\to\widehat{K}_0(X)\to K_0(X)\to 0, \tag{2}$$

with ρ the Beilinson regulator (up to a constant factor).

Then, the Chern character induces an isomorphism

$$\widehat{\mathrm{ch}}:\widehat{K}_0(X)_{\mathbb{Q}}\xrightarrow{\cong}\bigoplus_{p\geq 0}\widehat{CH}^p(X)_{\mathbb{Q}}.$$

As in the algebraic situation, this isomorphism relies on the graded decomposition of $\widehat{K}_0(X)$ given by the Adams operations. That is, $\widehat{K}_0(X)$ is endowed with a pre-λ-ring structure such that $\widehat{\mathrm{ch}}$ induces an isomorphism on the eigenspaces of $\widehat{K}_0(X)_{\mathbb{Q}}$ by the Adams operations:

$$\widehat{\mathrm{ch}}:\widehat{K}_0(X)_{\mathbb{Q}}^{(p)}\xrightarrow{\cong}\widehat{CH}^p(X)_{\mathbb{Q}}.$$

Gillet and Soulé, using the results of Bismut and his collaborators, proved an *arithmetic Grothendieck-Riemann-Roch theorem* (see [27] and [22]). Another approach to the Grothendieck-Riemann-Roch theorem is given by Faltings (see [19] and [20]).

Let $\widehat{\mathrm{Td}}$ denote the arithmetic Todd class of the tangent bundle over an arithmetic variety, let X,Y be arithmetic varieties and let $f:X\to Y$ be a projective, flat morphism of arithmetic varieties, which is smooth over the rational numbers. Then the arithmetic Grothendieck-Riemann-Roch theorem states that there is a commutative diagram:

$$\begin{tikzcd}
\widehat{K}_0(X)_{\mathbb{Q}} \arrow[r, "\widehat{\mathrm{Td}}(X)\cdot\widehat{\mathrm{ch}}(\cdot)"] \arrow[d, "f_*"'] & \bigoplus_{p\geq 0}\widehat{CH}^p(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\
\widehat{K}_0(Y)_{\mathbb{Q}} \arrow[r, "\widehat{\mathrm{Td}}(Y)\cdot\widehat{\mathrm{ch}}(\cdot)"] & \bigoplus_{p\geq 0}\widehat{CH}^p(Y)_{\mathbb{Q}}.
\end{tikzcd}$$

In [13], Burgos gave an alternative definition of the arithmetic Chow groups. It consists of considering a different space of Green forms associated with an algebraic cycle, by using *Deligne-Beilinson cohomology.* For projective schemes, Burgos definition of arithmetic Chow groups agrees with the one given by Gillet and Soulé.

Let us briefly review his definition. Let X be an arithmetic variety and consider $(\mathcal{D}^*_{\log}(X,p), d_{\mathcal{D}})$ to be the Deligne complex of differential forms on the associated real variety $X_{\mathbb{R}}$ with logarithmic singularities along infinity (see [16] or [13]). The cohomology of this complex gives the Deligne-Beilinson cohomology groups of $X_{\mathbb{R}}$, $H^*_{\mathcal{D}}(X,\mathbb{R}(p))$. For any codimension p irreducible subvariety of X, consider also the Deligne-Beilinson cohomology with supports in Z:

$$H^*_{\mathcal{D},Z}(X,\mathbb{R}(p)) = H^*(s(\mathcal{D}^*_{\log}(X,p) \to \mathcal{D}^*_{\log}(X \setminus Z,p))).$$

There is an isomorphism

$$cl : \mathbb{R}[Z] \xrightarrow{\cong} H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)),$$

called the cycle class map.

Let $\widetilde{\mathcal{D}}^*_{\log}(X,p)$ denote the quotient of $\mathcal{D}^*_{\log}(X,p)$ by the image of $d_{\mathcal{D}}$. Just as a remark, at degree $2p-1$ the differential $d_{\mathcal{D}}$ is $-2\partial\bar{\partial} = (4\pi i)dd^c$. A *Green form* for a codimension p irreducible subvariety Z is an element $(\omega,\tilde{g}) \in \mathcal{D}^{2p}_{\log}(X,p) \oplus \widetilde{\mathcal{D}}^{2p-1}_{\log}(X \setminus Z,p)$, such that $\omega = d_{\mathcal{D}}\tilde{g}$ and

$$cl(Z) = [(\omega,\tilde{g})] \in H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)).$$

Then, an *arithmetic cycle* is now a couple $(Z,(\omega,\tilde{g}))$, with $(\omega,\tilde{g})$ a Green form for Z. The *arithmetic Chow group of* X, $\widehat{CH}^p(X)$, is defined as the quotient of the free abelian group generated by the arithmetic cycles by an equivalence relation given by the group of arithmetic rational cycles.

The arithmetic Chow groups defined by Burgos satisfy the analogous properties *(i)-(iii)* stated above for the arithmetic Chow group defined by Gillet and Soulé. In particular, the exact sequence (1) is written as:

$$CH^{p-1,p}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0. \tag{3}$$

The Burgos definition of arithmetic Chow groups is the definition adopted in this study.

Later on, in [14], Burgos, Kramer and Kühn developed a formal theory of abstract arithmetic Chow rings, where the role of fibers at infinity is played by a complex of abelian groups that computes suitable cohomology theory. That is, the space of Green forms can be replaced by complexes with different properties in order to obtain arithmetic intersection theories enjoying suitable properties.

Higher arithmetic intersection theory. In a way, we could consider that the program of Arakelov intersection theory in the degree zero case is accomplished. To go further towards the goal of obtaining arithmetic analogues for the algebraic theories established, we would like to give the formalism of a higher intersection theory for arithmetic varieties. This should include the theory of higher arithmetic Chow groups, $\widehat{CH}^p(X,n)$, equipped with an intersection product, the definition of higher arithmetic K-groups, $\widehat{K}_n(X)$, characteristic class maps and Riemann- Roch theorems.

It has been suggested by Deligne and Soulé (see [16], Remark 5.4 and [54] §III.2.3.4) that the extension to higher degrees of the arithmetic K_0-group should be by means of extending the exact sequence (2) in order to obtain a long exact sequence

$$\cdots \to K_{n+1}(X) \xrightarrow{\rho} H^{2p-n-1}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{K}_n(X) \xrightarrow{\zeta} K_n(X) \to \cdots$$

$$\cdots \to K_1(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p) \xrightarrow{a} \widehat{K}_0(X) \xrightarrow{\zeta} K_0(X) \to 0.$$

The morphism ρ is the *Beilinson regulator*, that is, the Chern character taking values in real Deligne-Beilinson cohomology. Hence, the Archimedean component of the higher arithmetic K-groups should be handled by the Beilinson regulator:

$$\rho : K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}} \to \bigoplus_{p\geq 0} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Analogously, higher arithmetic Chow groups may be defined in order to extend the exact sequence (3) into a long exact sequence:

$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$

$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0.$$

The above long exact sequences can be obtained by considering the homotopy groups of the homotopy fiber of a simplicial representative of the Beilinson regulator.

Higher arithmetic Chow groups. Using these ideas, if X is proper, the arithmetic Chow groups have been extended in [30] by Goncharov.

Let $'\mathcal{D}^{2p-*}(X,p)$ be the Deligne complex of currents over X and let $E^{p,p}(X)(p)$ be the group of p-twisted differential forms of type (p,p). Denote by $'\widetilde{\mathcal{D}}^{2p-*}(X,p)$ the quotient of $'\mathcal{D}^{2p-*}(X,p)$ by the complex

$$\cdots \to 0 \to \cdots \to 0 \to E^{p,p}(X)(p) \to 0.$$

Let $Z^p(X,*)$ be the chain complex whose homology groups define $CH^p(X,*)$. Goncharov defined an explicit regulator morphism

$$Z^p(X,*) \xrightarrow{\mathcal{P}} {}'\widetilde{\mathcal{D}}^{2p-*}(X,p).$$

The *higher arithmetic Chow groups* of a regular complex variety X are given by the homology groups of the simple of the morphism $\mathcal{P}$:

$$\widehat{CH}^p(X,n) := H_n(s(\mathcal{P})).$$

For $n=0$, these groups agree with the ones given by Gillet and Soulé. However, this construction leaves the following questions open:

(1) Is the composition of the isomorphism $K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}$ with the morphism induced by $\mathcal{P}$ the Beilinson regulator?

(2) Can one define a product structure on $\bigoplus_{p,n} \widehat{CH}^p(X,n)$?

(3) Are there well-defined pull-back morphisms?

The main obstacle when we try to answer these questions is that we have to deal with the complex of currents, which does not behave well under pull-back or products. Moreover, the techniques on the comparison of regulators apply to morphisms defined for the class of quasi-projective varieties, which is not the case of $\mathcal{P}$.

Higher arithmetic K-theory. The first contribution in the direction of providing an explicit definition of higher arithmetic K-groups is the simplicial description of the Beilinson regulator given by Burgos and Wang in [15]. Let X be a complex manifold. Let $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$ be the complex of cubes of hermitian vector bundles on X. Its homology groups with rational coefficients are the rational algebraic K-groups of X, i.e., there is an isomorphism $H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X),\mathbb{Q}) \cong K_n(X)_{\mathbb{Q}}$ (see [47]). In [15], Burgos and Wang defined a chain morphism

$$\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \to \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p).$$

Here, $\widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p)$ is a complex built of differential forms on $X \times (\mathbb{P}^1)^{\cdot}$. It is quasi-isomorphic to the Deligne complex of differential forms on X with logarithmic singularities, $\mathcal{D}_{\log}^{2p-*}(X,p)$. Moreover, if X is compact, then there is an explicit inverse quasi-isomorphism $\widetilde{\mathcal{D}}_{\mathbb{P}}^{*}(X,p) \to \mathcal{D}^*(X,p)$ giving a morphism

$$\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \to \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p).$$

A result of Burgos and Wang shows that this morphism induces the Beilinson regulator in cohomology with rational coefficients.

The idea of the construction of the morphism "ch" is the following. To every n-cube E on X there is an associated locally free sheaf, $\mathrm{tr}_n(E)$, on $X \times (\mathbb{P}^1)^n$ which gives a deformation of the initial n-cube E by split cubes. Then, if "ch" is the Chern form given by the Weil formulae, $\mathrm{ch}(\mathrm{tr}_n(E))$ is a differential form on $\mathcal{D}_{\log}^{2p-n}(X \times (\mathbb{P}^1)^n, p)$. If X is compact, one can integrate this form along $(\mathbb{P}^1)^n$ against suitable differential forms T_n obtaining a differential form on X.

Let $\widehat{S}.(X)$ be the Waldhausen simplicial set for algebraic K-theory of the category of hermitian vector bundles on X and $\mathcal{K}.(\cdot)$ the Dold-Puppe functor from chain complexes to simplicial abelian groups. Then, the composition

$$\widehat{S}.(X) \xrightarrow{Hurewicz} \mathcal{K}.(\mathbb{Z}\widehat{S}_*(X)) \xrightarrow{\mathcal{K}(\mathrm{Cub})} \mathcal{K}.(\widetilde{\mathbb{Z}}\widehat{C}_*(X)) \xrightarrow{\mathrm{ch}} \mathcal{K}.\Big(\bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p)\Big)$$

is a simplicial representative of the Beilinson regulator.

Let $\widehat{\mathcal{D}}^*(X,p)$ be the bête truncation of the complex $\mathcal{D}^*(X,p)$ at degree greater than or equal to $2p$, and let

$$\widehat{\mathrm{ch}} : \widehat{S}.(X) \xrightarrow{\widehat{\mathrm{ch}}} \mathcal{K}.\Big(\bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p)\Big),$$

be the morphism induced by "ch". Then, following the ideas of Deligne and Soulé, one defines the *higher arithmetic K-groups* by

$$\widehat{K}_n(X) = \pi_{n+1}(\text{Homotopy fiber of } |\mathcal{K}(\widehat{\mathrm{ch}})|).$$

In this way, the desired long exact sequence extending (2) is obtained.

Observe that this definition of higher arithmetic K-groups treats the degree zero case in a different way from the rest. That is, the role of the differential forms in the non-zero degree groups is played only by those differential forms in the kernel of the differential $d_{\mathcal{D}}$, whereas no restriction is imposed in the degree zero group.

In order to avoid this difference, Takeda, in [57], has given an alternative definition of the *higher arithmetic K-groups* of X, by means of homotopy groups modified by the representative of the Beilinson regulator "ch". We denote these higher arithmetic K-groups by $\widehat{K}^T_n(X)$. The main characteristic of these groups is that instead of extending the exact sequence (2) to a long exact sequence, for every n there is an exact sequence

$$K_{n+1}(X) \xrightarrow{\rho} \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X,p) \xrightarrow{a} \widehat{K}^T_n(X) \xrightarrow{\zeta} K_n(X) \to 0,$$

analogous to the exact sequence for $\widehat{K}_0(X)$.

The two definitions do not agree, but, as proved by Takeda in [57], they are related by the characteristic class "ch":

$$\widehat{K}_n(X) \cong_{can} \ker(\mathrm{ch} : \widehat{K}^T_n(X) \to \widehat{\mathcal{D}}^{2p-n}(X,p)).$$

Overview of the results

The results of this thesis contribute to the program of developing a higher arithmetic intersection theory. These results constitute chapters 3 and 5. Chapters 2 and 4 consist of the preliminary results needed for chapters 3 and 5, in the area of homotopy theory of simplicial sheaves and algebraic K-theory.

In chapter 3, we develop a higher intersection theory on arithmetic varieties, à la Bloch. That is, we modify the higher Chow groups defined by Bloch by an explicit construction of the Beilinson regulator in terms of algebraic cycles.

We construct a representative of the Beilinson regulator using the Deligne complex of differential forms instead of the Deligne complex of currents. The regulator that we obtain turns out to be a minor modification of the regulator described by Bloch in [8].

Next, we develop a theory of higher arithmetic Chow groups, $\widehat{CH}^p(X,n)$, for any arithmetic variety X over a field. These groups are the homology groups of the simple of a diagram of complexes which represents the Beilinson regulator. We prove that there is a commutative and associative product structure on $\widehat{CH}^*(X,*) = \bigoplus_{p,n} \widehat{CH}^p(X,n)$, compatible with the algebraic intersection product. Therefore, we provide an arithmetic intersection product for arithmetic varieties over a field.

The advantages of our definition over Goncharov's definition are the following: the construction is valid for quasi-projective arithmetic varieties over a field, and not only over projective varieties; we can prove that our regulator is the Beilinson regulator; the groups we obtain are contravariant with respect to arbitrary maps; we can endow them with a product structure. All these improvements are mainly due to the fact that we avoid using the complex of currents.

The higher algebraic Chow groups defined by Bloch give a simple description of the motivic cohomology groups for smooth algebraic varieties over a field. One should view the higher arithmetic Chow groups as a simple description of a yet to be defined arithmetic motivic cohomology theory, valid for arithmetic varieties over a field.

We next focused on the relation between the higher arithmetic Chow groups and higher arithmetic K-theory. In order to follow the algebraic ideas, we should have a decomposition of the groups $\widehat{K}_n(X)_{\mathbb{Q}}$ given by eigenspaces of Adams operations $\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}}$. By the nature of the definition of $\widehat{K}_n(X)$, either by considering the homotopy fiber, or the modified homotopy groups of Takeda, it is apparently necessary to have a description of the Adams operations in algebraic K-theory in terms of a chain morphism, compatible with the representative of the Beilinson regulator "ch".

In chapter 4, we obtain a chain morphism inducing Adams operations on higher algebraic K-theory over the field of rational numbers. This definition is of combinatory nature. This chain morphism is designed to commute with the Beilinson regulator "ch" given by Burgos and Wang. Hence, one can appreciate that it has been strongly inspired by the definition of the Beilinson regulator and follows the same logical pattern.

In chapter 5 it is shown that this chain morphism indeed commutes with the representative of the Beilinson regulator "ch" and we use this fact to define Adams operations on the rational higher arithmetic K-groups.

Further studies in this direction will focus on determining if the Adams operations induce a graded decomposition $\widehat{K}_n(X)_{\mathbb{Q}} = \bigoplus_{p\geq 0} \widehat{K}_n(X)^{(p)}_{\mathbb{Q}}$ such that there is an isomorphism $\widehat{CH}^p(X,n)_{\mathbb{Q}} \cong \widehat{K}_n(X)^{(p)}_{\mathbb{Q}}$, as is the case in the algebraic setting. Notice that the arithmetic analogues of the algebraic theories discussed here rely on an explicit description of a certain morphism in the algebraic context. This is the case for the Beilinson regulator, in order to define higher arithmetic K-groups or Chow groups, and for the Adams operations, in order to define Adams operations on the higher arithmetic K-groups. In our view, the main difficulty to prove that there is an isomorphism

$$\widehat{CH}^p(X,n)_{\mathbb{Q}} \cong \widehat{K}_n(X)^{(p)}_{\mathbb{Q}}$$

is that, for the moment, there is no explicit representative of the algebraic analogue.

The development of this study required tools to compare morphisms from algebraic K-groups to a suitable cohomology theory or to the K-groups themselves. Indeed, we construct a chain morphism that is proved to induce the Beilinson regulator, and we construct a chain morphism that is proved to induce the Adams operations on algebraic K-theory. In chapter 2, we study these comparisons at a general level, providing theorems giving sufficient conditions for two morphisms to agree. The theory underlying the proofs is the homotopy theory of simplicial sheaves.

These theorems provide an alternative proof that the regulator defined by Burgos and Wang in [15] induces the Beilinson regulator. Moreover, we prove that the Adams operations defined by Grayson in [31] agree for any regular noetherian scheme of finite Krull dimension with the Adams operations defined by Gillet and Soulé in [28]. In particular, this implies that the Adams operations defined by Grayson satisfy the usual identities of a λ-ring, a fact that was left unproved in Grayson's work.

Results

We now explain the structure of the work and detail the main results.

Chapter 1 is of a preliminary nature. We briefly give the background needed for the understanding of the central work of the thesis. It also has the purpose of fixing the notation and definitions that will be used frequently in the forthcoming chapters. In the first section we discuss simplicial model categories, focusing on the category of simplicial sets and on the cubical abelian groups. In the second section we fix the notation on multi-indices, and discuss general facts on (co)chain complexes. We also discuss the relationship between simplicial or cubical abelian groups and chain complexes. In the third section we give the definition of algebraic K-theory in terms of the Quillen Q-construction and the Waldhausen construction. We also introduce the chain complex of cubes, which computes algebraic K-theory with rational coefficients and plays a central role in the definition of the Adams operations. Finally, in the last section of this chapter, we recall the definition of Deligne-Beilinson cohomology and state the main properties used in the study.

In **Chapter 2** we give theorems for the comparison of characteristic classes in algebraic K-theory. For a class of maps, named *weakly additive*, we give a criterion to decide whether two of them agree. All group morphisms induced by a map of simplicial sheaves are in this class, but these are not the only ones.

As mentioned already, in [15], Burgos and Wang defined a variant of the Chern character morphism from higher K-theory to real absolute Hodge cohomology,

$$\mathrm{ch} : K_n(X) \to \bigoplus_{p\geq 0} H^{2p-n}_{\mathcal{H}}(X, \mathbb{R}(p)),$$

for every smooth complex variety X. They proved that this morphism agrees with the already defined Beilinson regulator map. The proof relies only on the properties satisfied by the morphisms and by real absolute Hodge cohomology, and not on their definition. Hence, it is reasonable to think that there may be an axiomatic theorem for characteristic classes on higher K-theory. The proof of Burgos and Wang makes use of the bisimplicial scheme $B.P.$, introduced by Schechtman in [51]. This implies that a delooping in K-theory is necessary and hence, the method only applies to maps inducing group morphisms.

We use the techniques on the generalized cohomology theory described by Gillet and Soulé in [28]. Roughly speaking, the idea is that any good enough map from K-theory to K-theory or to a cohomology theory is characterized by its behavior over the K-groups of the simplicial scheme $B.GL_N$.

We give several characterization theorems. As a main consequence of these, we give a characterization of the *Adams and lambda operations* on higher K-theory and of the *Chern character* and *Chern classes* on a suitable cohomology theory.

More explicitly, let $\mathbf{C}$ be the big Zariski site over a noetherian finite dimensional scheme S. Denote by $B.GL_{N/S}$ the simplicial scheme $B.GL_N \times_{\mathbb{Z}} S$ and let $Gr(N,k)$ be the Grassmanian scheme over S. Let $S.\mathcal{P}$ be the Waldhausen simplicial sheaf computing algebraic K-theory and let $\mathbb{F}.$ be a simplicial sheaf. Note that $S.\mathcal{P}$ is an H-space. Let Ψ^k_{GS} be the Adams operations on higher algebraic K-theory defined by Gillet and Soulé in [28]. The two main consequences of our uniqueness theorem are the following.

Theorem 1 (Corollary 2.4.4)**.** *Let $\rho : S.\mathcal{P} \to S.\mathcal{P}$ be an H-space map in the homotopy category of simplicial sheaves on $\mathbf{C}$. If for some $k \geq 1$ there is a commutative square*

$$\begin{tikzcd}
K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"] \arrow[d, "\Psi^k"'] & K_0(B.GL_{N/S}) \arrow[d, "\rho"] \\
K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"'] & K_0(B.GL_{N/S}),
\end{tikzcd}$$

then ρ agrees with the Adams operation Ψ^k_{GS}, for all schemes X over S.

Theorem 2 (Theorem 2.5.5)**.** *Let $\mathcal{F}^*$ be a cochain complex of sheaves of abelian groups in $\mathbf{C}$. Let*

$$S.\mathcal{P} \longrightarrow \prod_{j\in\mathbb{Z}} \mathcal{K}.(\mathcal{F}(j)[2j])$$

be an H-space map in the homotopy category of simplicial sheaves on $\mathbf{C}$. The induced morphisms

$$K_m(X) \to \prod_{j\in\mathbb{Z}} H^{2j-m}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j))$$

agree with the Chern character defined by Gillet in [21] for every scheme X, if and only if, the induced map

$$K_0(X) \to \prod_{j\in\mathbb{Z}} H^{2j}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j))$$

is the Chern character for $X = Gr(N,k)$, for all N, k.

In particular:

- ▶ We prove that the Adams operations defined by Grayson in [31] agree with the ones defined by Gillet and Soulé in [28], for all noetherian schemes of finite Krull dimension. This implies that for this class of schemes, the operations defined by Grayson satisfy the usual identities of a λ-ring.

- ▶ We prove that the Adams operations defined in Chapter 4 agree with the ones defined by Gillet and Soulé in [28], for all noetherian schemes of finite Krull dimension.

- ▶ We give an alternative proof that the morphism defined by Burgos and Wang in [15] agrees with the Beilinson regulator.

Chapter 3 is devoted to the development of the theory of higher arithmetic Chow groups for arithmetic varieties. Since the theory of higher algebraic Chow groups given by Bloch, $CH^p(X,n)$, is only fully established for schemes over field, we have to restrict ourselves to arithmetic varieties over a field.

Let X be a complex algebraic manifold and let $H^*_{\mathcal{D}}(X,\mathbb{R}(p))$ denote the Deligne-Beilinson cohomology groups with real coefficients. For every $p\geq 0$, we define two cochain complexes, $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ and $\mathcal{D}^*_{\mathbb{A}}(X,p)_0$, constructed out of differential forms on $X\times(\mathbb{A}^1)^n$ with logarithmic singularities along infinity. The following isomorphisms are satisfied:

$$H^{2p-n}(\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0)\cong CH^p(X,n)_{\mathbb{R}},$$

and

$$H^r(\mathcal{D}^*_{\mathbb{A}}(X,p)_0)\cong H^r_{\mathcal{D}}(X,\mathbb{R}(p)),\qquad \text{for } r\leq 2p.$$

We show that the complex $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ enjoys the same properties as the complex $Z^p(X,n)_0$ defined by Bloch in [7]. We actually use its cubical analog, defined by Levine in [41], due to its suitability for describing the product structure on $CH^*(X,*)$. The subindex 0 means the normalized chain complex associated to a cubical abelian group.

Moreover, there is a natural chain morphism

$$\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0\xrightarrow{\rho}\mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0$$

which induces, after composition with the isomorphism

$$K_n(X)_{\mathbb{Q}}\cong\bigoplus_{p\geq 0}CH^p(X,n)_{\mathbb{Q}}$$

described by Bloch in [7], the *Beilinson regulator* (**Theorem 3.4.5**):

$$K_n(X)_{\mathbb{Q}}\cong\bigoplus_{p\geq 0}CH^p(X,n)_{\mathbb{Q}}\xrightarrow{\rho}\bigoplus_{p\geq 0}H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

An analogous construction using projective lines instead of affine lines can be developed. We define a chain complex, $\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p)$, analogous to the complex $\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$

and a chain complex $\widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p)$, analogous to the complex $\mathcal{D}_{\mathbb{A}}^{2p-*}(X,p)_0$. We also define a chain morphism

$$\widetilde{\mathcal{D}}_{\mathbb{P},\mathcal{Z}^p}^{2p-*}(X,p) \xrightarrow{\rho} \mathcal{D}_{\mathbb{P}}^{2p-*}(X,p).$$

In this case, if X is proper, following the methods of Burgos and Wang in [15], section 6, integration along projective lines induces a chain morphism

$$\widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p) \to \mathcal{D}^{2p-*}(X,p).$$

This gives a chain morphism

$$\widetilde{\mathcal{D}}_{\mathbb{P},\mathcal{Z}^p}^{2p-*}(X,p) \xrightarrow{\rho} \mathcal{D}^{2p-*}(X,p)$$

representing the Beilinson regulator. Observe that, when X is proper, this representative has the advantage of having as target precisely the Deligne complex of differential forms on X, and not a chain complex involving differential forms on $X \times (\mathbb{A}^1)^n$. This is needed in order to develop a theory of higher arithmetic Chow groups analogous to the higher arithmetic K-theory developed by Takeda in [57].

In the second part of this chapter we use the morphism ρ to define the higher arithmetic Chow group $\widehat{CH}^p(X,n)$, for any arithmetic variety X over a field. The formalism underlying our definition is the theory of diagrams of complexes and their associated simple complexes, developed by Beilinson in [5]. That is, one considers the diagram of chain complexes

$$\widehat{\mathcal{Z}}^p(X,*)_0 = \left(\begin{tikzcd}[column sep=small] & H_{\mathcal{D}}^{2p}(X\times\mathbb{A}^*,\mathbb{R}(p))_0 & & \widehat{\mathcal{D}}_{\mathbb{A}}^{2p-*}(X,p)_0 \\ Z^p(X,*)_0 \arrow[ur, "f_1"] & & \mathcal{D}_{\mathbb{A},\mathcal{Z}^p}^{2p-*}(X,p)_0 \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\rho"] & \end{tikzcd} \right).$$

Then, the higher arithmetic Chow groups of X are given by the homology groups of the simple of the diagram $\widehat{\mathcal{Z}}^p(X,*)_0$:

$$\widehat{CH}^p(X,n) := H_n(s(\widehat{\mathcal{Z}}^p(X,*)_0)).$$

The following properties are shown:

- **Theorem 3.6.11**: Let $\widehat{CH}^p(X)$ denote the arithmetic Chow group defined by Burgos. Then, there is a natural isomorphism

$$\widehat{CH}^p(X) \xrightarrow{\cong} \widehat{CH}^p(X,0).$$

- **Proposition 3.6.7**: There is a long exact sequence

$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$

$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \mathcal{D}_{\log}^{2p-1}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0.$$

- **Proposition 3.6.15** (*Pull-back*): Let $f : X \to Y$ be a morphism between two arithmetic varieties over a field. Then, there is a pull-back morphism

$$\widehat{CH}^p(Y,n) \xrightarrow{f^*} \widehat{CH}^p(X,n),$$

for every p and n, compatible with the pull-back maps on the groups $CH^p(X,n)$ and $H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p))$.

- **Corollary 3.6.19** (*Homotopy invariance*): Let $\pi : X \times \mathbb{A}^m \to X$ be the projection on X. Then, the pull-back map

$$\pi^* : \widehat{CH}^p(X,n) \to \widehat{CH}^p(X \times \mathbb{A}^m, n), \quad n \geq 1$$

is an isomorphism.

- **Theorem 3.9.7** (*Product*): There exists a product on

$$\widehat{CH}^*(X,*) := \bigoplus_{p\geq 0, n\geq 0} \widehat{CH}^p(X,n),$$

which is associative, graded commutative with respect to the degree n and commutative with respect to the degree p.

Finally, we briefly discuss an alternative approach for the definition of higher arithmetic Chow groups, which follows the ideas of Takeda in [57], for the definition of the higher arithmetic K-groups of a proper arithmetic variety. To this end, we use the definition of the regulator by means of projective lines, restricting ourselves to proper arithmetic varieties over a field.

The following two questions remain open:

- Do the groups constructed here agree with the definition of higher arithmetic Chow groups of Goncharov?
- Can we extend the definition to arithmetic varieties over an arithmetic ring?

In **Chapter 4**, we construct a representative of the Adams operations on higher algebraic K-theory. Let X be any scheme and let $\mathcal{P}(X)$ be the exact category of locally free sheaves of finite rank on X. The algebraic K-groups of X, $K_n(X)$, are defined as the Quillen K-groups of the category $\mathcal{P}(X)$.

These groups can be equipped with a λ-ring structure. Then, the *Adams operations* on each $K_n(X)$ are obtained from the λ-operations by a polynomial formula on the λ-operations. In the literature there are several definitions of the Adams operations on the higher K-groups of a scheme X. By means of the homotopy theory of simplicial sheaves (as recalled in chapter 2), Gillet and Soulé defined Adams operations for any noetherian scheme of finite Krull dimension. Grayson, in [31], constructed a simplicial map inducing Adams operations on the K-groups of any category endowed with a suitable tensor product, symmetric power and exterior power. In particular, he constructed Adams operations for the algebraic K-groups of any scheme X. Following the methods of Schechtman in [51], Lecomte, in [40], defined Adams operations for the rational K-theory of any scheme X equipped with an ample family of invertible sheaves. They are induced by map in the homotopy category of infinite loop spectra.

Our aim is to construct an explicit chain morphism which induces the Adams operations on rational algebraic K-theory. It is our hope that this construction will improve our understanding of the eigenvalue spaces for the Adams operations.

Consider the chain complex of cubes associated to the category $\mathcal{P}(X)$. McCarthy in [47], showed that the homology groups of this complex, with rational coefficients, are isomorphic to the rational algebraic K-groups of X (see section 1.3.3).

We first attempted to find a homological version of Grayson's simplicial construction, but this seems particularly difficult from the combinatorial point of view.

The current approach is based on a simplification obtained by using the *transgressions of cubes by affine or projective lines*, at the price of having to reduce to regular noetherian schemes. This was Burgos and Wang's idea in [15], in order to define a chain morphism representing Beilinson's regulator.

In order to commute with the representative of the Beilinson regulator "ch", the desired morphism should be of the form

$$E \mapsto \Psi^k(\mathrm{tr}_n(E)),$$

with Ψ^k a description of the k-th Adams operation at the level of vector bundles. Unfortunately, for the known choices of Ψ^k, this map does not define a chain morphism. The key obstruction is that while, for any two hermitian vector bundles $\overline{E}, \overline{F}$, we have the equality

$$\mathrm{ch}(\overline{E} \oplus \overline{F}) = \mathrm{ch}(\overline{E}) + \mathrm{ch}(\overline{F}),$$

it is not true that for any two vector bundles E, F, we have the equality

$$\Psi^k(E \oplus F) = \Psi^k(E) \oplus \Psi^k(F).$$

It is true, however, at the level of $K_0(X)$.

The root of the problem is that the map

$$E \mapsto \mathrm{tr}_n(E)$$

is not a chain morphism. However, adding to this map a collection of cubes which have the property of being split in all directions, we obtain a chain morphism. The fact that the added cubes are split in all directions implies that they are cancelled after applying "ch". Therefore, we will still have commutativity of Ψ^k with "ch".

With this strategy, we first assign to a cube on X a collection of cubes defined either on $X \times (\mathbb{P}^1)^*$ or on $X \times (\mathbb{A}^1)^*$, which have the property of being split in all directions (**Proposition 4.3.17**). These cubes are called *split cubes.* This gives a morphism which we call the transgression morphism.

Then, by a purely combinatorial formula on the Adams operations of locally free sheaves, we give a formula for the Adams operations on split cubes (**Corollary 4.2.39**). The key point is to use Grayson's idea of considering the secondary Euler characteristic class of the Koszul complex associated to a locally free sheaf of finite rank.

The composition of the transgression morphism with the Adams operations for split cubes gives a chain morphism representing the Adams operations for any regular noetherian scheme of finite Krull dimension (**Theorem 4.4.2**).

The two constructions, with projective lines or with affine lines, are completely analogous. One may choose the more suitable one in each particular case. For instance, to define Adams operations on the K-groups of a regular ring R, one may consider the definition with affine lines so as to remain in the category of affine schemes. On the other hand, if for instance our category of schemes is the category of projective regular schemes, then the construction with projective lines may be the appropriate one.

The main application of our construction of Adams operations is the definition of a (pre)-λ-ring structure on the rational arithmetic K-groups of an arithmetic variety X.

In **Chapter 5**, we give a pre-λ-ring structure to both definitions of higher arithmetic K-groups tensored by the rational numbers $\mathbb{Q}$, $\widehat{K}_n(X)_{\mathbb{Q}}$ and $\widehat{K}_n^T(X)_{\mathbb{Q}}$. It is compatible with the λ-ring structure on the algebraic K-groups, $K_n(X)$, defined by Gillet and Soulé in [28], and with the canonical λ-ring structure on $\bigoplus_{p\geq 0}\widetilde{\mathcal{D}}^{2p-*}(X,p)$, given by the graduation by p (see lemma 1.3.28). Moreover, for $n=0$ we recover the λ-ring structure of $\widehat{K}_0(X)\otimes\mathbb{Q}$.

More concretely, we construct *Adams operations*

$$\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}}, \qquad k\geq 0,$$

which, since we have tensored by $\mathbb{Q}$, induce λ-operations on $\widehat{K}_n(X)_{\mathbb{Q}}$.

In order to deal with $\widehat{K}_n^T(X)_{\mathbb{Q}}$, we introduce the *modified homology groups*, which are the analogue in homology of the modified homotopy groups. Then, the homology groups modified by "ch" give a homological description of $\widehat{K}_n(X)_{\mathbb{Q}}$ (**Theorem 5.3.11**).

In this chapter we show that the construction of Adams operations of chapter 4 commutes strictly with "ch" (**Theorem 5.4.11**), and we deduce the pre-λ-ring structure for $\widehat{K}_n(X)_{\mathbb{Q}}$ and $\widehat{K}_n^T(X)_{\mathbb{Q}}$ (**Corollary 5.4.14 and Corollary 5.4.16**).

For the time being, we have not been able to prove that it is a λ-ring.

Chapter 1

Preliminaries

This first chapter gives the background needed for the rest of the work. We introduce the definitions and main properties of the different theories that underlie our results.

Specifically, the required preliminaries are divided into four sections:

- *Closed simplicial model categories.* The definitions and main properties of closed simplicial model categories are given. We focus on the category of simplicial sets. The topics covered here are mainly used in chapter 2, for the uniqueness theorems on characteristic classes.

- *Homological algebra.* Basic definitions and results on homological algebra are given. We mainly discuss iterated (co)chain complexes and their associated simple complexes, the relationship between simplicial sets and chain complexes, and the analogous relationship between cubical sets and chain complexes. These concepts are required in all chapters, mainly in chapters 4, 5 and 3.

- *Algebraic K-theory.* We give a brief introduction to algebraic K-theory of exact categories, using the Quillen and Waldhausen construction. We proceed with the definition of the complex of cubes, which will play a key role in chapters 4 and 5.

- *Deligne-Beilinson cohomology.* In this final section of the preliminaries, we give the definitions and main properties of Deligne-Beilinson cohomology. This includes the Deligne complex for Deligne-Beilinson cohomology and the Deligne complex for cohomology with supports. These concepts are needed in chapters 5 and 3.

1.1 Closed simplicial model categories

Quillen, in [49], defined the notion of *closed model categories*. These are categories equipped with three families of morphisms, called *weak equivalences (W), fibrations (F) and cofibrations (C)* satisfying a list of axioms which allows one to do "homotopy theory", i.e. to talk about homotopic morphisms. With an additional structure, namely if the category is enriched over the category of simplicial sets, the homotopy theory becomes easier to handle. In this section we give the definitions and basic facts about these categories, focusing on the categories which are of interest to us. The main reference is the original source [49], as well as several books on model categories, such as [46], [29] and [36].

Let $\mathcal{U}$ be a universe, in the sense of [1], Exposé I, §0. In the sequel, a set, group, topological space etc., mean a $\mathcal{U}$-set, a $\mathcal{U}$-group, a $\mathcal{U}$-topological space etc.

1.1.1 Simplicial and cosimplicial objects over a category

Let $\mathbf{\Delta}$ be the category whose objects are the ordered finite sets $[0] = \{0\}$, $[1] = \{0, 1\}$, $\ldots$, $[n] = \{0, \ldots, n\}$, $\ldots$, and whose morphisms are the ordered maps. Every such morphism can be expressed as a composition of maps of the following type

$$\begin{array}{lll} \partial^i : [n-1] \to [n] & \text{for} \quad 0 \le i \le n & \textit{(cofaces)}, \\ s^i : [n+1] \to [n] & \text{for} \quad 0 \le i \le n & \textit{(codegeneracies)}, \end{array} \tag{1.1.1}$$

where

$$\partial^i(m) = \begin{cases} m & \text{if} \quad m < i, \\ m+1 & \text{if} \quad m \ge i, \end{cases} \qquad s^i(m) = \begin{cases} m & \text{if} \quad m \le i, \\ m-1 & \text{if} \quad m > i. \end{cases}$$

These morphisms satisfy the *cosimplicial identities*:

$$\begin{array}{rcll} \partial^j \partial^i & = & \partial^i \partial^{j-1} & \text{for} \quad i < j, \\ s^j \partial^i & = & \begin{cases} \partial^i s^{j-1} & \\ id & \\ \partial^{i-1} s^j & \end{cases} & \begin{array}{l} \text{for} \quad i < j, \\ \text{for} \quad i = j, j+1, \\ \text{for} \quad i > j+1, \end{array} \\ s^j s^i & = & s^i s^{j+1} & \text{for} \quad i \le j. \end{array} \tag{1.1.2}$$

Simplicial objects. Let $\mathbf{\Delta}^{op}$ be the category opposite to $\mathbf{\Delta}$ and let $\mathcal{C}$ be any category.

Definition 1.1.3. A *simplicial object $X.$ over the category* $\mathcal{C}$ is a functor

$$\mathbf{\Delta}^{op} \to \mathcal{C}.$$

Equivalently, a simplicial object $X.$ is a collection of objects in $\mathcal{C}$, $\{X_n\}_{n \ge 0}$, together with maps

$$\begin{array}{lll} \partial_i : X_n \to X_{n-1} & \text{for} \quad 0 \le i \le n & \textit{(faces)}, \\ s_i : X_n \to X_{n+1} & \text{for} \quad 0 \le i \le n & \textit{(degeneracies)}, \end{array}$$

satisfying the *simplicial identities* (which are dual to the cosimplicial identities):

$$\begin{array}{rcll} \partial_i \partial_j & = & \partial_{j-1} \partial_i & \text{for} \quad i < j, \\ \partial_i s_j & = & \begin{cases} s_{j-1} \partial_i & \\ id & \\ s_j \partial_{i-1} & \end{cases} & \begin{array}{l} \text{for} \quad i < j, \\ \text{for} \quad i = j, j+1, \\ \text{for} \quad i > j+1, \end{array} \\ s_i s_j & = & s_{j+1} s_i & \text{for} \quad i \le j. \end{array} \tag{1.1.4}$$

A *simplicial map* between two simplicial objects $X., Y.$ in $\mathcal{C}$, consists of a collection of maps

$$f_n : X_n \to Y_n, \qquad n \geq 0$$

in $\mathcal{C}$, commuting with the face and degeneracy maps.

A simplicial object over the category of sets is called a *simplicial set.* This category is denoted specifically by **SSets**. Generally, a simplicial object over the category of groups, rings, schemes, etc. is called a *simplicial group, ring, scheme*, etc.

Definition 1.1.5. Let $\mathbf{\Delta}^k$ denote the k-fold cartesian product of $\mathbf{\Delta}$.

(i) A *k-simplicial object over* $\mathcal{C}$ is a functor

$$(\mathbf{\Delta}^k)^{op} \to \mathcal{C}.$$

(ii) Let $X.$ be a k-simplicial object in $\mathcal{C}$. Then the *diagonal* of $X.$, $\operatorname{diag}(X).$, is the simplicial object in $\mathcal{C}$ given by

$$\operatorname{diag}(X)_n = X_{n,\ldots,n}.$$

Denote by $\partial^{n_1,\ldots,n_k}_{i_1,\ldots,i_k}$ the $(i_1, \ldots, i_k)$-th face map in $X_{n_1,\ldots,n_k}$. Then, the i-th face map in $\operatorname{diag}(X)_n$ is $\partial^{n,\ldots,n}_i$. The degeneracy maps are defined analogously.

Definition 1.1.6. Let $X.$ be an object of **SSets**.

(i) An element $x \in X_n$ is called an *n-simplex* of $X.$. We say that x is a *vertex* if it is a 0-simplex.

(ii) An element $x \in X_n$ is called *degenerate* if for some i and $y \in X_{n-1}$, $s_i(y) = x$. Otherwise, x is called *non-degenerate.*

(iii) The *n-th skeleton* of $X.$, denoted by $\operatorname{sk}_n X.$, is the subobject generated by all the simplices of $X.$ of dimension $\leq n$.

Cosimplicial objects.

Definition 1.1.7. A *cosimplicial object* $X^{\cdot}$ over $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$.

A functor from $\mathbf{\Delta}$ to $\mathcal{C}$ is equivalent to a collection of objects in $\mathcal{C}$, $\{X^n\}_{n\geq 0}$, together with maps

$$\begin{array}{lll} \partial^i : X^{n-1} \to X^n & \text{for} \quad 0 \leq i \leq n & \textit{(cofaces)}, \\ s^i : X^{n+1} \to X^n & \text{for} \quad 0 \leq i \leq n & \textit{(codegeneracies)}, \end{array}$$

satisfying the *cosimplicial identities* in (1.1.2). A *cosimplicial map* between two cosimplicial objects $X^{\cdot}$, $Y^{\cdot}$ in $\mathcal{C}$ consists of a collection of maps

$$f^n : X^n \to Y^n, \qquad n \geq 0$$

in $\mathcal{C}$, commuting with the coface and codegeneracy maps.

A cosimplicial object over the category of sets is called a *cosimplicial set.* Similarly, a cosimplicial object over the category of groups, rings, etc. is called a *cosimplicial group, ring*, etc.

We finish this subsection with the main examples of simplicial and cosimplicial sets, that will be used in the sequel.

Example 1.1.8 (Standard simplex)**.** For every $n \geq 0$, the *standard n-simplex,* $\Delta^n_\cdot$, is the simplicial set given by the functor represented by $[n]$, i.e.

$$\Delta^n_\cdot = \mathrm{Hom}_{\mathbf{\Delta}}(\cdot, [n]).$$

The simplicial set $\Delta^n_\cdot$ has exactly one non-degenerate n-simplex, denoted by i_n. Its main properties are:

- *Universal property*: for every simplicial set $X.$ and every n-simplex $x \in X_n$, there is a unique simplicial map
$$\Delta_x : \Delta^n_\cdot \to X.$$
with $\Delta_x(i_n) = x$.

- *Yoneda's lemma*: for every simplicial set $X.$,
$$\mathrm{Hom}_{\mathbf{SSets}}(\Delta^n_\cdot, X.) \cong X_n.$$

Example 1.1.9 (Simplicial sphere)**.** The *n-simplicial sphere* $S^n_\cdot$ is the simplicial set with only two non-degenerate simplices, a 0-simplex, $*$, and an n-simplex, σ. It is the quotient simplicial set

$$\Delta^n_\cdot / \mathrm{sk}_{n-1}\, \Delta^n_\cdot.$$

The simplicial sphere $S^n_\cdot$ is naturally pointed by its unique non-degenerate vertex.

Example 1.1.10 (Cosimplicial standard simplex)**.** The basic example of a cosimplicial object in **SSets** is the *cosimplicial standard simplex* $\Delta^\cdot_\cdot$. It is the cosimplicial simplicial set which, in codimension n, is the standard n-simplex (see example 1.1.8) and with coface and codegeneracy maps given by the universal property

(i) $\partial^j = \Delta_{\partial_j i_n} : \Delta^{n-1}_\cdot \to \Delta^n_\cdot \quad \text{for} \quad 0 \leq j \leq n,$

(ii) $s^j = \Delta_{s_j i_n} : \Delta^{n+1}_\cdot \to \Delta^n_\cdot \quad \text{for} \quad 0 \leq j \leq n.$

Example 1.1.11 (Nerve of a small category)**.** Let $\mathcal{C}$ be a small category (i.e. a category whose objects form a set). The *nerve* of $\mathcal{C}$, $N.\mathcal{C}$, is the simplicial set with $N_n\mathcal{C}$ the set of the compositions of n morphisms in $\mathcal{C}$,

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n.$$

The i-th face map is obtained by omitting X_i and composing $f_i \circ f_{i-1}$. The i-th degeneracy map is given by the insertion of the identity morphism at X_i. The assignment $\mathcal{C} \mapsto N.\mathcal{C}$ is a functor from the category of small categories to **SSets**.

Example 1.1.12 (Nerve of an abelian group)**.** Let G be an abelian group and let $\mathcal{G}$ be the category with only one object and having G as group of morphisms. Then, the *nerve of* G is by definition the nerve of the category $\mathcal{G}$.

Example 1.1.13 (Nerve of an open cover)**.** Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X. Then, the nerve of $\mathcal{U}$, $N.\mathcal{U}$ is the simplicial set with

$$N_n\mathcal{U} = \coprod_{i_1,\ldots,i_n \in\ I} U_{i_1} \cap \cdots \cap U_{i_n}.$$

For explanations of the relation between the nerve of an open cover, as defined here, and the general definition of the nerve of a category, see [52].

Example 1.1.14 (Associated simplicial abelian group)**.** Let $X.$ be a simplicial set. The associated simplicial abelian group, $\mathbb{Z}X.$, is defined by setting $\mathbb{Z}X_n$ to be the free abelian group on the elements of X_n. The face and degeneracy maps are induced by those of $X.$.

1.1.2 Homotopy theory of simplicial sets

In this section, three classes of maps in **SSets** are introduced. They are named weak equivalences, fibrations and cofibrations.

We start by defining the homotopy groups of a simplicial set. These groups are simply the topological homotopy groups of the realization of the simplicial set.

The realization functor and homotopy groups. Denote by **Top** the category of topological spaces. Let $|\Delta^n|$ be the topological n-simplex, i.e.

$$|\Delta^n| := \{(t_0,\ldots,t_n) \in \mathbb{R}^{n+1}|\ \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\},$$

equipped with the subspace topology induced by the Euclidean topology in $\mathbb{R}^{n+1}$. In fact, $|\Delta^*|$ is a cosimplicial topological space, with coface and codegeneracy maps given, for all $i = 0,\ldots,n$, by

$$\begin{aligned} \partial^i(t_0,\ldots,t_{n-1}) &= (t_0,\ldots,t_{i-1},0,t_i,\ldots,t_{n-1}) \\ s^i(t_0,\ldots,t_{n+1}) &= (t_0,\ldots,t_i+t_{i+1},\ldots,t_{n+1}). \end{aligned}$$

Definition 1.1.15. The *realization functor* $|\cdot| :$ **SSets** $\to$ **Top** is the functor defined as follows.

(i) For every object $X.$ in **SSets**, equip X_n with the discrete topology and let $|X.|$ be the topological space

$$|X.| := \coprod_n X_n \times |\Delta^n| \big/ \sim$$

where $\sim$ is the equivalence relation given by

$$(\partial_i x, u) \sim (x, \partial^i u), \quad (s_i y, u) \sim (y, s^i u),$$

for all $x \in X_{n+1},\ y \in X_{n-1}$, and $u \in |\Delta^n|$.

(ii) If $f : X. \to Y.$ is a simplicial map, the map

$$\coprod_n X_n \times |\Delta^n| \xrightarrow{f \times id} \coprod_n Y_n \times |\Delta^n|$$

is compatible with the equivalence relation $\sim$. Hence, there is an induced continuous map $|f| : |X.| \to |Y.|$.

There is an equivalent definition of the realization of simplicial sets in a categorical frame involving colimits (see [29], § I.2).

It follows from the definition that the realization of the standard n-simplex $\Delta^n_.$ is exactly the topological n-simplex $|\Delta^n|$.

Lemma 1.1.16. *For every simplicial set $X.$, $|X.|$ is a CW-complex with one n-cell for every non-degenerate n-simplex of $X.$.*

Proof. See [46], § III.14. □

Example 1.1.17 (Classifying space)**.** Let $\mathcal{C}$ be a small category. The *classifying space* of a small category $\mathcal{C}$ is the realization of its nerve (see example 1.1.11), i.e.

$$B\mathcal{C} := |N.\mathcal{C}|.$$

The assignment $\mathcal{C} \mapsto B\mathcal{C}$ is a functor from the category of small categories to the category of topological spaces. Moreover, an equivalence of small categories $F : \mathcal{C} \to \mathcal{D}$ induces a homotopy equivalence $BF : B\mathcal{C} \to B\mathcal{D}$ of topological spaces.

Remark 1.1.18. In the literature one also finds the notation $B.\mathcal{C}$ for $N.\mathcal{C}$. We will use this notation in chapter §2, for the simplicial set $B.Q\mathcal{P}$, computing algebraic K-theory of the category $\mathcal{P}$ (see 1.3.1).

Definition 1.1.19. Let $X.$ be a simplicial set and $* \in X_0$ a base point. Denote by $*$ the corresponding point in $|X.|$. For $n \geq 0$, the *n-th homotopy set* of $X.$ with respect to $*$ is defined by

$$\pi_n(X., *) := \pi_n(|X.|, *).$$

When there is no source of confusion, the base point $*$ is usually dropped from the notation. Observe that by definition, $\pi_n(\cdot, *)$ is a functor from **SSets** to the category of sets for $n = 0$, to the category of groups for $n = 1$, and to the category of abelian groups for $n \geq 2$.

Weak equivalences, fibrations and cofibrations Let $\Delta^n_\cdot$ be the standard n-simplex of example 1.1.8. For every $0 \leq k \leq n$ the k-th *horn*, $\Lambda^n_k \subset \Delta^n_\cdot$, is the simplicial set generated by all $\partial_j(i_n)$ except for $\partial_k(i_n)$.

Definition 1.1.20. Let $f : X_\cdot \to Y_\cdot$ be a simplicial map.

(i) The map f is a *weak equivalence* if the induced map $\pi_0(X_\cdot) \xrightarrow{f_*} \pi_0(Y_\cdot)$ is a bijection and for every $n \geq 1$ and any choice of vertex $* \in X_0$, the induced map

$$\pi_n(X_\cdot, *) \xrightarrow{f_*} \pi_n(Y_\cdot, f(*))$$

is an isomorphism.

(ii) The map f is a *fibration*, or a *Kan fibration*, if for all $n \geq 0$, $0 \leq k \leq n$, and any commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "i"'] & X_\cdot \arrow[d, "p"] \\ \Delta^n_\cdot \arrow[r] \arrow[ur, dotted] & Y_\cdot, \end{tikzcd}$$

there exists a diagonal dotted arrow, making the whole diagram commute.

(iii) The map f is a *cofibration* if it is a monomorphism.

1.1.3 Closed simplicial model categories

Closed model categories. Let $\mathcal{C}$ be a category equipped with three families of morphisms, called *weak equivalences (W), fibrations (F) and cofibrations (C).* One says that a morphism f is a *trivial fibration* (resp. *trivial cofibration*) if it is both a fibration (resp. cofibration) and a weak equivalence. Then, $\mathcal{C}$ together with the families W, F and C is a *closed model category* if the following five axioms are satisfied:

CM1. $\mathcal{C}$ is closed under finite direct limits and finite inverse limits.

CM2. *(Two of three)* If f and g are such that gf is defined, then, if two of f, g and gf are weak equivalences, so is the third.

CM3. *(Retraction axiom)* Let f be a retract of g, i.e., in the category of maps, there are maps $a : f \to g$ and $b : g \to f$ such that $ba = id_f$. Then, if g is a weak equivalence, a fibration or a cofibration, so is f.

CM4. *(Lifting axiom)* Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where i is a cofibration, p a fibration, and either i or p is a weak equivalence, then there exists a diagonal arrow $B \to X$ making the whole diagram commute.

CM5. *(Factorization axiom)* Any map f can be factored in two ways

- $\triangleright$ $f = p \circ i$, with i a cofibration and p a trivial fibration.
- $\triangleright$ $f = p \circ i$, with i a trivial cofibration and p a fibration.

Remark 1.1.21. These are the axioms initially proposed by Quillen. Later, some modifications have been introduced. Namely,

(1) In axiom CM1, one requires the existence of all direct and inverse limits, not only the finite ones.

(2) In axiom CM5, one requires the factorizations to be functorial. That is, if we denote by $\mathrm{Map}(\mathcal{C})$ the category of morphisms in $\mathcal{C}$, one requires the existence of four functors

$$\alpha, \beta, \gamma, \delta : \mathrm{Map}(\mathcal{C}) \to \mathrm{Map}(\mathcal{C})$$

such that

$$f = \beta(f) \circ \alpha(f), \quad \text{and} \quad f = \delta(f) \circ \gamma(f),$$

with $\alpha(f)$ a trivial fibration, $\beta(f)$ a cofibration, $\gamma(f)$ a fibration and $\delta(f)$ a trivial cofibration.

The usual closed model categories satisfy this modified version of the Quillen axioms.

Immediate consequences of the five axioms CM1-CM5 are the following:

(1) Axiom CM1 ensures the existence of an *initial object*, denoted by $\emptyset$, and a *final object*, denoted by $*$, in $\mathcal{C}$.

(2) The class of (co)fibrations and the class of trivial (co)fibrations are both closed under composition and (co)base change, and contain all isomorphisms.

Theorem 1.1.22. *With the class of weak equivalences, fibrations and cofibrations defined in 1.1.20,* **SSets** *is a closed model category.*

Proof. See [49], § II.3.14, Theorem 3, [29], § I Theorem 11.3. □

An initial object of **SSets** is the empty simplicial set $\emptyset$ and a final object is $* = \Delta^0_\cdot$.

Fibrant and cofibrant resolutions.

Definition 1.1.23. Let $\mathcal{C}$ be a closed model category and $X \in \mathrm{Obj}\,\mathcal{C}$.

(1) The object X is *fibrant* if the unique map $X \to *$ is a fibration.

(2) The object X is *cofibrant* if the unique map $\emptyset \to X$ is a cofibration.

Axiom CM5 ensures the existence of (functorial) fibrant and cofibrant resolutions. Namely, for every object X in $\mathcal{C}$, there exist weak equivalences

$$X \to X^{\sim}, \qquad X^{*} \to X$$

where $X^{\sim}$ is fibrant and X^{*} is cofibrant.

Example 1.1.24. (1) By definition, every simplicial set is cofibrant.

(2) (Moore) The underlying simplicial set of a simplicial group is fibrant (see [49], §II.3.12 or [29], §I Lemma 3.4 for a proof).

(3) For $n \geq 2$, $\Delta^{n}_{\cdot}$ is not fibrant.

The five axioms CM1-CM5 imply that for a category $\mathcal{C}$, once two of the three required families of morphisms are given, and satisfy the axioms, then the third family exists and it is characterized as follows.

Definition 1.1.25. A map $f : X \to Y$ in $\mathcal{C}$ has the *right lifting property* (RLP) with respect to $g : A \to B$ in $\mathcal{C}$, (or g has the *left lifting property* (LLP) with respect to f) if for every commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

the dotted diagonal arrow exists.

If the class of weak equivalences and fibrations (resp. cofibrations) is given, then cofibrations (resp. fibrations) are characterized by being the morphisms having the LLP (resp. RLP) with respect to all trivial fibrations (resp. cofibrations).

From the fifth axiom, we see that if fibrations and cofibrations are defined, then the weak equivalences are the morphisms f such that $f = u \circ v$, with v having the left lifting property with respect to the class of fibrations and u having the right lifting property with respect to the class of cofibrations.

If $\mathcal{C}$ is a closed model category, it is possible to define *left and right homotopy* for maps between objects which are both fibrant and cofibrant. Moreover, one can form the homotopy category $\operatorname{Ho}\mathcal{C}$ by localizing with respect to the weak equivalences (for the definition of localization of a category, see [34] or [49], § I.1). However, these concepts become easier in a closed simplicial model category. We discuss this approach in this section. For details on the "non-simplicial" approach of homotopy theory, see for instance [49] and [29].

Closed simplicial model categories

Definition 1.1.26. A category $\mathcal{C}$ is called *simplicial* if there exists a functor (called the *function complex*)

$$\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{SSets},$$

such that for all objects A and B in $\mathcal{C}$, the following axioms are satisfied.

SC1. $\mathrm{Hom}_{\mathcal{C}}(A, B)_0 = \mathrm{Map}_{\mathcal{C}}(A, B)$.

SC2. The functor $\mathrm{Hom}_{\mathcal{C}}(A, \cdot) : \mathcal{C} \to \mathbf{SSets}$ has a left adjoint

$$A \otimes \cdot : \mathbf{SSets} \to \mathcal{C}$$

such that $A \otimes (K \times L) \cong (A \otimes K) \otimes L$, with the isomorphism being natural in A, K and L.

SC3. The functor $\mathrm{Hom}_{\mathcal{C}}(\cdot, B) : \mathcal{C}^{op} \to \mathbf{SSets}$ has a left adjoint

$$\mathbf{hom}(\cdot, B) : \mathbf{SSets} \to \mathcal{C}^{op}.$$

Remark 1.1.27. It follows from 1.1.8 and axiom SC2, that for all $n \geq 0$,

$$\mathrm{Hom}_{\mathcal{C}}(A, B)_n \cong \mathrm{Map}_{\mathbf{SSets}}(\Delta^n_{\cdot}, \mathrm{Hom}_{\mathcal{C}}(A, B)) \cong \mathrm{Map}_{\mathcal{C}}(A \otimes \Delta^n_{\cdot}, B).$$

This equality is usually taken as the definition of the function complex, once a functor $\otimes$ is defined.

We are interested in the categories $\mathcal{C}$ which are closed model categories and, in addition, simplicial categories. This is the notion of *closed simplicial model categories.*

Definition 1.1.28. Let $\mathcal{C}$ be a category which is both a closed model category and a simplicial category. Following Quillen, one says that $\mathcal{C}$ is a *closed simplicial model category* if the next axiom is satisfied:

SM7. For every cofibration $j : A \to B$, and fibration $q : X \to Y$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(B, X) \xrightarrow{(j^*, q_*)} \mathrm{Hom}_{\mathcal{C}}(A, X) \times_{\mathrm{Hom}_{\mathcal{C}}(A,Y)} \mathrm{Hom}_{\mathcal{C}}(B, Y)$$

is a fibration in $\mathbf{SSets}$, which is trivial if either j or q is trivial.

The following lemma follows from axiom SM7.

Lemma 1.1.29. *Let A, B, X and Y be objects in $\mathcal{C}$.*

(i) If $f : X \to Y$ is a fibration and B is cofibrant, then the induced map

$$f_* : \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(B, Y)$$

is a fibration in $\mathbf{SSets}$.

(ii) If X is fibrant and $g : A \to B$ is a cofibration, then the induced map

$$g^* : \mathrm{Hom}_{\mathcal{C}}(B, X) \to \mathrm{Hom}_{\mathcal{C}}(A, X)$$

is a fibration in **SSets**.

Definition 1.1.30. Let $X.$ and $Y.$ be simplicial sets. The *function complex* or *function space* $\mathrm{Hom}_{\mathbf{SSets}}(X., Y.)$ is the simplicial set with

$$\mathrm{Hom}_{\mathbf{SSets}}(X., Y.)_n = \{\text{simplicial maps } \Delta^n_. \times X. \to Y.\},$$

and face and degeneracy maps induced by (1.1.1).

For simplicity, denote

$$\underline{\mathrm{Hom}}(\cdot, \cdot) = \mathrm{Hom}_{\mathbf{SSets}}(\cdot, \cdot).$$

Proposition 1.1.31. *With the function complex* $\underline{\mathrm{Hom}}$*, the closed model category* **SSets** *is a closed simplicial model category.*

The homotopy relation. We give here the definition of homotopy equivalence of maps in any simplicial category $\mathcal{C}$. Then, for any closed simplicial model category, axiom SM7 implies that the diagonal arrow of axiom CM4 is unique up to homotopy (see [29], § II Proposition 3.8).

Let $X.$ be a simplicial set. We say that $x, y \in X_0$ are *strictly homotopic* if there exists $z \in X_1$ with $\partial_1 z = x$ and $\partial_0 z = y$. We say that they are *homotopic* if they are equivalent with respect to the equivalence relation generated by "to be strictly homotopic to".

Proposition 1.1.32. *If $X.$ is a fibrant simplicial set, "to be strictly homotopic" is an equivalence relation. Moreover, in this case*

$$\pi_0(X.) = \{\textit{homotopy classes of elements in } X_0\}.$$

Proof. See [46], Proposition 3.2. □

Definition 1.1.33. Let $f, g : X \to Y$ be two morphisms in a closed simplicial model category $\mathcal{C}$. We say that f is *strictly homotopic* (resp. *homotopic*) to g if this is the case for $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)_0 = \mathrm{Map}_{\mathcal{C}}(X, Y)$.

If f and g are homotopic, one writes $f \simeq g$. Observe that by the last proposition and lemma 1.1.29, if X is cofibrant and Y is fibrant, then $\simeq$ is an equivalence relation.

The next corollary follows from 1.1.32.

Corollary 1.1.34. *Let $\mathcal{C}$ be a closed simplicial model category. If X is a cofibrant object in $\mathcal{C}$ and Y is a fibrant object in $\mathcal{C}$, then*

$$\pi_0 \mathrm{Hom}_{\mathcal{C}}(X, Y) = \{\textit{homotopy classes of maps } X \to Y\}.$$

In this terminology, one obtains a generalization of Whitehead's classical theorem in algebraic topology.

Theorem 1.1.35 (Whitehead)**.** *Let $\mathcal{C}$ be a closed model category. Let $X, Y \in \operatorname{Obj} \mathcal{C}$ be both fibrant and cofibrant. Then, any weak equivalence $f : X \to Y$ is a homotopy equivalence.*

Proof. See [29], §II Theorem 1.10. □

Whitehead's classical theorem asserts that any weak equivalence $X \to Y$ of CW-complexes is in fact a homotopy equivalence. Since CW-complexes are both fibrant and cofibrant, this result follows from the theorem above.

The homotopy category. Let $\mathcal{C}$ be a closed simplicial model category. The homotopy category associated to $\mathcal{C}$, $\operatorname{Ho}\mathcal{C}$ is formed by formally inverting the weak equivalences. For any two spaces X, Y, one denotes by $[X, Y]$ the set of maps between X and Y in this category.

If $Y \to Y^{\sim}$ is any fibrant resolution of Y, then

$$[X, Y] = \pi_0 \operatorname{Hom}(X, Y^{\sim}) = \{\text{homotopy classes of maps } X \to Y\}.$$

Suppose that $Y \to \hat{Y}$ is a weak equivalence (not necessarily a cofibration) and $\hat{Y}$ is fibrant. Then, if $Y^{\sim}$ is any fibrant resolution, there exists a weak equivalence $Y^{\sim} \to \hat{Y}$. Therefore, by [36], 9.5.12,

$$[X, Y] = \pi_0 \operatorname{Hom}(X, \hat{Y}).$$

Consider $X, Y \in \operatorname{Obj}(\mathcal{C})$ and $f : X \to Y$ a morphism. Suppose that Y is fibrant and let $X^{\sim}$ be a fibrant resolution of X. Then f factorizes uniquely up to homotopy through $X^{\sim}$, i.e. there exists a map in $\mathcal{C}$, $f^{\sim} : X^{\sim} \to Y$, unique up to homotopy under X, such that the following diagram is commutative

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"', "\sim"] & Y \\ X^{\sim}. \arrow[ur, "f^{\sim}"'] & \end{tikzcd}$$

See [36], 8.1.6 for a proof. Therefore, there is a well-defined bijective map

$$[X, Y] \to [X^{\sim}, Y].$$

The pointed category. A small modification in the definition of weak equivalences, fibrations and cofibrations in **SSets** leads to a closed model category structure in the pointed category $\mathbf{SSets}_*$. The category $\mathbf{SSets}_*$ consists of pairs $(X., *)$, where $X.$ is a simplicial set and $* \in X_0$. Morphisms are point-preserving simplicial maps.

Consider the base-point forgetful functors

$$\mathbf{SSets}_* \to \mathbf{SSets}.$$

By definition, a map $f : X_. \to Y_.$ in $\mathbf{SSets}_*$ is a *weak equivalence, fibration or cofibration*, if this is the case in **SSets**.

Let $X_., Y_.$ be two pointed simplicial sets. The *pointed function space*, denoted by $\underline{\mathrm{Hom}}_*(X_., Y_.)$, is the simplicial set with n-simplices the set of maps

$$(\Delta^n_. \times X_.)/(\Delta^n_. \times *) \to Y_.$$

in $\mathbf{SSets}_*$. Face and degeneracy maps are induced by the coface and codegeneracy maps of $\Delta^\cdot$. The *smash product* of two pointed simplicial sets $X_., Y_.$ is the pointed simplicial set

$$X_. \wedge Y_. := (X_. \times Y_.)/(* \times Y_.) \cup (X_. \times *).$$

We set the functor $\otimes$ to be the smash product $\wedge$.

Theorem 1.1.36. *With these definitions,* $\mathbf{SSets}_*$ *is a closed simplicial model category.*

It follows that if $Y_.$ is a fibrant pointed simplicial set, then

$$\pi_0 \operatorname{Hom}_*(X_., Y_.) = \{\text{homotopy classes of pointed maps } X_. \to Y_.\}.$$

1.1.4 Simplicial abelian groups

Let **Ab** be the category of abelian groups and let $s\mathbf{Ab}$ be the category of simplicial abelian groups. Observe that, forgetting the group structure, every simplicial abelian group is a simplicial set.

Definition 1.1.37. Let $f : A_. \to B_.$ be a morphism of simplicial abelian groups.

(1) The morphism f is a *weak equivalence* or *fibration* if this is the case for the underlying simplicial set map.

(2) The morphism f is a *cofibration* if it has the left lifting property with respect to all trivial fibrations.

(3) The homotopy groups of a simplicial abelian group are the homotopy groups of its underlying simplicial set.

Theorem 1.1.38. *With the definitions above,* $s\mathbf{Ab}$ *is a closed model category.*

Proof. See [29], § III, Theorem 2.8. □

The category $s\mathbf{Ab}$ also has a closed simplicial model category structure, given as follows. If $K_.$ is a simplicial set and $A_.$ is a simplicial abelian group, the simplicial abelian group $A_. \otimes K_.$ is defined by

$$(A_. \otimes K_.)_n = \bigoplus_{\sigma \in K_n} A_n.$$

Face and degeneracy maps are induced by those of $K.$ and $A.$.

Let $A.$ and $B.$ be two simplicial abelian groups. The *function complex* is the simplicial set with n-simplices

$$\mathrm{Hom}_{s\mathbf{Ab}}(A., B.)_n := \mathrm{Map}_{s\mathbf{Ab}}(A. \otimes \Delta^n_., B.)$$

and induced face and degeneracy maps.

Then, with this function complex, the category $s\mathbf{Ab}$ is a closed simplicial model category.

Lemma 1.1.39. *The free abelian group functor* $\mathbf{SSets} \to s\mathbf{Ab}$*, of example 1.1.14, preserves weak equivalences.*

Proof. See [29], § III Proposition 2.16. □

1.2 Some results on homological algebra

In this second section of the preliminaries, we state the main definitions and results in homological algebra needed for the forthcoming chapters. We start by fixing the notation for multi-indices. They are used mainly in chapter § 4. We proceed with the discussion of iterated (co)chain complexes. At the end, we recall the relationship between simplicial abelian groups and chain complexes, and the analogous relationship between cubical abelian groups and chain complexes.

General references for homological algebra are [43].

1.2.1 Notation for multi-indices

We give here some notations on multi-indices that will be used in the sequel.

Let $\mathfrak{I}$ be the set of all multi-indices of finite length, i.e.

$$\mathfrak{I} = \{\boldsymbol{i} = (i_1, \dots, i_n) \in \mathbb{N}^n,\ n \in \mathbb{N}\} = \bigcup_{k>0} \mathbb{N}^k.$$

For every $m \geq 0$, consider the set $[0, m] := \{0, \dots, m\}$. If $a \in [0, m]$ and $l = 1, \dots, n$, let $a_l \in [0, m]^n$ be the multi-index

$$(0, \dots, 0, a, 0, \dots, 0),$$

that is, the multi-index where the only non-zero entry is a in the l-th position. We write $\mathbf{1} = (1, \dots, 1)$ and more generally, if $r_1 \leq r_2$, we define $\mathbf{1}_{r_1}^{r_2}$ to be the multi-index with

$$(\mathbf{1}_{r_1}^{r_2})_i = \begin{cases} 1 & \text{if } \ r_1 \leq i \leq r_2, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.2.1. Let $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^n$. We fix the following notations for multi-indices:

(1) The *length* of $\boldsymbol{i}$ is the integer $\mathrm{length}(\boldsymbol{i}) := n$.

(2) The *characteristic* of $\boldsymbol{i}$ is the multi-index $\nu(\boldsymbol{i}) \in \{0,1\}^n$, defined by

$$\nu(\boldsymbol{i})_j = \begin{cases} 0 & \text{if } i_j = 0, \\ 1 & \text{otherwise.} \end{cases}$$

(3) The *norm* of $\boldsymbol{i}$ is defined by $|\boldsymbol{i}| = i_1 + \cdots + i_n$. If $1 \leq l \leq n$, we denote $|\boldsymbol{i}|_l = i_1 + \cdots + i_l$.

(4) *Order on the set of multi-indices:*

- We write $\boldsymbol{i} \geq \boldsymbol{j}$, if for all r, $i_r \geq j_r$. Otherwise we write $\boldsymbol{i} \ngeq \boldsymbol{j}$.
- We denote by $\preceq$ the *lexicographic order* on multi-indices. By $\boldsymbol{i} \prec \boldsymbol{j}$ we mean $\boldsymbol{i} \preceq \boldsymbol{j}$ and $\boldsymbol{i} \neq \boldsymbol{j}$.

(5) Let $1 \leq l \leq n$ and $m \in \mathbb{N}$. Then, we define

$$\begin{array}{lrcl} \textit{Faces:} & \partial_l(\boldsymbol{i}) & := & (i_1, \dots, \hat{i_l}, \dots, i_n). \\ \textit{Degeneracies:} & s_l^m(\boldsymbol{i}) & := & (i_1, \dots, i_{l-1}, m, i_l, \dots, i_n). \\ \textit{Substitution:} & \sigma_l^m(\boldsymbol{i}) & := & s_l^m \partial_l(\boldsymbol{i}) = (i_1, \dots, i_{l-1}, m, i_{l+1}, \dots, i_n). \end{array}$$

In general, for any $\boldsymbol{l} = (l_1, \dots, l_s)$ with $1 \leq l_1 < \cdots < l_s \leq n$ and $\mathbf{m} = (m_1, \dots, m_s) \in \mathbb{N}^s$, we write

$$\partial_{\boldsymbol{l}}(\boldsymbol{i}) = \partial_{l_1} \dots \partial_{l_s}(\boldsymbol{i}), \quad s_{\boldsymbol{l}}^{\mathbf{m}}(\boldsymbol{i}) = s_{l_1}^{m_1} \dots s_{l_s}^{m_s}(\boldsymbol{i}), \quad \text{and} \quad \sigma_{\boldsymbol{l}}^{\mathbf{m}}(\boldsymbol{i}) = \sigma_{l_1}^{m_1} \dots \sigma_{l_s}^{m_s}(\boldsymbol{i}).$$

(6) If $\operatorname{length}(\boldsymbol{i}) = l$ and $\operatorname{length}(\boldsymbol{j}) = r$, the *concatenation* of $\boldsymbol{i}$ and $\boldsymbol{j}$ is the multi-index of length $l + r$ given by

$$\boldsymbol{ij} = (i_1, \dots, i_l, j_1, \dots, j_r).$$

(7) Assume that $\boldsymbol{i} \in \{0,1\}^n$. The *complementary multi-index* of $\boldsymbol{i}$ is the multi-index $\boldsymbol{i}^c := \mathbf{1} - \boldsymbol{i}$, i.e.

$$(\boldsymbol{i}^c)_r = \begin{cases} 0 & \text{if } i_r = 1, \\ 1 & \text{if } i_r = 0. \end{cases}$$

(8) Assume that $\boldsymbol{i}, \boldsymbol{j} \in \{0,1\}^n$. We define their *intersection* by

$$\boldsymbol{i} \cap \boldsymbol{j} = (i_1 \cdot j_1, \dots, i_n \cdot j_n),$$

and their *union* $\boldsymbol{i} \cup \boldsymbol{j}$ by

$$(\boldsymbol{i} \cup \boldsymbol{j})_r = \max\{i_r, j_r\}.$$

1.2.2 Iterated (co)chain complexes

Let $\mathcal{P}$ be an additive category.

Definition 1.2.2. (i) A *k-iterated cochain complex* $A^* = (A^*, d_1, \dots, d_k)$ over $\mathcal{P}$ is a k-graded object together with k endomorphisms $d_1, \dots, d_k$ of multidegrees $1_1, ..., 1_k$, such that, for all i, j, $d_i d_i = 0$ and $d_i d_j = d_j d_i$. The endomorphism d_i is called the i-th differential of A^*.

(ii) A *k-iterated chain complex* $A_* = (A_*, d^1, \ldots, d^k)$ over $\mathcal{P}$ is a k-graded object together with k endomorphisms $d^1, \ldots, d^k$ of multidegrees $-1_1, ..., -1_k$, such that, for all i, j, $d^i d^i = 0$ and $d^i d^j = d^j d^i$. The endomorphism d^i is called the i-th differential of A_*.

(iii) A *(co)chain morphism between two k-iterated (co)chain complexes* is a morphism of k-graded objects commuting with all the differentials.

Let $\mathcal{I}^k$ be the category of k-iterated cochain complexes and let $\mathcal{I}_k$ be the category of k-iterated chain complexes.

Remark 1.2.3. If A^* is a k-iterated cochain complex and $\boldsymbol{i}$ is a multi-index of length $k-1$, then $A^{s_l^*(\boldsymbol{i})}$ is a cochain complex. In this way, if P is a property of cochain complexes, we will say that A^* *satisfies the property P in the l-th direction*, if, for all multi-indices $\boldsymbol{i}$ of length $k-1$, the cochain complex $A^{s_l^*(\boldsymbol{i})}$ satisfies P.

For instance, let A^* be a cochain complex. The complex A^* has *length l* if $A^n = 0$ for all $n < 0$ and $n > l$, and $A^l \neq 0$. Then, we say that a k-iterated cochain complex A^* has lengths $l_1, \ldots, l_k$, if for all multi-indices $\boldsymbol{i}$ of length $k-1$, the cochain complex $A^{s_j^*(\boldsymbol{i})}$ has length l_j.

Simple complex associated to an iterated complex. We describe here the functor "*associated simple complex*":

$$s(-) : \mathcal{I}^k \to \mathcal{I}^1.$$

Definition 1.2.4. Let $(A^*, d_1, \ldots, d_k)$ be a k-iterated cochain complex. The *simple complex* of A^* is the cochain complex $s(A)^*$ whose graded groups are

$$s(A)^n = \bigoplus_{|\boldsymbol{j}|=n} A^{\boldsymbol{j}},$$

and whose differential $s(A)^n \xrightarrow{d_s} s(A)^{n+1}$ is defined by, for every $a_{\boldsymbol{j}} \in A^{\boldsymbol{j}}$ with $|\boldsymbol{j}| = n$,

$$d_s(a_{\boldsymbol{j}}) = \sum_{l=1}^{k} (-1)^{|\boldsymbol{j}|_{l-1}} d_l(a_{\boldsymbol{j}}) \in s(A)^{n+1}.$$

Here one understands that for $l = 1$, the sign in the sum is $+$.

The elements $a \in s(A)^n$ with $d_s(a) = 0$ are called *cycles*. If $f : A^* \to B^*$ is a morphism of k-iterated cochain complexes, there is an induced morphism between the associated simple complexes

$$f : s(A)^* \to s(B)^*.$$

Therefore, there is a functor

$$\mathcal{I}^k \xrightarrow{s(-)} \mathcal{I}^1.$$

Example 1.2.5. If A^{**} is a 2-iterated cochain complex, the associated simple complex is the cochain complex whose n-th graded group is

$$s(A)^n := \bigoplus_{n_1+n_2=n} A^{n_1,n_2},$$

and whose differential is

$$\begin{array}{ccc} A^{n_1,n_2} & \xrightarrow{d_s} & A^{n_1+1,n_2} \oplus A^{n_1,n_2+1} \\ a & \mapsto & d_1(a) + (-1)^{n_1} d_2(a). \end{array}$$

Example 1.2.6 (Tensor product)**.** Assume that in $\mathcal{P}$ there is a notion of tensor product. In our applications, $\mathcal{P}$ will be the category of abelian groups or the category of locally free sheaves on a scheme. Let (A^*, d_A) and (B^*, d_B) be two cochain complexes. The tensor product $(A \otimes B)^*$ is the 2-iterated cochain complex with

$$(A \otimes B)^{n,m} = A^n \otimes B^m,$$

and differentials $(d_A \otimes id_B, id_A \otimes d_B)$. By abuse of notation, the associated simple complex will also be denoted by $(A \otimes B)^*$.

Observe that the definition of the simple of a k-iterated complex A^*, depends on the fixed order of the directions of A^*. The proof of the following lemma is an easy exercise.

Lemma 1.2.7. *Let (A^*, d_1, d_2) be a 2-iterated cochain complex and consider (A_t^*, d_1^t, d_2^t) the 2-iterated cochain complex with $A_t^{n,m} = A^{m,n}$ and $d_1^t = d_2$ and $d_2^t = d_1$. Then, the morphism*

$$\begin{array}{ccc} s(A)^* & \to & s(A_t)^* \\ a \in A^{n,m} & \mapsto & (-1)^{nm} a \in A_t^{m,n}, \end{array}$$

is an isomorphism of complexes.

This lemma can be easily generalized to arbitrary k-iterated cochain complexes.

Lemma 1.2.8. *Let (A^*, d_A) be a cochain complex and consider the cochain complex $A_-^* = (A^*, -d_A)$. Then, the morphism*

$$A^n \xrightarrow{(-1)^n id_A} A_-^n$$

is an isomorphism of cochain complexes.

For the rest of this section, we assume that $\mathcal{P}$ is an abelian category.

Quasi-isomorphisms and cochain homotopies. We recall briefly the definition of a quasi-isomorphism and the homotopy equivalence relation.

Definition 1.2.9. Let (A^*, d_A) and (B^*, d_B) be two cochain complexes in $\mathcal{P}$.

(i) Let $f : A^* \to B^*$ be a cochain morphism. Then, f is a *quasi-isomorphism* if the induced morphisms
$$H^*(f) : H^n(A) \to H^n(B)$$
are isomorphisms for all n.

(ii) Two cochain morphisms $f, g : A^* \to B^*$ are said to be *quasi-isomorphic*, if $H^*(f) = H^*(g)$.

(iii) Let $f, g : A^* \to B^*$ be two cochain morphisms. We say that f and g are *homotopy equivalent* (written $f \simeq g$), if for all n, there exists a morphism
$$h^n : A^n \to B^{n-1},$$
such that $f - g = d_B h_n + h_{n+1} d_A$.

(iv) A morphism $f : A^* \to B^*$ is a *homotopy equivalence*, if there exists a morphism $g : B^* \to A^*$, such that $fg \simeq id_B$, and $gf \simeq id_A$.

One can easily check that if f is a homotopy equivalence, then f is a quasi-isomorphism.

Translation and truncation of a cochain complex. Let (A^*, d_A) be a cochain complex in $\mathcal{P}$.

- Let $m \in \mathbb{Z}$. The *translation of A^* by m*, $A[m]^*$, is the cochain complex with
$$A[m]^n := A^{n+m},$$
and differential $(-1)^m d_A$. In general, if $(A^*, d_1, \ldots, d_k)$ is a k-iterated cochain complex and $\boldsymbol{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, one defines the translation of A^* by $\boldsymbol{m}$ by setting
$$A[\boldsymbol{m}]^{\boldsymbol{n}} = A^{\boldsymbol{n}+\boldsymbol{m}}, \quad \text{and} \quad d_i^{A[\boldsymbol{m}]} = (-1)^{m_i} d_i.$$

- Let $n \in \mathbb{Z}$. The *canonical truncation of A^* at degree n*, $\tau_{\leq n} A^*$, is the cochain complex given by
$$(\tau_{\leq n} A)^r = \begin{cases} A^r & r < n, \\ \ker(d_A : A^n \to A^{n+1}) & r = n, \\ 0 & r > n, \end{cases}$$
and differential induced by d_A. Observe that it follows from the definition that
$$H^r(\tau_{\leq n} A^*) = \begin{cases} H^r(A^*) & r \leq n, \\ 0 & r > n. \end{cases}$$

- Let $n \in \mathbb{N}$. The *bête truncation of A^* at degree n*, $\sigma_{\leq n}A^*$, is the cochain complex given by
$$(\sigma_{\leq n}A)^r = \begin{cases} A^r & r \leq n, \\ 0 & r > n, \end{cases}$$
and differential induced by d_A. Observe that it follows from the definition that
$$H^r(\sigma_{\leq n}A^*) = \begin{cases} H^r(A^*) & r < n, \\ A^n/\operatorname{im} d_A & r = n, \\ 0 & r > n. \end{cases}$$

The canonical and bête truncations $\tau_{\geq n}$ and $\sigma_{\geq n}$ are defined analogously.

Lemma 1.2.10. *Let A^{**} and B^{**} be two 2-iterated cochain complexes located on the first or third quadrant. Let $f : A^{**} \to B^{**}$ be a cochain morphism.*

(i) If $f^n : A^{n*} \to B^{n*}$ is a quasi-isomorphism of cochain complexes for all n, or,

(ii) If $f^m : A^{*m} \to B^{*m}$ is a quasi-isomorphism of cochain complexes for all m,

then the induced morphism
$$f : s(A)^* \to s(B)^*$$
is a quasi-isomorphism.

Proof. It follows from a spectral sequence argument. □

The simple of a morphism. Let (A^*, d_A) and (B^*, d_B) be two cochain complexes in $\mathcal{P}$. Then, any cochain morphism $f : A^* \to B^*$ can be viewed as a 2-iterated cochain complex with differentials $(f, (d_A, d_B))$. The simple of f is then defined as the simple associated to this 2-iterated complex.

Definition 1.2.11. Let (A^*, d_A) and (B^*, d_B) be two cochain complexes and let $f : A^* \to B^*$ be a cochain morphism. The *simple of f* is the cochain complex $s(f)^*$ with

- $s(f)^n = A^n \oplus B^{n-1}$.
- If $(a, b) \in s(f)^n$, then $d_s(a, b) = (d_A a, f(a) - d_B b)$.

This complex is the cone of a morphism shifted by 1.
The morphisms
$$\begin{array}{rcl} B[-1]^n & \to & s(f)^n \\ b & \mapsto & (0, b), \end{array} \qquad \begin{array}{rcl} s(f)^n & \to & A^n \\ (a, b) & \mapsto & a, \end{array}$$
fit in a short exact sequence
$$0 \to B[-1]^* \to s(f)^* \to A^* \to 0,$$
which yields to a long exact sequence
$$\cdots \to H^n(s(f)^*) \to H^n(A^*) \xrightarrow{f} H^n(B^*) \to H^{n+1}(s(f)^*) \to \cdots \tag{1.2.12}$$

Proposition 1.2.13. *Let $f : A^* \to B^*$ be a surjective cochain morphism. Then, there is a quasi-isomorphism*

$$\begin{array}{ccc} \ker f & \xrightarrow{i} & s(-f)^* \\ x & \mapsto & (x,0). \end{array}$$

Proof. It follows from the long exact sequences associated to the short exact sequences

$$0 \to B[-1]^* \to s(-f)^* \to A^* \to 0, \quad 0 \to \ker f \to A^* \xrightarrow{f} B^* \to 0$$

and the five lemma. □

Remark 1.2.14. Let

$$\varepsilon : A_0^* \xrightarrow{f_0} A_1^* \xrightarrow{f_1} \dots \xrightarrow{f_{k-1}} A_k^*$$

be a sequence of cochain morphisms, i.e. $f_i \circ f_{i-1} = 0$ for all i. Then, the *simple complex associated to* ε is defined analogously. Specifically,

$$s(\varepsilon)^n = A_0^n \oplus A_1^{n-1} \oplus \cdots \oplus A_k^{n-k},$$

and for all $(a_0, \dots, a_k) \in s(\varepsilon)^n$,

$$d_s(a_0, \dots, a_k) = (d_{A_0}(a_0), f_0(a_0) - d_{A_1}(a_1), \dots, f_{k-1}(a_{k-1}) + (-1)^k d_{A_k}(a_k)).$$

Chain complexes versus cochain complexes. For every k, there is an isomorphism of categories

$$\mathcal{I}_k \xrightarrow{\cong} \mathcal{I}^k. \tag{1.2.15}$$

For simplicity, we give the details only for the case $k = 1$.

Definition 1.2.16. (i) Let (A_*, d) be a chain complex. The *cochain complex associated to* A_*, denoted by A^*, is given by

$$A^n := A_{-n}, \qquad d^n = d_{-n}, \qquad n \in \mathbb{Z}.$$

(ii) Let (A^*, d) be a cochain complex. The *chain complex associated to* A^*, denoted by A_*, is given by

$$A_n := A^{-n}, \qquad d_n = d^{-n}, \qquad n \in \mathbb{Z}.$$

These assignments give an isomorphism between the category of chain complexes and the category of cochain complexes. Under this isomorphism, the concepts already defined for cochain complexes can be transferred to the chain complex case. However, a few remarks need to be made.

(i) *Translation of a chain complex.* Let (A_*, d) be a chain complex and let $m \in \mathbb{Z}$. The translation of A_* by m is the chain complex with

$$A[m]_n := A_{n-m}, \quad \text{and} \quad d^{A[m]} = (-1)^m d.$$

Then, the definition is compatible with the isomorphism of categories of (1.2.15), i.e. there is a commutative diagram of functors

$$\begin{tikzcd} \mathcal{I}_1 \arrow[r] \arrow[d, "{[m]}"'] & \mathcal{I}^1 \arrow[d, "{[m]}"] \\ \mathcal{I}_1 \arrow[r] & \mathcal{I}^1. \end{tikzcd}$$

(ii) *Simple of a chain morphism.* Given $A_* \xrightarrow{f} B_*$ a morphism of chain complexes, the associated simple complex is the chain complex given by

- $s(f)_n = A_n \oplus B_{n+1}$,
- For $(a, b) \in s(f)_n$, $d_s(a, b) = (d_A a, f(a) - d_B b)$.

Let f_* be a chain morphism. Then, the cochain complex associated to the simple of f_* agrees with the simple of the cochain morphism f^* associated to f_*.

1.2.3 Simplicial abelian groups and chain complexes

Recall that **Ab** denotes the category of abelian groups. Let $\mathcal{I} = \mathcal{I}^1$ be the category of chain complexes defined in the previous section. Denote by $\mathcal{I}_+$ the subcategory consisting of the chain complexes which are zero in negative degrees.

The Dold-Puppe correspondence. Most of the results of this section are due to Dold and Puppe (see [17]). However, our treatment basically follows [29], §$III.2$. The Dold-Puppe correspondence consists of two functors

$$N : s\mathbf{Ab} \to \mathcal{I}_+ \quad \text{and} \quad \mathcal{K} : \mathcal{I}_+ \to s\mathbf{Ab},$$

which are equivalences of categories.

Definition 1.2.17 (The functor N)**.** Let $X.$ be a simplicial abelian group. The *normalized chain complex* associated to $X.$, NX_*, is the chain complex with

- $NX_n := \begin{cases} \bigcap_{i=0}^{n-1} \ker(\partial_i : X_n \to X_{n-1}) & \text{if } n \geq 1, \\ X_0 & \text{if } n = 0. \end{cases}$
- The differential $d : NX_n \to NX_{n-1}$ is $d := \partial_n$.

The simplicial identities guarantee that NX_* is a chain complex. Moreover, given a simplicial map $f : X. \to Y.$, the restriction map

$$Nf : NX_n \to NY_n,$$

is a chain morphism. Hence, N is an additive functor from the category of simplicial abelian groups to the category of chain complexes which are zero in negative degrees.

Definition 1.2.18 (The functor $\mathcal{K}$)**.** Let (A_*, d) be a chain complex in $\mathcal{I}_+$. The *associated simplicial abelian group*, $\mathcal{K}.(A)$, is given as follows. For every $n \geq 0$,

$$\mathcal{K}_n(A) := \bigoplus_{q \leq n} \bigoplus_{\eta} A_q$$

where η runs over all surjective monotonic maps $\eta : \mathbf{n} \twoheadrightarrow \mathbf{q}$. For every monotonic map $\alpha : \mathbf{m} \to \mathbf{n}$, the map

$$\alpha : \mathcal{K}_n(A) \to \mathcal{K}_m(A)$$

is defined, in the component corresponding to $\eta : \mathbf{n} \twoheadrightarrow \mathbf{q}$ as follows. Let $\eta' : \mathbf{m} \twoheadrightarrow \mathbf{p}$, and $\epsilon : \mathbf{p} \hookrightarrow \mathbf{q}$, for some p, be the only monotonic maps in which $\eta \circ \alpha = \epsilon \circ \eta'$. Then,

$$\alpha|_{A_q} = \begin{cases} id_{A_q} & \text{if } p = q, \\ d & \text{if } p = q - 1 \text{ and } \epsilon(0) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, if $f : A_* \to B_*$ is a chain morphism, there is an induced simplicial map

$$\mathcal{K}(f) : \mathcal{K}.(A) \to \mathcal{K}.(B).$$

The following result is proved in [17] §3.

Proposition 1.2.19. *The functor $N\mathcal{K} : \mathcal{I}_+ \to \mathcal{I}_+$ is the identity. The functor $\mathcal{K}N$: $s\mathbf{Ab} \to s\mathbf{Ab}$ is naturally equivalent to the identity functor. Therefore, the categories $s\mathbf{Ab}$ and $\mathcal{I}_+$ are equivalent.*

Remark 1.2.20. Let A^* be a cochain complex concentrated in non-positive degrees. Then, the associated chain complex A_* is concentrated in non-negative degrees and hence the simplicial abelian group $\mathcal{K}.(A)$ is defined.

It follows that the category $\mathcal{I}_+$ inherits a closed model structure from the category of simplicial abelian groups. In the propositions that follow, the class of weak equivalences, fibrations and cofibrations of $\mathcal{I}_+$ are described.

Proposition 1.2.21. *Let $X.$ be a simplicial abelian group and let $x \in X_0$. Then, for all $n \geq 0$,*

$$\pi_n(X., x) \cong H_n(NX_*).$$

Proof. This is proved in [29], § III.2. $\square$

Proposition 1.2.22. *(i) A simplicial map $f : X. \to Y.$ is a weak equivalence if and only if it induces a quasi-isomorphism $Nf : NX_* \to NY_*$.*

(ii) A chain map $f : A_* \to B_*$ is a quasi-isomorphism if and only if it induces a weak equivalence $\mathcal{K}(f) : \mathcal{K}.(A) \to \mathcal{K}.(B)$.

(iii) A simplicial map $f : X. \to Y.$ is a fibration if and only if the morphisms $f : NX_n \to NY_n$ are surjective for all $n \geq 1$.

Proof. The first two statements follow from proposition 1.1.32 and the equivalence of the functors N and $\mathcal{K}$. For the third statement see [29], § III Lemma 2.11. □

We deduce that the category $\mathcal{I}_+$ has a closed model structure with:

- *Weak equivalences* are quasi-isomorphisms.
- *Fibrations* are the chain maps which are surjective in degree $n \geq 1$.
- *Cofibrations* are the chain maps which have the left lifting property with respect to all trivial fibrations.

The Normalized chain complex and the Moore complex.

Definition 1.2.23. Let $X.$ be a simplicial abelian group. The *Moore complex* of $X.$, X_*, is the chain complex whose n-th graded group is X_n and whose differential $d : X_n \to X_{n-1}$ is given by $d = \sum_{i=0}^{n}(-1)^i\partial_i$.

At this point, we have introduced two chain complexes associated to a simplicial abelian group: the Moore complex and the normalized chain complex. It is natural to ask whether there is any relation between them. The next proposition tells us that they are in fact homotopy equivalent.

Proposition 1.2.24. *Let $X.$ be a simplicial abelian group and let $i : NX_* \to X_*$ be the natural inclusion of chain complexes. Then, i is a homotopy equivalence. In particular,*

$$H_*(NX_*) \overset{i_*}{\cong} H_*(X_*).$$

Moreover, this equivalence is natural with respect to the simplicial abelian group $X.$.

Proof. See [29], § III Theorem 2.4. □

Let $DX_n \subset X_n$ be the subgroup consisting of the degenerate simplices. It follows from the simplicial identities, that $d(DX_n) \subset d(DX_{n-1})$. Then, there is an induced differential

$$d : X_n/DX_n \to X_{n-1}/DX_{n-1}.$$

Let $\widetilde{X}_*$ be the resulting chain complex X_*/DX_*.

Proposition 1.2.25. *The composition*

$$NX_* \xrightarrow{i} X_* \xrightarrow{p} \widetilde{X}_*$$

is an isomorphism of chain complexes.

Proof. See [29], § III Theorem 2.1. □

Remark 1.2.26. Observe that if follows from propositions 1.2.24 and 1.2.25, that the complex of degenerate simplices DX_* is acyclic.

Remark 1.2.27. By example 1.1.14, to every simplicial set $X.$, there is a simplicial abelian group $\mathbb{Z}X.$ associated. The Moore complex of $\mathbb{Z}X.$ gives a chain complex $\mathbb{Z}X_*$ associated to $X.$.

Remark 1.2.28. Instead of taking homological chain complexes and simplicial groups, one could have considered cohomological chain complexes and cosimplicial groups. Then, we can define the Moore complex associated to a cosimplicial group in the same way, just with the arrows reversed.

The Hurewicz morphism. Let T be a topological space. Then, for $m \geq 1$ there is a morphism (*the Hurewicz morphism*)

$$h_m : \pi_m(T) \to H_m(T)$$

defined as follows. Choose a generator $[a]$ of $H_m(S^m) \cong \mathbb{Z}$. Since every $x \in \pi_m(T)$ corresponds to a homotopy class of based morphisms $x : S^m \to T$, we define

$$h_m(x) = x_*(a) \in H_m(T).$$

The definition is independent of the choice of the generator $[a]$. There is an analogous morphism for simplicial sets. Namely, let $X.$ be a simplicial set and consider the simplicial map

$$X. \xrightarrow{h} \mathbb{Z}X.$$

which sends $x \in X_n$ to $x \in \mathbb{Z}X_n$. Let $x \in X_0$ be a base point. Then, the induced morphism

$$h_* : \pi_n(X., x) \to \pi_n(\mathbb{Z}X., x) \cong H_n(\mathbb{Z}X_*) \tag{1.2.29}$$

is called the *Hurewicz morphism.*

The homotopy fiber and the simple. The following is a well-known fact. We give a sketch of the proof for completeness.

Proposition 1.2.30. *Let (A_*, d_A), (B_*, d_B) be two chain complexes. Let $f : A_* \to B_*$ be a chain morphism and let $\mathcal{K}(f) : \mathcal{K}.(A) \to \mathcal{K}.(B)$ be the induced morphism. Let* $\mathrm{HoFib}(f)$ *denote the homotopy fiber of the topological realization of $\mathcal{K}(f)$. Then, for every $n \geq 1$, there is an isomorphism*

$$\pi_n(\mathrm{HoFib}(f)) \to H_n(s_*(f))$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\pi_{n+1}(\mathcal{K}.(B)) \arrow[r] \arrow[d, "\cong"'] & \pi_n(\mathrm{HoFib}(f)) \arrow[r] \arrow[d] & \pi_n(\mathcal{K}.(A)) \arrow[d, "\cong"'] \\
H_{n+1}(B_*) \arrow[r] & H_n(s(f)) \arrow[r] & H_n(A_*).
\end{tikzcd}$$

Proof. We sketch here the proof of the proposition. The details are left to the reader. Consider the diagram

$$\begin{tikzcd}
\pi_{n+1}(\mathcal{K}.(A)) \arrow[r] \arrow[d, "\cong"'] & \pi_{n+1}(\mathcal{K}.(B)) \arrow[r] \arrow[d, "\cong"'] & \pi_n(\mathrm{HoFib}(f)) \arrow[r] & \pi_n(\mathcal{K}.(A)) \arrow[r] \arrow[d, "\cong"'] & \pi_n(\mathcal{K}.(B)) \arrow[d, "\cong"'] \\
H_{n+1}(A_*) \arrow[r] & H_{n+1}(B_*) \arrow[r] & H_n(s(f)) \arrow[r] & H_n(A_*) \arrow[r] & H_n(B_*).
\end{tikzcd}$$

By the five lemma, we only have to construct a morphism

$$\pi_n(\mathrm{HoFib}(f)) \to H_n(s(f))$$

making the whole diagram commutative.

Let $F.$ be the cartesian product of the following diagram

$$\begin{tikzcd}
& \underline{\mathrm{Hom}}(\Delta^1_., \mathcal{K}.(B)) \arrow[d, "i_1"] \\
\mathcal{K}.(A) \arrow[r, "\mathcal{K}(f)"'] & \mathcal{K}.(B),
\end{tikzcd}$$

where i_1 sends a map $g \in \underline{\mathrm{Hom}}(\Delta^1_., \mathcal{K}.(B))$ to $g(1)$. Let $\pi_1 : F. \to \mathcal{K}.(A)$ and $\pi_2 : F. \to \underline{\mathrm{Hom}}(\Delta^1_., \mathcal{K}.(B))$ denote the natural projections.

The homotopy fiber of $\mathcal{K}(f) : |\mathcal{K}.(A)| \to |\mathcal{K}.(B)|$ is homotopically equivalent to the topological realization of $F.$. Hence,

$$\pi_n(\mathrm{HoFib}(f)) \cong \pi_n(F.).$$

Since F is a fibrant simplicial set, $\pi_n(F.)$ consists of homotopy classes of pointed simplicial maps

$$\Delta^n_./\partial\Delta^n_. \to F.$$

Observe that any map $f : \Delta^n_\cdot/\partial\Delta^n_\cdot \to \underline{\mathrm{Hom}}(\Delta^1_\cdot, \mathcal{K}.(B))$ induces a map

$$f : \Delta^n_\cdot \times \Delta^1_\cdot/(\partial\Delta^n_\cdot \times \Delta^1_\cdot \cup \Delta^n_\cdot \times \{0\}) \to \mathcal{K}.(B).$$

Let s_{n+1} denote the fundamental class of $\Delta^n_\cdot \times \Delta^1_\cdot$ and σ_n denote the fundamental class of $\Delta^n_\cdot$. If C_* is any chain complex and $\mathcal{K}_*(C)$ is the chain complex associated to the simplicial abelian group $\mathcal{K}.(C)$, let

$$\phi : H_n(\mathcal{K}_*(C)) \to H_n(C_*)$$

be the morphism induced by the natural morphism $\mathcal{K}_*(C) \to C_*$.

Then, we define a morphism

$$\begin{array}{rcl} \pi_n(\mathrm{HoFib}(f)) \cong \pi_n(F.) & \to & H_n(s(f)) \\ [g] & \mapsto & (\phi((\pi_1)_*(g)(\sigma_n)), \phi((\pi_2)_*(g)(s_{n+1}))). \end{array}$$

One checks that this is a well-defined morphism of groups and commutes with the isomorphisms above. □

1.2.4 Cubical objects and chain complexes

Although it is common in (co)homology theories to base the constructions on a simplicial setting, most theories can be rewritten using a cubical setting. The advantage of the cubical constructions is that while the product of two simplices is not a simplex, the product of two cubes is a cube. This makes cubical theory more suitable to define products. There are many examples of theories based on the use of cubical objects. For instance, we find them on the singular homology theory described by Massey in [45], on the theory of cubical hyperresolutions of a scheme described by Guillen, Navarro, Pascual-Gainza and Puerta in [33], or on the cubical higher Chow groups given by Levine in [41].

In this section we discuss general facts about cubical abelian groups and their relation to chain complexes. This relation is analogous to the relationship between simplicial abelian groups and chain complexes discussed in the previous section.

Let $\langle \mathbf{1} \rangle$ be the category whose objects are the finite sets $\{0,1\}^n$, for $n \geq 0$ and morphisms are morphism expressed as a composition of the following maps:

- *Cofaces*: for every $i = 1, \dots, n+1$ and $j = 0, 1$, let $\delta^i_j : \{0,1\}^n \to \{0,1\}^{n+1}$, be defined as
$$\delta^i_j(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, j, x_i, \dots, x_n).$$
- *Codegeneracies:* for every $i = 1, \dots, n$, let $\sigma^i : \{0,1\}^n \to \{0,1\}^{n-1}$ be defined as
$$\sigma^i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Definition 1.2.31. A *cubical abelian group $C.$* is a functor

$$C. : \langle \mathbf{1} \rangle^{op} \to \mathbf{Ab}.$$

It follows that a cubical abelian group $C.$ consists of a collection of abelian groups $\{C_n\}_{n\geq 0}$ together with maps

- *Faces:* $\delta_i^j : C_n \to C_{n-1}$, for $i = 1, \ldots, n$ and $j = 0, 1$.
- *Degeneracies:* $\sigma_i : C_n \to C_{n+1}$, for $i = 1, \ldots, n+1$,

satisfying the *cubical identities*: for all $l, k \in \{0, 1\}$,

$$\begin{array}{rcll} \delta_i^l \delta_j^k & = & \delta_{j-1}^k \delta_i^l & \text{for} \quad i < j, \\ \delta_i^l \sigma_j & = & \begin{cases} \sigma_{j-1}\delta_i^l & \\ id & \\ \sigma_j \delta_{i-1}^l & \end{cases} & \begin{array}{l} \text{for} \quad i < j, \\ \text{for} \quad i = j, \\ \text{for} \quad i > j, \end{array} \\ \sigma_i \sigma_j & = & \sigma_{j+1}\sigma_i & \text{for} \quad i \leq j. \end{array} \tag{1.2.32}$$

Definition 1.2.33. Let $C.$ be a cubical abelian group. The *associated chain complex* C_*, is the chain complex whose n-th graded piece is C_n and whose differential $\delta : C_n \to C_{n-1}$ is given by

$$\delta = \sum_{i=1}^{n} \sum_{j=0,1} (-1)^{i+j} \delta_i^j.$$

Let $D_n \subset C_n$ be the subgroup of *degenerate elements* of C_n, i.e. the elements which are in the image of σ_i for some i. By the cubical identities, we have an inclusion $\delta(D_n) \subseteq D_n$. Therefore, the quotient

$$\widetilde{C}_* := C_*/D_*$$

is a chain complex, whose differential is induced by δ. Moreover, there is an epimorphism

$$C_* \twoheadrightarrow \widetilde{C}_*.$$

Definition 1.2.34. Let $C.$ be a cubical abelian group and fix $l = 0$ or 1. The *normalized chain complex* associated to $C.$, $N^l C_*$, is the chain complex whose n-th graded group is

$$N^l C_n := \bigcap_{i=1}^{n} \ker \delta_i^l,$$

and whose differential is the one induced by the inclusion $N^l C_n \subset C_n$, i.e.

$$\delta = \sum_{i=1}^{n} (-1)^{i+l'} \delta_i^{l'}, \qquad \text{where} \quad \begin{cases} l' = 1 & \text{if } l = 0 \\ l' = 0 & \text{otherwise.} \end{cases}$$

Observe that by the cubical identities, δ induces a differential on $N^l C_*$. The following lemma is well known.

Lemma 1.2.35. *Let $C.$ be a cubical abelian group. There is a decomposition of chain complexes*

$$C_* = N^l C_* \oplus D_*.$$

Proof. Let $x \in C_n$ such that $\delta^l_r(x) = \cdots = \delta^l_n(x) = 0$. Then, consider $y = x - \sigma_{r-1}\delta^l_{r-1}(x)$. By the cubical identities,

$$\delta^l_{r-1}(y) = \delta^l_{r-1}(x) - \delta^l_{r-1}\sigma_{r-1}\delta^l_{r-1}(x) = \delta^l_{r-1}(x) - \delta^l_{r-1}(x) = 0.$$

By induction on r, we see that any $x \in C_n$ can be decomposed as $x = y + z$ with $y \in N^l C_n$ and $z \in D_n$.

All that remains is to see that the decomposition is direct. Assume that $y = \sigma_i(x) \in N^l C_n$, for some i. Then, $0 = \delta^l_i \sigma_i(x) = x$ and hence $y = 0$. We proceed by induction. Assume that if $y = \sum_{k=1}^{r-1} \sigma_k(x_k)$ and y belongs to $N^l C_n$, then $y = 0$. Let y be of the form $y = \sum_{k=1}^{r} \sigma_k(x_k)$, for some x_k, and assume that $y \in N^l C_n$. Then, applying δ^l_r we obtain

$$0 = \delta^l_r(y) = \sum_{k=1}^{r} \delta^l_r \sigma_k(x_k) = \sum_{k=1}^{r-1} \delta^l_r \sigma_k(x_k) + x_k.$$

Hence, $x_k = -\sum_{k=1}^{r-1} \delta^l_r \sigma_k(x_k)$ and so

$$y = \sum_{k=1}^{r-1} \sigma_k(x_k) - \sum_{k=1}^{r-1} \sigma_r \delta^l_r \sigma_k(x_k) = \sum_{k=1}^{r-1} \sigma_k(x_k - \delta_r \sigma_{r-1}(x_k)).$$

By the induction hypothesis, $y = 0$. $\square$

As a consequence, we obtain the following lemma.

Lemma 1.2.36. *The composition*

$$\phi : N^l C_* \hookrightarrow C_* \twoheadrightarrow \widetilde{C}_*$$

is an isomorphism of chain complexes.

Therefore, the subcomplexes

$$\bigcap_{i=1}^{*} \ker \delta^0_i, \ \bigcap_{i=1}^{*} \ker \delta^1_i \subset C_*$$

are both isomorphic to $\widetilde{C}_*$.

From now on, we fix the normalized complex to be constructed with $l = 1$, i.e.

$$NC_n := N^l C_n = \bigcap_{i=1}^{n} \ker \delta^1_i, \qquad \text{and} \qquad \delta = \sum_{i=1}^{n} (-1)^i \delta^0_i. \tag{1.2.37}$$

Example 1.2.38. Let C. be an abelian group. Then, the associated *trivial cubical abelian group* is constructed as follows. For every n, set $C_n = C$, and for every i, l, set $\delta^l_i = \sigma_i = id$.

1.3 Algebraic K-theory of exact categories

In this section we give the definition of the algebraic K-groups of a small exact category, following Quillen in [48]. We also describe the construction of K-groups of an exact category due to Waldhausen in [59]. In fact, Waldhausen gave a generalization of the definition of K-groups of exact categories to the *Waldhausen categories*. These are categories which have a zero object and are equipped with a family of morphisms called *cofibrations* and a family of morphisms called *weak equivalences* (see [59] for details). Every exact category becomes a Waldhausen category in a natural way. Therefore, for the sake of simplicity, and since we are only interested in the application to exact categories, we restrict our discussion of the Waldhausen construction to exact categories.

We proceed with the definition of the chain complex of cubes, whose homology groups with rational coefficients are isomorphic to the algebraic K-groups. This isomorphism is proved by McCarthy in [47].

In the second part of this section, we restrict ourselves to the exact category of locally free sheaves on a scheme X, and to the category of coherent sheaves.

Finally, we review the theory of λ-rings, for later use.

1.3.1 Exact categories and the Q-construction

Exact categories.

Definition 1.3.1. An *exact category* is an additive category $\mathcal{P}$ equipped with a class $\mathbf{E}$ of sequences called the *short exact sequences* of $\mathcal{P}$:

$$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0, \tag{1.3.2}$$

such that the following axioms are satisfied:

EX1. Any sequence of $\mathcal{P}$ isomorphic to a sequence in $\mathbf{E}$ is in $\mathbf{E}$. For any pair of objects M', M'' in $\mathcal{P}$, the sequence

$$0 \to M' \xrightarrow{(id,0)} M' \oplus M'' \xrightarrow{pr_2} M'' \to 0$$

is in $\mathbf{E}$. For any sequence of the type of (1.3.2), i is a kernel for j and j is a cokernel for i. We call i an *admissible monomorphism* and it is denoted by $M' \rightarrowtail M$. We call j an *admissible epimorphism* and it is denoted by $M \twoheadrightarrow M''$.

EX2. The class of admissible epimorphisms (resp. monomorphisms) is closed under composition and under base-change (resp. cobase-change) by arbitrary maps in $\mathcal{P}$.

EX3. Let $M \to M''$ be a morphism possessing a kernel in $\mathcal{P}$. Then, if there is a morphism $N \to M$ in $\mathcal{P}$ such that the composition $N \to M \to M''$ is an admissible epimorphism, then the morphism $M \to M''$ is an admissible epimorphism. Dually for admissible monomorphisms.

An *exact functor* $F : \mathcal{P} \to \mathcal{P}'$ between exact categories is an additive functor carrying exact sequences of $\mathcal{P}$ into exact sequences of $\mathcal{P}'$.

Example 1.3.3. For details about the following examples of exact categories, see [48], §2.

(1) Every abelian category is an exact category, with the usual class of short exact sequences.

(2) Suppose that $\mathcal{P}$ is an additive category embedded as a full (additive) subcategory of an abelian category $\mathcal{A}$ such that if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of $\mathcal{A}$ and M', M'' are objects of $\mathcal{P}$, then M is isomorphic to an object in $\mathcal{P}$. Let $\mathbf{E}$ be the class of the exact sequences of $\mathcal{A}$ whose terms lie in $\mathcal{P}$. Then, the category $\mathcal{P}$ with the class $\mathbf{E}$ is an exact category. In fact, any exact category can be realized as a category of this type.

(3) Any additive category can be made into an exact category in at least one way, taking $\mathbf{E}$ to be the family of split exact sequences.

Quillen Q-construction. Let $\mathcal{P}$ be an exact category. A new "category" $Q\mathcal{P}$ is defined as follows:

- $\operatorname{Obj} Q\mathcal{P} = \operatorname{Obj} \mathcal{P}$.
- Let $C, D \in \operatorname{Obj} \mathcal{P}$ and consider all the diagrams in $\mathcal{P}$ of the form

$$C \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} D \tag{1.3.4}$$

where j is an admissible epimorphism and i an admissible monomorphism. Two diagrams $C \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} D$ and $C \overset{j'}{\twoheadleftarrow} N' \overset{i'}{\rightarrowtail} D$ are isomorphic if there exists an isomorphism $q : N \to N'$ in $\mathcal{P}$ such that $j = j' \circ q$ and $i = i' \circ q$. Then, a morphism between the objects C and D in $Q\mathcal{P}$ is an isomorphism class of diagrams of this type.

- Let $C \overset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} D$ and $D \overset{j'}{\twoheadleftarrow} M \overset{i'}{\rightarrowtail} F$ be two morphisms in $Q\mathcal{P}$ and consider the diagram

$$\begin{tikzcd}
C & & \\
N \arrow[u, two heads, "j"] \arrow[r, tail, "i"] & D & \\
N \times_D M \arrow[u, "pr_1"] \arrow[r, "pr_2"] & M \arrow[u, two heads, "j'"] \arrow[r, tail, "i'"] & F
\end{tikzcd}$$

By axiom **EX2**, $j \circ pr_1$ and $i' \circ pr_2$ are, respectively, an admissible epimorphism and an admissible monomorphism. Then, the composition diagram is given by

$$C \xleftarrow{j \circ pr_1} N \times_D M \xrightarrow{pr_2 \circ i'} F.$$

One can check that the isomorphism class of this diagram depends only on the isomorphism class of the data diagrams. Moreover, the composition is associative.

Observe that the Q-construction is functorial on the category $\mathcal{P}$.

Let $\mathcal{U}$ be a universe. If the diagrams of the form (1.3.4) form a $\mathcal{U}$-set (this is the case when $\mathcal{P}$ is a small $\mathcal{U}$-category), then $Q\mathcal{P}$ is a category, which is also $\mathcal{U}$-small. It follows that we can consider the classifying space of $Q\mathcal{P}$ (see example 1.1.17). So, from now on, $\mathcal{P}$ will be a **small exact category**, for a fixed universe $\mathcal{U}$.

Remark 1.3.5. Given an exact category $\mathcal{P}$ having a set of isomorphism classes of objects, one can always consider a small subcategory $\mathcal{P}'$ equivalent to $\mathcal{P}$ and then define $Q\mathcal{P}$ as $Q\mathcal{P}'$. This is defined up to equivalence of categories since, given two equivalent small exact categories $\mathcal{P}$ and $\mathcal{P}'$, $Q\mathcal{P}$ and $Q\mathcal{P}'$ are also equivalent.

Algebraic K-theory. Let $\mathcal{P}$ be an exact category. The *Grothendieck group* $K_0(\mathcal{P})$ is defined by

$$K_0(\mathcal{P}) := \mathcal{Z}/\mathcal{R}$$

where $\mathcal{Z}$ is the free abelian group on the objects of $\mathcal{P}$ and $\mathcal{R}$ is the subgroup generated by classes $[M] - [M'] - [M'']$ for each exact sequence in $\mathbf{E}$

$$0 \to M' \to M \to M'' \to 0.$$

Proposition 1.3.6. *Let $\mathcal{P}$ be a small exact category and let $0 \in \operatorname{Obj}\mathcal{P} = \operatorname{Obj} Q\mathcal{P}$ be a zero object (hence $\{0\}$ is a point of $BQ\mathcal{P}$). Then, there is a natural isomorphism*

$$\pi_1(BQ\mathcal{P}, \{0\}) \cong K_0(\mathcal{P}).$$

Proof. See [48], § 2 Theorem 1. □

Hence, it is natural to define the higher algebraic K-groups of a small exact category as follows.

Definition 1.3.7. Let $\mathcal{P}$ be a small exact category. Then, for every $i \geq 0$, the i-th K-group is defined as

$$K_m(\mathcal{P}) = \pi_{m+1}(BQ\mathcal{P}, \{0\}).$$

Remark 1.3.8. (i) Up to a unique isomorphism, the definition does not depend on the choice of the zero object in $\mathcal{P}$. Indeed, given another zero object $0'$, there is a canonical isomorphism $0 \to 0'$ in $\mathcal{P}$ which gives a canonical choice of a path joining $\{0\}$ and $\{0'\}$ in $BQ\mathcal{P}$.

(ii) Since the realization, the nerve and the Q-construction are functors of categories, given an exact functor $F : \mathcal{P} \to \mathcal{P}'$ between two small exact categories, there is an induced homomorphism

$$K_m(\mathcal{P}) \to K_m(\mathcal{P}'), \quad \text{for all } m.$$

1.3.2 Waldhausen K-theory

Let M_n be the category of morphisms of $[n]$, i.e.

$$\operatorname{Obj} M_n = \{(i,j) \in \mathbb{N}^2 |\ 0 \le i \le j \le n\},$$

and $\operatorname{Hom}_{M_n}((i,j),(k,l))$ contains a unique element if $i \le k$ and $j \le l$ and is empty otherwise.

Let $\mathcal{P}$ be a small exact category and fix a zero object 0 of $\mathcal{P}$. Let $\underline{S}_n(\mathcal{P})$ be the category of functors $E_{\cdot,\cdot} : M_n \to \mathcal{P}$, that satisfy

- $E_{i,i} = 0$ for all i.
- $E_{i,j} \to E_{i,k} \to E_{j,k}$ is a short exact sequence, for all $i \le j \le k$.

Denote by $S_n(\mathcal{P})$ the set of objects of $\underline{S}_n(\mathcal{P})$. Observe that an element of $S_n(\mathcal{P})$ is a sequence of admissible monomorphisms in $\mathcal{P}$,

$$0 = E_{0,0} \rightarrowtail E_{0,1} \rightarrowtail \cdots \rightarrowtail E_{0,n},$$

with a compatible choice of quotients

$$E_{i,j} \cong E_{0,j}/E_{0,i}, \qquad \text{for } 0 < i \le j \le n.$$

In particular,

$$S_0(\mathcal{P}) = \{0\},\ S_1(\mathcal{P}) = \operatorname{Obj}(\mathcal{P}),\ \text{and } S_2(\mathcal{P}) = \{\text{short exact sequences of } \mathcal{P}\}.$$

The assignment

$$\underline{S}.(\mathcal{P}) : [n] \to \underline{S}_n(\mathcal{P})$$

is a functor. Hence, $\underline{S}.(\mathcal{P})$ is a simplicial category and, in particular, $S.(\mathcal{P})$ is a simplicial set.

Proposition 1.3.9. *There is a homotopy equivalence*

$$S.(\mathcal{P}) \simeq N.Q\mathcal{P}.$$

Therefore, for all $m \geq 0$, there is an isomorphism

$$K_m(\mathcal{P}) = \pi_{m+1}(|S.(\mathcal{P})|, \{0\}).$$

Proof. See [21], Lemma 6.3. □

1.3.3 The complex of cubes

Let $\langle 0,1,2\rangle$ be the category associated to the ordered set $\{0,1,2\}$. Let $\langle 0,1,2\rangle^n$ be the n-th cartesian power. For every $i=1,\dots,n$ and $j=0,1,2$, there is a morphism, called *coface*

$$\begin{array}{ccc} \langle 0,1,2\rangle^{n-1} & \xrightarrow{\partial_i^j} & \langle 0,1,2\rangle^n \\ (j_1,\dots,j_{n-1}) & \mapsto & (j_1,\dots,j_{i-1},j,j_i,\dots,j_{n-1}). \end{array}$$

Definition 1.3.10. Let $\mathcal{P}$ be a small exact category. An *n-cube* E in $\mathcal{P}$ is a functor

$$\langle 0,1,2\rangle^n \xrightarrow{E^{\cdot}} \mathcal{P},$$

such that, for all $\boldsymbol{j}\in\{0,1,2\}^{n-1}$ and $i=1,\dots,n$,

$$E^{s_i^0(\boldsymbol{j})} \to E^{s_i^1(\boldsymbol{j})} \to E^{s_i^2(\boldsymbol{j})}$$

is a short exact sequence of $\mathcal{P}$.

These cubes are usually called exact cubes. But since there is no source of confusion, we just drop the word exact. For every $n\geq 0$, we define the set

$$C_n(\mathcal{P}) = \{\text{n-cubes}\},$$

and let $\mathbb{Z}C_n(\mathcal{P})$ be the free abelian group on the n-cubes. The coface maps induce face maps,

$$\partial_i^j : \mathbb{Z}C_n(\mathcal{P}) \to \mathbb{Z}C_{n-1}(\mathcal{P}), \quad \text{for every } i=1,\dots,n, \text{ and } j=0,1,2.$$

$\mathbb{Z}C_n(\mathcal{P})$

Remark 1.3.11. Let

$$\varepsilon : 0 \to E_0 \to E_1 \to E_2 \to 0$$

be an exact sequence of $(n-1)$-cubes. That is, for every $\boldsymbol{j}\in\{0,1,2\}^{n-1}$, the sequence

$$0 \to E_0^{\boldsymbol{j}} \to E_1^{\boldsymbol{j}} \to E_2^{\boldsymbol{j}} \to 0$$

is exact. Then, for all $i=1,\dots,n$, there is an n-cube $\widetilde{E}$, with

$$\partial_i^j \widetilde{E} = E_j.$$

This cube is called the *cube obtained from ε along the i-th direction.*

For every $i=1,\dots,n$ and $j=0,1$, one defines *degeneracies*

$$s_i^j : \mathbb{Z}C_{n-1}(\mathcal{P}) \to \mathbb{Z}C_n(\mathcal{P}),$$

by setting for every $E \in C_{n-1}(\mathcal{P})$,

$$s_i^j(E)_{\boldsymbol{j}} = \begin{cases} 0 & j_i \neq j, j+1 \\ E_{\partial_i(\boldsymbol{j})} & j_i = j, j+1. \end{cases}$$

That is, $s_i^j(E)$ is the n-cube obtained from the exact sequences of n-cubes

$$\begin{array}{ll} 0 \to E \xrightarrow{=} E \to 0 \to 0, & \text{if } j = 0, \\ 0 \to 0 \to E \xrightarrow{=} E \to 0, & \text{if } j = 1, \end{array}$$

along the i-th direction. An element $F \in C_n(\mathcal{P})$ is called *degenerate* if for some i and j, $F \in \operatorname{im} s_i^j$.

Observe that for any $k, l \in \{0, 1, 2\}$, the following equalities are satisfied:

$$\partial_i^l \partial_j^k = \begin{cases} \partial_j^k \partial_{i+1}^l & \text{if } j \leq i, \\ \partial_{j-1}^k \partial_i^l & \text{if } j > i. \end{cases}$$

Then, if we set

$$d = \sum_{i=1}^{n} \sum_{j=0}^{2} (-1)^{i+j} \partial_i^j, \tag{1.3.12}$$

$(\mathbb{Z}C_*(\mathcal{P}), d)$ is a chain complex.

Let

$$\mathbb{Z}D_n(\mathcal{P}) = \sum_{i=1}^{n} s_i^0(\mathbb{Z}C_{n-1}(\mathcal{P})) + s_i^1(\mathbb{Z}C_{n-1}(\mathcal{P})) \subset \mathbb{Z}C_n(\mathcal{P}).$$

Since the differential of a degenerate cube is also degenerate, the differential of $\mathbb{Z}C_*(\mathcal{P})$ induces a differential on $\mathbb{Z}D_*(\mathcal{P})$ making the inclusion arrow

$$\mathbb{Z}D_*(\mathcal{P}) \hookrightarrow \mathbb{Z}C_*(\mathcal{P})$$

a chain morphism. The quotient complex

$$\widetilde{\mathbb{Z}}C_*(\mathcal{P}) = \mathbb{Z}C_*(\mathcal{P}) / \mathbb{Z}D_*(\mathcal{P})$$

is called the *chain complex of cubes* in $\mathcal{P}$.

The morphism Cub**.** Let $\mathbb{Z}S_*(\mathcal{P})$ be the Moore complex associated to the simplicial set $S.(\mathcal{P})$. We recall here the construction of a chain morphism

$$\mathbb{Z}S_*(\mathcal{P})[-1] \xrightarrow{\mathrm{Cub}} \widetilde{\mathbb{Z}}C_*(\mathcal{P}),$$

which is a quasi-isomorphism in homology with $\mathbb{Q}$-coefficients. This construction appears in Wang's thesis ([61]) and in [47]. We refer the reader to [47] for proofs and details.

The morphism Cub is defined inductively on n. Let $\{E_{ij}\}_{0 \leq i, j \leq 1} \in S_1\mathcal{P}$. Then, one defines

$$\mathrm{Cub}(\{E_{ij}\}_{0 \leq i, j \leq 1}) = E_{01} \in C_0(\mathcal{P}).$$

Assume that the morphism Cub is defined for $n-1$. Then, for every $E = \{E_{ij}\}_{0\leq i,j\leq n+1} \in S_{n+1}\mathcal{P}$, $\operatorname{Cub} E$ is defined to be the n-cube such that

$$\begin{array}{rcl} \partial_1^0 \operatorname{Cub} E &=& s_{n-1}^0 \cdots s_1^0(E_{01}), \\ \partial_1^1 \operatorname{Cub} E &=& \operatorname{Cub}(\partial_1 E), \\ \partial_1^2 \operatorname{Cub} E &=& \operatorname{Cub}(\partial_0 E). \end{array}$$

For example, if $n = 1$, then the image by Cub of $\{E_{ij}\}_{0\leq i,j\leq 2}$ is the short exact sequence

$$E_{01} \to E_{02} \to E_{12}.$$

For $n = 2$, we obtain

$$\begin{tikzcd} E_{01} \arrow[r] \arrow[d] & E_{02} \arrow[r] \arrow[d] & E_{12} \arrow[d] \\ E_{01} \arrow[r] \arrow[d] & E_{03} \arrow[r] \arrow[d] & E_{13} \arrow[d] \\ 0 \arrow[r] & E_{23} \arrow[r] & E_{23} \end{tikzcd}$$

Proposition 1.3.13. *Let $E \in S_n(\mathcal{P})$. Then, for $i = 1, \ldots, n-1$, we have*

$$\begin{array}{rcl} \partial_i^0 \operatorname{Cub} E &=& s_{n-2}^0 \cdots s_i^0 \operatorname{Cub} \partial_{i+1} \cdots \partial_n E, \\ \partial_i^1 \operatorname{Cub} E &=& \operatorname{Cub} \partial_i E, \\ \partial_i^2 \operatorname{Cub} E &=& s_{i-1}^1 \cdots s_1^1 \operatorname{Cub} \partial_0 \cdots \partial_{i-1} E. \end{array}$$

It follows from the lemma that for every $E \in S_n(\mathcal{P})$,

$$\sum_{i=1}^{n-1} \sum_{j=0}^{2} (-1)^{i+j} \partial_i^j \operatorname{Cub} E = \sum_{i=0}^{n} (-1)^{i+1} \operatorname{Cub} \partial_i E + \text{degenerate cubes}.$$

Proposition 1.3.14. *There is a morphism of chain complexes*

$$\begin{array}{rcl} \mathbb{Z}S_*(\mathcal{P})[-1] & \xrightarrow{\operatorname{Cub}} & \widetilde{\mathbb{Z}}C_*(\mathcal{P}) \\ E & \mapsto & \operatorname{Cub} E. \end{array}$$

The composition of the Hurewicz morphism of (1.2.29) with the morphism induced by Cub in homology, gives a morphism

$$\operatorname{Cub} : K_n(\mathcal{P}) = \pi_{n+1}(S.(\mathcal{P})) \xrightarrow{\text{Hurewicz}} H_n(\mathbb{Z}S_*(\mathcal{P})[-1]) \xrightarrow{\operatorname{Cub}} H_n(\widetilde{\mathbb{Z}}C_*(\mathcal{P})).$$

Theorem 1.3.15 (McCarthy)**.** *Let $\mathcal{P}$ be a small exact category. Then, for all $n \geq 0$, the morphism*

$$K_n(\mathcal{P})_{\mathbb{Q}} \xrightarrow{\operatorname{Cub}} H_n(\widetilde{\mathbb{Z}}C_*(\mathcal{P}), \mathbb{Q})$$

is an isomorphism.

Proof. See [47]. □

1.3.4 Algebraic K-theory of schemes

In this section we discuss the algebraic K-groups of a scheme X. All the proofs can be found in [48], section § 7. Another good reference is [55], § 5.

Let X be an arbitrary scheme and consider the following exact categories:

- $\mathcal{P}(X)$ the category of locally free sheaves of finite rank. It is a full subcategory of the abelian category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on X.
- If X is noetherian, $\mathcal{M}(X)$ the abelian category of coherent sheaves on X.

In both cases, either considering small full subcategories equivalent to them, or considering a big enough universe $\mathcal{U}$, the following groups are defined:

- $K_m(X) := K_m(\mathcal{P}(X))$.
- $G_m(X) := K_m(\mathcal{M}(X))$ if X is noetherian.

Proposition 1.3.16. *If X is regular, $K_m(X) \cong G_m(X)$, for all $m \geq 0$.*

We list here the main properties of the algebraic K-groups of schemes:

1. (Pull-back). Let $f : X \to Y$ be a morphism of schemes. Then, there is an exact functor $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$, inducing, for all $m \geq 0$, morphisms
$$f^* : K_m(Y) \to K_m(X).$$
Hence, K_m is a contravariant functor from schemes to abelian groups. If $f : X \to Y$ is a flat morphism of noetherian schemes, then there is an exact functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$, inducing for all $m \geq 0$ morphisms
$$f^* : G_m(Y) \to G_m(X).$$
Hence, G_m is a contravariant functor from the category of noetherian schemes and flat morphisms to the category of abelian groups.

2. (Products). The tensor product of $\mathcal{O}_X$-modules gives an exact functor
$$\mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$$
which induces morphisms
$$K_m(X) \times K_n(X) \to K_{m+n}(X), \qquad n, m \geq 0.$$
Then, $K_*(X) := \bigoplus_{m\geq 0} K_m(X)$ is a graded anti-commutative ring. If X is noetherian, the tensor product of sheaves
$$\mathcal{P}(X) \times \mathcal{M}(X) \to \mathcal{M}(X),$$

gives morphisms

$$K_m(X) \times G_n(X) \to G_{m+n}(X), \qquad \text{for all } n, m \geq 0.$$

Then, $G_*(X) := \bigoplus_{m \geq 0} G_m(X)$ is a graded module over $K_*(X)$. In particular, $G_m(X)$ is a $K_0(X)$-module. Moreover, the morphism

$$K_*(X) \to G_*(X)$$

is a module homomorphism.

3. (Mayer-Vietoris). Let U and V be open subschemes of a noetherian scheme X. Then there is a long exact sequence

$$\cdots \to G_{m+1}(U \cap V) \to G_m(U \cup V) \to G_m(U) \oplus G_m(V) \to G_m(U \cap V) \to \cdots .$$

4. (Homotopy invariance). Let $f : P \to X$ be a flat map whose fibers are affine spaces. Then the morphism

$$f^* : G_m(X) \to G_m(P)$$

is an isomorphism for all $m \geq 0$.

5. (Projective bundle). Let ξ be a vector bundle of rank r over a noetherian scheme X. Let $\mathbb{P}(\xi) = \mathrm{Proj}(S(\xi))$ be the associated projective bundle, where $S(\xi)$ is the symmetric algebra of ξ. Let $f : \mathbb{P}(\xi) \to X$ be the structure map. Then there is an isomorphism of $K_0(\mathbb{P}(\xi))$-modules

$$K_0(\mathbb{P}(\xi)) \otimes_{K_0(X)} G_m(X) \xrightarrow{\cong} G_m(\mathbb{P}(\xi)),$$

for every $m \geq 0$, given by $y \otimes x \to y \cdot f^* x$. If $z \in K_0(\mathbb{P}(\xi))$ is the class of the tautological bundle $\mathcal{O}(-1)$ of $\mathbb{P}(\xi)$, then the above isomorphism can be rewritten as

$$G_m(X)^r \xrightarrow{\cong} G_m((\mathbb{P}(\xi)))$$

$$(x_i)_{0 \leq i < r} \mapsto \sum_{i=0}^{r-1} z^i \cdot f^* x_i$$

1.3.5 λ-rings

In this subsection we recall the basic facts on λ-rings. We have followed the treatment of [4].

Definition 1.3.17. Let R be a commutative ring with unity. The ring R is a *pre-λ-ring*, if there exists a collection of maps

$$\lambda^n : R \to R, \qquad n \geq 0,$$

such that, for all $x, y \in R$,

$$\begin{aligned}
\lambda^0 &= 1, \\
\lambda^1(x) &= x, \\
\lambda^n(x+y) &= \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y).
\end{aligned}$$

Let $\xi_1, \ldots, \xi_q$, $\eta_1, \ldots, \eta_r$ be indeterminacies and let s_i and σ_i be the i-th elementary symmetric functions in the variables ξ_j and η_j respectively. Two universal polynomials are defined as follows.

(i) The polynomial
$$P_n(s_1, \ldots, s_n; \sigma_1, \ldots, \sigma_n)$$
is the coefficient of t^n in $\prod_{i,j}(1 + \xi_i \eta_j t)$.

(ii) The polynomial
$$P_{n,m}(s_1, \ldots, s_{nm})$$
is the coefficient of t^n in $\prod_{i_1 < \cdots < i_m}(1 + \xi_{i_1} \cdot \ldots \cdot \xi_{i_m} t)$.

Definition 1.3.18. A pre-λ-ring R is a *λ-ring*, if for all $x, y \in R$,

$$\begin{aligned}
\lambda^n(xy) &= P_n(\lambda^1(x), \ldots, \lambda^n(x); \lambda^1(y), \ldots, \lambda^n(y)), \\
\lambda^m(\lambda^n(x)) &= P_{m,n}(\lambda^1(x), \ldots, \lambda^{mn}(x)).
\end{aligned}$$

Remark 1.3.19. In [4], a pre-λ-ring is called a λ-ring; a λ-ring is called a special λ-ring.

Example 1.3.20. The main example of a λ-ring is the ring of integers $\mathbb{Z}$ with λ-operations
$$\lambda^k(n) = \left(\begin{array}{c} n \\ k \end{array} \right).$$
It is called the *canonical λ-ring structure* in $\mathbb{Z}$.

Example 1.3.21. Let X be a scheme and let $K_0(X)$ denote the Grothendieck group of vector bundles on X. The tensor product of vector bundles on X induces a product structure on $K_0(X)$. Then, defining

$$\lambda^k(E) = \bigwedge^k E, \qquad k \geq 0, \tag{1.3.22}$$

$K_0(X)$ is a λ-ring.

Definition 1.3.23. Let R be a pre-λ-ring. The *Adams operations*

$$\Psi^k : R \to R, \qquad k \geq 0,$$

are defined by the recursive formula

$$\Psi^k - \Psi^{k-1}\lambda^1 + \cdots + (-1)^{k-1}\Psi^1\lambda^{k-1} + (-1)^k k\lambda^k = 0. \tag{1.3.24}$$

Alternatively, if N_k is the k-th Newton polynomial, one can check that

$$\Psi^k = N_k(\lambda^1, \dots, \lambda^k).$$

From the definition of pre-λ-ring and equation (1.3.24), it follows that Ψ^k is a group morphism for every k, i.e.

$$\Psi^k(x+y) = \Psi^k(x) + \Psi^k(y).$$

If R is a λ-ring, then the following statements are also satisfied:

$$\left\{ \begin{array}{l} \Psi^k : R \to R \text{ is a ring morphism,} \\ \Psi^k \circ \Psi^l = \Psi^l \circ \Psi^k = \Psi^{kl}. \end{array} \right. \tag{1.3.25}$$

Remark 1.3.26. Let Ψ^k denote the Adams operations corresponding to the λ-ring structure on $K_0(X)$ given in 1.3.21. They are obtained combining definition (1.3.22) with the formula (1.3.24). Grayson, in [31], showed that in the group $K_0(X)$, Ψ^k agrees with the secondary Euler characteristic class of the Koszul complex, i.e.

$$\Psi^k(E) = \sum_{p\geq 0} (-1)^{k-p+1}(k-p) S^p E \otimes \bigwedge^{k-p} E \in K_0(X).$$

Lemma 1.3.27. *Let R be a commutative $\mathbb{Q}$-algebra with unit. Assume that for every $k \geq 0$, there are group morphisms*

$$\Psi^k : R \to R$$

with $\Psi^1(x) = x$. Then, R is a pre-λ-ring, with λ-operations being defined by equation (1.3.24)*. If moreover, Ψ^k satisfy the statements in* (1.3.25)*, then R is a λ-ring.*

Lemma 1.3.28. *Let $R = \bigoplus_{p\geq 0} R_p$ be a graded $\mathbb{Q}$-algebra,with $R_i{\cdot}R_j \subset R_{i+j}$. Then, there is a canonical λ-ring structure on R with Adams operations given by*

$$\Psi^k(x) = k^p, \qquad \textit{if } x \in R_p.$$

1.4 Deligne-Beilinson cohomology

We recall here the definitions and main properties of Deligne-Beilinson cohomology that will be needed in the sequel. The main reference is [38]. A good reference for this subsection is also [14], chapter 5. Observe, however, that we restrict ourselves to the Deligne-Beilinson cohomology truncated at the degree $2p$ (where p is the twist).

One denotes $\mathbb{R}(p) = (2\pi i)^p \cdot \mathbb{R} \subset \mathbb{C}$.

1.4.1 Real Deligne-Beilinson cohomology and the Deligne complex

Real Deligne-Beilinson cohomology. Let X be a complex algebraic manifold and let $\overline{X}$ be a smooth compactification of X with $D = \overline{X} \setminus X$ a normal crossing divisor. Denote by $j : X \to \overline{X}$ the inclusion map.

Let $E^*_{\overline{X}}$ be the complex of smooth differential forms on $\overline{X}$. The complex $E^*_{\overline{X}}(\log D)$ of *differential forms with logarithmic singularities along* D, is the subcomplex of $j_* E^*_X$ which is locally generated by the sections

$$\log z_i \bar{z}_i, \ \frac{dz_i}{z_i}, \ \frac{d\bar{z}_i}{\bar{z}_i},$$

if $z_1 \cdot \ldots \cdot z_m = 0$ is a local equation for D.

Let Ω^*_X be the sheaf of holomorphic differentials on X and let $\Omega^*_{\overline{X}}(\log D)$ be the sheaf of holomorphic differential forms on $\overline{X}$ with logarithmic singularities along D. Let $F^p\Omega^*_{\overline{X}}(\log D) = \bigoplus_{q \geq p} \Omega^q_{\overline{X}}(\log D)$ and consider the complex of sheaves on $\overline{X}$

$$\mathbb{R}(p)_{\mathcal{D}} = s(Rj_*\mathbb{R}(p) \oplus F^p\Omega^*_{\overline{X}}(\log D) \xrightarrow{u} j_*\Omega^*_X)$$

where $u(a, f) = -a + f$.

Then, the *Deligne-Beilinson cohomology groups* are given by the hypercohomology groups of $\mathbb{R}(p)_{\mathcal{D}}$,

$$H^*_{\mathcal{D}}(X, \mathbb{R}(p)) := \mathbb{H}^*(\overline{X}, \mathbb{R}(p)_{\mathcal{D}}).$$

One can see that this definition does not depend on the chosen compactification, up to a unique isomorphism.

Let I be the category of pairs $(\overline{X}_\alpha, j_\alpha)$ with $\overline{X}_\alpha$ a smooth compactification of X, and $j_\alpha : X \to \overline{X}_\alpha$ an immersion such that $D_\alpha = \overline{X}_\alpha \setminus j_\alpha(X)$ is a normal crossing divisor. Morphisms in I are the maps between the smooth compactifications that compatible with the immersions j_α. The opposite category I° is directed. Then the complex of *differential forms with logarithmic singularities along infinity* is defined as

$$E^*_{\log}(X) := \lim_{\substack{\longrightarrow \\ \alpha \in I^\circ}} E^*_{\overline{X}_\alpha}(\log D_\alpha).$$

Observe that if X is a proper complex algebraic manifold, then $E^*_{\log}(X)$ is identified with the complex of differential forms on X, $E^*(X)$.

Let $E^n_{\log,\mathbb{R}}(X)(p) := (2\pi i)^p \cdot E^n_{\log,\mathbb{R}}(X) \subset E^*_{\log}(X)$.

The Deligne complex. We describe here the (truncated) Deligne complex which computes Deligne-Beilinson cohomology. Let

$$F^{p,q}E^*_{\log}(X) := \bigoplus_{p' \geq p, \ q' \geq q} E^{p',q'}_{\log}(X)$$

and consider the projection

$$\begin{array}{rcl} F^{p,q} : E^*_{\log}(X) & \to & E^*_{\log}(X) \\ x & \mapsto & \displaystyle\sum_{p'\geq p,\ q'\geq q} x^{p',q'}. \end{array}$$

The (truncated) *Deligne complex* for $E^*_{\log}(X)$ is the graded complex given by

$$\mathcal{D}^n_{\log}(X,p) = \begin{cases} E^{n-1}_{\log,\mathbb{R}}(X)(p-1) \cap F^{n-p,n-p} E^{n-1}_{\log}(X) & n \leq 2p-1, \\ \ker(d : E^{p,p}_{\log,\mathbb{R}}(X)(p) \to E^{2p+1}_{\log,\mathbb{R}}(X)(p)) & n = 2p, \end{cases}$$

and differential given by, for $x \in \mathcal{D}^n(X,p)$,

$$d_{\mathcal{D}} x = \begin{cases} -F^{n-p+1,n-p+1} dx & n < 2p-1, \\ -2\partial\overline{\partial} x & n = 2p-1. \end{cases}$$

Up to degree $2p$, the cohomology of this complex is the Deligne-Beilinson cohomology with p-twist:

$$H^n(\mathcal{D}^*_{\log}(X,p)) = H^n_{\mathcal{D}}(X, \mathbb{R}(p)), \qquad n \leq 2p.$$

Graphically, one has

$$\begin{tikzcd} & & & \ker d \subset E^{p,p}_{\log,\mathbb{R}}(X)(p) \\ E^{0,p-1}_{\log,\mathbb{R}}(X)(p-1) \arrow[r, "-\partial"] & \cdots \arrow[r, "-\partial"] & E^{p-1,p-1}_{\log,\mathbb{R}}(X)(p-1) \arrow[ur, "-2\partial\bar{\partial}"] & \\ \vdots \arrow[u, "-\bar{\partial}"] & & \vdots \arrow[u, "-\bar{\partial}"'] & \\ E^{0,0}_{\log,\mathbb{R}}(X)(p-1) \arrow[u, "-\bar{\partial}"] \arrow[r, "-\partial"'] & \cdots \arrow[r, "-\partial"'] & E^{p-1,0}_{\log,\mathbb{R}}(X)(p-1) \arrow[u, "-\bar{\partial}"'] & \end{tikzcd}$$

with $E^{0,0}_{\log,\mathbb{R}}(X)(p-1)$ sitting in degree 1.

1.4.2 Product structure

Deligne-Beilinson cohomology has a product structure, which can be described by a cochain morphism on the Deligne complex.

Let $x \in \mathcal{D}^n(X,p)$ and $y \in \mathcal{D}^m(X,q)$. Then the product is defined as

$$x \bullet y = \begin{cases} (-1)^n(\partial x^{p-1,n-p} - \bar{\partial} x^{n-p,p-1}) \wedge y + \\ \qquad + x \wedge (\partial y^{q-1,m-q} - \bar{\partial} y^{m-q,q-1}) & \text{if } n,m < 2p, \\ x \wedge y & \text{if } n = 2p \text{ or } m = 2p. \end{cases}$$

This product satisfies the expected relations:

- ▶ Graded commutativity: $x \bullet y = (-1)^{nm} y \bullet x.$
- ▶ Leibniz rule: $d_{\mathcal{D}}(x \bullet y) = d_{\mathcal{D}} x \bullet y + (-1)^n x \bullet d_{\mathcal{D}} y.$

Therefore, the Deligne product is strictly graded commutative.

Proposition 1.4.1. *The Deligne product $\bullet$ is associative up to a natural homotopy, i.e. there exists*

$$h : \mathcal{D}^r(X,p) \otimes \mathcal{D}^s(X,q) \otimes \mathcal{D}^t(X,l) \to \mathcal{D}^{r+s+t}(X, p+q+l)$$

such that

$$d_{\mathcal{D}} h(\omega_1 \otimes \omega_2 \otimes \omega_3) + h d_{\mathcal{D}}(\omega_1 \otimes \omega_2 \otimes \omega_3) = (\omega_1 \bullet \omega_2) \bullet \omega_3 - \omega_1 \bullet (\omega_2 \bullet \omega_3).$$

If $\omega_1 \in \mathcal{D}^{2p}(X,p)$, $\omega_2 \in \mathcal{D}^{2q}(X,q)$ and $\omega_3 \in \mathcal{D}^{2l}(X,l)$ satisfy $d_{\mathcal{D}}\omega_i = 0$ for all i, then

$$h(\omega_1 \otimes \omega_2 \otimes \omega_3) = 0. \tag{1.4.2}$$

Proof. This is [13], Theorem 3.3. □

1.4.3 Deligne-Beilinson cohomology and currents

We next introduce the complex of currents which can be used to describe Deligne-Beilinson cohomology. We discuss only the case where X is a proper complex algebraic manifold, although the results can be extended, not directly, to non-proper complex algebraic manifolds.

Let ${}'E^n(X)$ be the group of currents of degree n on X, i.e. the topological dual of $E^{-n}(X)$. Observe that it comes equipped with a bigrading induced by the bigrading on $E^{-n}(X)$. The differential

$$d : {}'E^n(X) \to {}'E^{n+1}(X)$$

is defined by setting

$$(dT)(\alpha) = (-1)^n T(d\alpha)$$

for every current T of degree n and differential form α of degree $n+1$. Then ${}'E^*(X)$ is a cochain complex concentrated in negative degrees.

There is a pairing

$$\begin{array}{rcl} E^n(X) \otimes {}'E^m(X) & \to & {}'E^{n+m}(X) \\ \alpha \otimes T & \mapsto & \alpha \wedge T \end{array}$$

with $\alpha \wedge T$ the current defined by

$$(\alpha \wedge T)(\beta) := T(\alpha \wedge \beta).$$

Assume that X is equidimensional of complex dimension d. Consider the cochain complex of currents, now in positive degrees, given by

$$D^*(X) := {}'E_X^{*-2d}(-d).$$

A bigrading in $D^*(X)$ is fixed by setting $D^{p,q}(X)$ to be the topological dual of $E^{d-p,d-q}(X)$. The twist by p is given by

$$D^*_{X,\mathbb{R}}(p) = (2\pi i)^{p-d}\ {}'E^{*-2d}_{X,\mathbb{R}}.$$

There is a natural morphism of cochain complexes

$$\begin{array}{ccc} E^*(X) & \xrightarrow{[\cdot]} & D^*(X) \\ \alpha & \mapsto & [\alpha] \end{array}$$

where

$$[\alpha](\eta) := \frac{1}{(2\pi i)^d}\int_X \eta \wedge \alpha. \tag{1.4.3}$$

If g is a locally integrable differential form on X, the current $[g]$ is also defined by equality (1.4.3).

Consider the Deligne complex associated to this complex of currents on X, ${}'\mathcal{D}^*(X,p)$, changing in our initial definition the complex $E^*_{\log}(X)$ by the complex $D^*(X)$. Then, the map $[\cdot]$ induces an isomorphism

$$H^n_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{\cong} H^n({}'\mathcal{D}^*(X,p)).$$

1.4.4 Functorial properties

Let $f : X \to Y$ be a morphism between two complex algebraic manifolds. The pull-back of a differential form on Y, with logarithmic singularities, is well defined. Hence, it induces a morphism of complexes

$$E^*_{\log}(Y) \xrightarrow{f^*} E^*_{\log}(X),$$

which is compatible with the definition of $\mathcal{D}^*_{\log}(\cdot,p)$. Therefore, we obtain a morphism of complexes

$$\mathcal{D}^*_{\log}(Y,p) \xrightarrow{f^*} \mathcal{D}^*_{\log}(X,p).$$

We deduce the *contravariant functoriality of Deligne-Beilinson cohomology*

$$H^*_{\mathcal{D}}(Y,\mathbb{R}(p)) \xrightarrow{f^*} H^*_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Given a proper morphism $f : X \to Y$ between two proper equidimensional complex algebraic manifolds, there is a morphism of complexes

$$f_*({}'E^m(X)) \xrightarrow{f_!} {}'E^m(Y)$$

defined by

$$(f_! T)(\alpha) = T(f^*\alpha).$$

If f has relative dimension e, there are induced morphisms in cohomology, giving the *covariant functoriality of Deligne-Beilinson cohomology*

$$H^n_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{f_!} H^{n-2e}_{\mathcal{D}}(Y,\mathbb{R}(p-e)).$$

1.4.5 Cohomology with supports

Let Z be a closed subvariety of a complex algebraic manifold X. Consider the complex

$$\mathcal{D}^*_{\log}(X \setminus Z, p),$$

i.e. the Deligne complex of differential forms in $X \setminus Z$ with logarithmic singularities along Z and infinity. The inclusion $X \setminus Z \hookrightarrow X$ induces an injective map

$$i : \mathcal{D}^*_{\log}(X, p) \to \mathcal{D}^*_{\log}(X \setminus Z, p).$$

Definition 1.4.4. The (truncated) *Deligne complex with supports in* Z is defined to be the simple of i truncated at degree $2p$,

$$\mathcal{D}^*_{\log,Z}(X, p) = \tau_{\leq 2p} s(i).$$

The (truncated) *Deligne-Beilinson cohomology with supports in* Z is defined as the cohomology groups of the Deligne complex for cohomology with supports in Z:

$$H^n_{\mathcal{D},Z}(X, \mathbb{R}(p)) := H^n(\mathcal{D}^*_{\log,Z}(X, p)), \quad n \leq 2p.$$

By the long exact sequence associated to the simple of i, there is a long exact sequence

$$\begin{aligned} \cdots \to H^n_{\mathcal{D},Z}(X, \mathbb{R}(p)) \to H^n_{\mathcal{D}}(X, \mathbb{R}(p)) \to H^n_{\mathcal{D}}(X \setminus Z, \mathbb{R}(p)) \to H^{n+1}_{\mathcal{D},Z}(X, \mathbb{R}(p)) \to \cdots \\ \cdots \to H^{2p}_{\mathcal{D},Z}(X, \mathbb{R}(p)) \to \ker(H^{2p}_{\mathcal{D}}(X, \mathbb{R}(p)) \to H^{2p}_{\mathcal{D}}(X \setminus Z, \mathbb{R}(p))) \to 0. \end{aligned} \tag{1.4.5}$$

Lemma 1.4.6. *Let Z, W be two closed subvarieties of a complex algebraic manifold X. Then there is a short exact sequence of Deligne complexes,*

$$0 \to \mathcal{D}^*_{\log}(X \setminus Z \cap W, p) \xrightarrow{i} \mathcal{D}^*_{\log}(X \setminus Z, p) \oplus \mathcal{D}^*_{\log}(X \setminus W, p) \xrightarrow{j} \mathcal{D}^*_{\log}(X \setminus Z \cup W, p) \to 0,$$

where $i(\alpha) = (\alpha, \alpha)$ and $j(\alpha, \beta) = -\alpha + \beta$.

Proof. It follows from [11], Theorem 3.6. □

In addition, Deligne-Beilinson cohomology with supports satisfies a *semipurity property.* Namely, let Z be a codimension p subvariety of an equidimensional complex manifold X, and let $Z_1, \ldots, Z_r$ be its irreducible components. Then

$$H^n_{\mathcal{D},Z}(X, \mathbb{R}(p)) = \begin{cases} 0 & n < 2p, \\ \bigoplus_{i=1}^r \mathbb{R}[Z_i] & n = 2p. \end{cases} \tag{1.4.7}$$

For $n = 2p$, the isomorphism is described below (in section 1.4.6).

The definition of the cohomology with support in a subvariety can be extended to the definition of the cohomology with support in a set of subvarieties of X. We explain here the case used in the sequel. Let $\mathcal{Z}^p$ be a subset of the set of codimension p closed subvarieties of X, ordered by the inclusion. We define the complex

$$\mathcal{D}^*_{\log}(X \setminus \mathcal{Z}^p, p) := \varinjlim_{Z \in \mathcal{Z}^p} \mathcal{D}^*_{\log}(X \setminus Z, p), \tag{1.4.8}$$

which is provided with an injective map

$$\mathcal{D}^*_{\log}(X, p) \xrightarrow{i} \mathcal{D}^*_{\log}(X \setminus \mathcal{Z}^p, p).$$

As above, we denote

$$\mathcal{D}^*_{\log,\mathcal{Z}^p}(X, p) = \tau_{\leq 2p} s(i)$$

and the cohomology groups by

$$H^n_{\mathcal{D},\mathcal{Z}^p}(X, \mathbb{R}(p)) := H^n(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X, p)), \quad n \leq 2p.$$

From the definition, there is a long exact sequence analogous to (1.4.5), relating the cohomology with support in $\mathcal{Z}^p$, the cohomology of X, and the cohomology of $X \setminus \mathcal{Z}^p$.

1.4.6 The class of a cycle and the first Chern class of a line bundle

The class of a cycle. Let X be a proper equidimensional complex algebraic manifold of dimension d, and Z a codimension p irreducible subvariety of X. Consider $\iota : \tilde{Z} \to Z$ a resolution of singularities of Z. Then, the *current integration along* Z is defined as

$$\delta_Z(\eta) := \frac{1}{(2\pi i)^{d-p}} \int_{\tilde{Z}} \iota^* \eta.$$

This definition is extended by linearity to any algebraic cycle z. The corresponding current will be denoted by δ_z and called the *class of the cycle* z. Let $CH^p(X)$ denote the codimension p Chow group of X. Then, cl induces a morphism

$$\begin{array}{ccc} CH^p(X) & \xrightarrow{cl} & H^{2p}_{\mathcal{D}}(X, \mathbb{R}(p)), \\ [Z] & \mapsto & \delta_Z. \end{array}$$

Let X be an equidimensional complex algebraic manifold and Z a codimension p irreducible subvariety of X. Let $j : X \to \overline{X}$ be a smooth compactification of X (with a normal crossing divisor as its complement) and $\overline{Z}$ the closure of Z in $\overline{X}$.

Proposition 1.4.9. *Let X be an equidimensional complex algebraic manifold and Z a codimension p irreducible subvariety of X. Let $j : X \to \overline{X}$ be a smooth compactification of X (with a normal crossing divisor as its complement) and $\overline{Z}$ the closure of Z in $\overline{X}$. The isomorphism*

$$cl : \mathbb{R}[Z] \xrightarrow{\cong} H^{2p}_{\mathcal{D},Z}(X, \mathbb{R}(p))$$

sends $[Z]$ to $[(j^* w, j^* g)]$, for any $[(w, g)] \in H^{2p}_{\mathcal{D},\overline{Z}}(\overline{X}, \mathbb{R}(p))$ satisfying the relation of currents in $\overline{X}$

$$-2\partial\bar{\partial}[g] = [w] - \delta_{\overline{Z}}. \tag{1.4.10}$$

Proof. See [14], proposition 5.58. □

In particular, assume that $Z = \operatorname{div}(f)$ is a principal divisor, where f is a rational function on X. Then $[Z]$ is represented by the couple

$$(0, -\frac{1}{2}\log(f\bar{f})) \in H^{2p}_{\mathcal{D},Z}(X, \mathbb{R}(p)).$$

In view of the last proposition, the morphism cl is described in the following form:

$$\begin{array}{rcl} CH^p(X) & \xrightarrow{cl} & H^{2p}_{\mathcal{D}}(X, \mathbb{R}(p)), \\ {[Z]} & \mapsto & [\omega] \end{array}$$

for any couple $[(j^*w, j^*g)]$ as in the proposition.

First Chern class. There are morphisms

$$\begin{array}{rcl} c_1 : \operatorname{Pic}(X) & \to & H^2(X, \mathbb{R}(1)), \\ c_1 : \Gamma(X, \mathbb{G}_m) & \to & H^1(X, \mathbb{R}(1)), \end{array} \tag{1.4.11}$$

given as follows. On $\operatorname{Pic}(X)$, c_1 is the first Chern class given by the Chern-Weil formulae, with the normalization factor $(2\pi i)$. On $\Gamma(X, \mathbb{G}_m)$, c_1 maps an invertible function f to the real function $\frac{1}{2}\log(f\bar{f})$.

1.4.7 Real varieties

A *real variety* X consists of a couple $(X_{\mathbb{C}}, F_\infty)$, with $X_{\mathbb{C}}$ a complex algebraic manifold and F_∞ an antilinear involution of $X_{\mathbb{C}}$.

Definition 1.4.12. Let $X = (X_{\mathbb{C}}, F_\infty)$ be a real variety. The *real Deligne-Beilinson cohomology of* X is defined by

$$H^n_{\mathcal{D}}(X, \mathbb{R}(p)) := H^n_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(p))^{\overline{F}^*_\infty = id}.$$

The real cohomology of X is expressed as the cohomology of the real Deligne complex

$$\mathcal{D}^n_{\log}(X, p) := \mathcal{D}^n_{\log}(X_{\mathbb{C}}, p)^{\overline{F}^*_\infty = id}, \tag{1.4.13}$$

i.e. there is an isomorphism

$$H^n_{\mathcal{D}}(X, \mathbb{R}(p)) \cong H^n(\mathcal{D}^n_{\log}(X, p), d_{\mathcal{D}}).$$

Chapter 2

Uniqueness of characteristic classes

In section 1.1 we introduced simplicial model categories and their associated homotopy categories. The main example was the category of simplicial sets. In this chapter, we deal with the category of simplicial sheaves over schemes in a Zariski site. This category can be endowed with a simplicial model category structure. Then, the algebraic K-groups are representable in the homotopy category of simplicial sheaves. That is, there exists a simplicial sheaf $\mathbb{K}.$, such that for every noetherian scheme of finite Krull dimension X,

$$K_m(X) \cong [S^m \wedge X_+, \mathbb{K}.].$$

Using this fact, we give a uniqueness theorem for characteristic classes, which includes a comparison of different definitions of Adams operations on higher algebraic K-theory and of the Chern character.

The chapter is organized as follows. The first two sections are dedicated to review part of the theory developed by Gillet and Soulé in [28]. More specifically, in Section 1 we recall the main concepts about the homotopy theory of simplicial sheaves and generalized cohomology theories. In Section 2 we explain how K-theory can be given in this setting. We introduce the sheaves $\mathbb{K}_.^N = \mathbb{Z} \times \mathbb{Z}_\infty B.GL_N$ and $\mathbb{K}. = \mathbb{Z} \times \mathbb{Z}_\infty B.GL$.

In Section 3, we consider compatible systems of maps $\Phi_N : \mathbb{K}_.^N \to \mathbb{F}.$, for $N \geq 1$ and $\mathbb{F}$ a simplicial sheaf. We introduce the class of *weakly additive* system of maps, which are the ones characterized in this paper. Roughly speaking, they are the systems for which all the information can be obtained separately from the composition $\mathbb{Z}_\infty B.GL_N \hookrightarrow \mathbb{K}_.^N \to \mathbb{F}.$ and from the composition $\mathbb{Z} \hookrightarrow \mathbb{K}_.^N \to \mathbb{F}.$. They are named *weakly additive* due to the fact that when $\mathbb{F}.$ is an H-space, they are the maps given by the sum of this two mentioned compositions. The main example are the systems of the type $\mathbb{K}_.^N \hookrightarrow \mathbb{K}. \xrightarrow{\Phi} \mathbb{F}.$, inducing group morphisms on cohomology. We discuss then the comparison of two different weakly additive systems of maps. We end this section by applying the general discussion to generalized cohomology theories on a Zariski site.

In the last two sections we develop the application of the characterization results to K-theory and to cohomology theories.

Section 4 is devoted to maps from K-theory to K-theory, specifically to the Adams and to the lambda operations on higher algebraic K-theory. A characterization of these operations is given.

In section 5, we consider maps from the K-groups to sheaf cohomology. We give a characterization of the Chern character and of the Chern classes for the higher algebraic K-groups of a scheme.

2.1 The homotopy category of simplicial sheaves

We review here the main definitions and properties of homotopy theory of simplicial sheaves. For more details about this topic see [28].

Let $\mathbf{C}$ be a site and let $\mathbf{T} = T(\mathbf{C})$ be the (Grothendieck) topos of sheaves on $\mathbf{C}$. We will always suppose that $\mathbf{T}$ has enough points (see [1], § IV 6.4.1).

Let $\mathbf{sT}$ be the category of simplicial objects in $\mathbf{T}$. One identifies $\mathbf{sT}$ with the category of sheaves of simplicial sets on $\mathbf{C}$. An object of $\mathbf{sT}$ is called a *space*.

2.1.1 Simplicial model category structure

The category $\mathbf{sT}$ is endowed with a simplicial model category structure in the sense of Quillen [49] and as recalled in section 1.1.3. This result is due to Joyal; a proof of it can be found in [39], corollary. 2.7. Here we recall the definitions that give this structure to $\mathbf{sT}$.

The structure of *model category* of $\mathbf{sT}$ is given as follows. Let $X.$ be a space in $\mathbf{sT}$. One defines $\pi_0(X.)$ to be the sheaf associated to the presheaf

$$U \mapsto \pi_0(X.(U)), \quad \text{for } U \in \mathrm{Obj}(\mathbf{C}).$$

Let $\mathbf{C}|U$ be the site of objects over U as described in [1], § III 5.1, and let $\mathbf{T}|U$ denote the corresponding topos. For every object $X.$ in $\mathbf{sT}$, let $X.|U$ be the restriction of $X.$ in $\mathbf{sT}|U$. Then, for every $U \in \mathrm{Obj}(\mathbf{C})$, $x \in X_0(U)$ a vertex of the simplicial set $X.(U)$, and every integer $n > 0$, one defines $\pi_n(X.|U, x)$ to be the sheaf associated to the presheaf

$$V \mapsto \pi_n(X.(V), x), \quad \text{for } V \in \mathrm{Obj}(\mathbf{C}|U).$$

Let $X., Y.$ be two spaces and let $f : X. \to Y.$ be a map.

(i) The map f is called a *weak equivalence* if the induced map $f_* : \pi_0(X.) \to \pi_0(Y.)$ is an isomorphism and, for all $n > 0$, $U \in \mathrm{Obj}(\mathbf{C})$ and $x \in X_0(U)$, the natural maps

$$f_* : \pi_n(X.|U, x) \to \pi_n(Y.|U, f(x))$$

are isomorphisms.

(ii) The map f is called a *cofibration* if for every $U \in \mathrm{Obj}(\mathbf{C})$, the induced map

$$f(U) : X.(U) \to Y.(U)$$

is a cofibration of simplicial sets, i.e. it is a monomorphism.

(iii) The map f is called a *fibration* if it has the right lifting property with respect to trivial cofibrations.

Observe that since the unique map $\emptyset \to X.$ is always a monomorphism, all objects $X.$ in **sT** are cofibrant.

The structure of *simplicial category* of **sT** is given by the following definitions:

(i) There is a functor **SSets** $\to$ **sT**, which sends every simplicial set $K.$ to the sheafification of the constant presheaf that takes the value $K.$ for every U in **C**.

(ii) For every space $X.$ and every simplicial set $K.$, the direct product $X. \times K.$ in **sT** is the simplicial set given by

$$[n] \mapsto \coprod_{\sigma \in K_n} X_n,$$

and induced face and degeneracy maps.

(iii) Let $X.$, $Y.$ be two spaces and let $\Delta^n_.$ be the standard n-simplex in **SSets**. The simplicial set $\underline{\mathrm{Hom}}(X., Y.)$ is the functor

$$[n] \mapsto \mathrm{Hom}_{\mathbf{sT}}(X. \times \Delta^n_., Y.).$$

By definition, a map is a cofibration of spaces if and only if it is a section-wise cofibration of simplicial sets. For fibrations and weak equivalences, this is not always true. However, it follows from the definition that a section-wise weak equivalence is a weak equivalence of spaces.

The category of simplicial presheaves. Let **sPre(C)** be the category of simplicial presheaves on **C**, i.e. the category of functors $\mathbf{C}^{op} \to$ **SSets**. Then, one defines:

- ▷ *weak equivalences* and *cofibrations* of simplicial presheaves exactly as for simplicial sheaves, and,
- ▷ *fibrations* to be the maps satisfying the right lifting property with respect to trivial cofibrations.

As shown by Jardine in [39], these definitions equip **sPre(C)** with a model category structure. The sheafification functor

$$\mathbf{sPre}(\mathbf{C}) \xrightarrow{(\cdot)^s} \mathbf{sT}$$

induces an equivalence between the respective homotopy categories, sending weak equivalences to weak equivalences. Moreover, the natural map $X. \to X^s_.$ is a weak equivalence of simplicial presheaves ([39], Lemma 2.6).

The pointed setting. All the results above can be applied to the pointed category of simplicial sheaves, $\mathbf{sT}_*$. In this situation, if $X.$ and $Y.$ are pointed simplicial sheaves of sets, then one writes $[X., Y.]_0$ for the morphisms in the pointed homotopy category. However, since from now on we will always work in the pointed category, we will just write $[X., Y.]$ for any pointed spaces $X., Y.$.

Let $*$ be the base point of $\mathbf{sT}_*$. Then, if $X.$ is a simplicial sheaf, one considers its associated pointed object to be $X._+ = X. \sqcup *$.

2.1.2 Generalized cohomology theories

Let $X.$ be any space in $\mathbf{sT}_*$. The *suspension* of $X.$, $S \wedge X.$, is defined to be the space $X. \wedge \Delta^1/\sim$, where $\sim$ is the equivalence relation generated by $(x, 0) \sim (x, 1)$ and where $\wedge$ is the pointed product. The *loop space functor* Ω is the right adjoint functor of S in the homotopy category.

Let $A.$ be any space in $\mathbf{sT}_*$. For every space $X.$ in $\mathbf{sT}_*$, one defines the *cohomology of $X.$ with coefficients in $A.$* as

$$H^{-m}(X., A.) = [S^m \wedge X., A.], \quad m \geq 0.$$

This is a pointed set for $m = 0$, a group for $m > 0$ and an abelian group for $m > 1$.

An infinite loop spectrum $A^*_.$ is a collection of spaces $\{A^i_.\}_{i \geq 0}$, together with the given weak equivalences $A^i_. \xrightarrow{\sim} \Omega A^{i+1}_.$. *The cohomology with coefficients in the spectrum $A^*_.$* is defined as

$$H^{n-m}(X., A^*_.) = [S^m \wedge X., A^n_.], \quad m, n \geq 0.$$

Due to the adjointness relation between the loop space functor and the suspension, these sets depend only on the difference $n - m$. Therefore, all of them are abelian groups.

Let $A.$ be a simplicial sheaf and assume that there is an infinite loop spectrum $A^*_.$ with $A^0_. = A.$. Then the cohomology groups with coefficients in $A.$ are also defined with positive indices, with respect to this infinite loop spectrum. By abuse of notation, when there is no source of confusion, or when we are not interested in which the loop spectrum is, we will write $H^m(X, A.)$, for the generalized cohomology with positive indices, instead of writing $H^m(X, A^*_.)$.

When $X.$ is a non-pointed space in $\mathbf{sT}$, we define

$$H^{-m}(X., A.) = [S^m \wedge X._+, A.], \quad m \geq 0.$$

Induced morphisms. Let $A., B.$ be two pointed spaces. Every element $f \in [A., B.]$ induces functorial maps

$$[X., A.] \xrightarrow{f_*} [X., B.] \quad \text{and} \quad [B., X.] \xrightarrow{f^*} [A., X.],$$

for every space $X.$. Therefore, there are induced maps between the generalized cohomology groups

$$H^{-*}(X., A.) \xrightarrow{f_*} H^{-*}(X., B.) \quad \text{and} \quad H^{-*}(B., X.) \xrightarrow{f^*} H^{-*}(A., X.).$$

Using simplicial resolutions, these maps can be described as follows. If $B^{\sim}_{\cdot}$ is any fibrant resolution of $B.$, then f is given by a homotopy class of maps $A. \to B^{\sim}_{\cdot}$. This map factorizes, uniquely up to homotopy, through a fibrant resolution of $A.$, $A^{\sim}_{\cdot}$. Therefore there is a map

$$f^{\sim} : A^{\sim}_{\cdot} \to B^{\sim}_{\cdot}$$

which induces, for every $m \geq 0$, a map

$$H^{-m}(X., A.) = \pi_0 \operatorname{Hom}(S^m \wedge X., A^{\sim}_{\cdot}) \to \pi_0 \operatorname{Hom}(S^m \wedge X., B^{\sim}_{\cdot}) = H^{-m}(X., B.).$$

The description of f^* is analogous.

2.1.3 Zariski topos

By the *big Zariski site*, ZAR, we refer to the category of all noetherian schemes of finite Krull dimension, equipped with the Zariski topology.

Given any scheme X, one can consider the category formed by the inclusion maps $V \to U$ with U and V open subsets of X and then define the covers of $U \subseteq X$ to be the open covers of U. This is called the *small Zariski site* of X, $\mathrm{Zar}(X)$. By the *big Zariski site* of X, $\mathrm{ZAR}(X)$, we mean the category of all schemes of finite type over X equipped with the Zariski topology.

The corresponding topos are named the small or big Zariski topos (over X) respectively.

Generally, one also considers subsites of the big and small Zariski sites. For instance, the site of all noetherian schemes of finite Krull dimension which are also smooth, regular, quasi-projective or projective is a subsite of ZAR. Similar subsites can be defined in $\mathrm{ZAR}(X)$ and $\mathrm{Zar}(X)$, depending on the properties of X.

At any of these sites, one associates to every scheme X in the underlying category $\mathbf{C}$, the constant pointed simplicial sheaf

$$U \mapsto \operatorname{Map}_{\mathbf{C}}(U, X) \cup \{*\}, \qquad U \in \operatorname{Obj} \mathbf{C}.$$

This simplicial sheaf is also denoted by X. For any simplicial sheaf $\mathbb{F}.$ and any scheme X in $\mathbf{C}$, the equality of simplicial sets

$$\mathbb{F}.(X) = \operatorname{Hom}(X., \mathbb{F}.)$$

is satisfied.

Definition 2.1.1. A space $X.$ is said to be *constructed from schemes* if, for all $n \geq 0$, X_n is representable by a scheme in the site plus a disjoint base point. If P is a property of schemes, one says that $X.$ satisfies the property P, if this is the case for its schematic parts.

Any simplicial scheme gives rise to a space constructed from schemes, but the converse is not true (see [37], §B.1).

If $X.$ is a space constructed from schemes, we can write $X_n = * \coprod X'_n$, with X'_n a scheme. For every pointed simplicial sheaf $\mathbb{F}.$ in $\mathbf{sT}_*$, set $\mathbb{F}.(X_n) := \mathbb{F}.(X'_n)$. Then, one defines

$$\mathbb{F}.(X.) = \underset{\overleftarrow{n}}{\operatorname{holim}}\, \mathbb{F}.(X_n),$$

where holim is the homotopy limit functor defined in [9].

2.1.4 Pseudo-flasque presheaves

We fix $\mathbf{T}$ to be the topos associated to any Zariski site $\mathbf{C}$ as in the previous section. The next definition is at the end of §2 in [10].

Definition 2.1.2. Let $\mathbb{F}.$ be a pointed simplicial presheaf on $\mathbf{C}$. It is called a *pseudo-flasque* presheaf, if the following two conditions hold:

(i) $\mathbb{F}.(\emptyset) = 0$.

(ii) For every pair of open subsets U, V of some scheme X, the square

$$\begin{tikzcd} \mathbb{F}.(U \cap V) \arrow[r] \arrow[d] & \mathbb{F}.(V) \arrow[d] \\ \mathbb{F}.(U) \arrow[r] & \mathbb{F}.(U \cup V) \end{tikzcd}$$

is homotopy cartesian.

A pseudo-flasque presheaf $\mathbb{F}.$ satisfies the Mayer-Vietoris property, i.e. for any scheme X in the site $\mathbf{C}$ and any two open subsets U, V of X, there is a long exact sequence

$$\cdots \to H^i(\mathbb{F}.(U \cup V)) \to H^i(\mathbb{F}.(U) \oplus \mathbb{F}.(V)) \to H^i(\mathbb{F}.(U \cap V)) \to H^{i+1}(\mathbb{F}.(U \cup V)) \to \cdots$$

The importance of pseudo-flasque presheaves relies on the following proposition, due to Brown and Gestern (see [10], Theorem 4).

Proposition 2.1.3. *Let $\mathbb{F}.$ be a pseudo-flasque presheaf. For every scheme X in $\mathbf{C}$, the natural map*

$$\pi_i(\mathbb{F}.(X)) \to H^{-i}(X, \mathbb{F}.^+)$$

is an isomorphism.

Observe that this proposition is already true for any fibrant sheaf. In fact, any fibrant space is pseudo-flasque.

2.2 K-theory as a generalized cohomology

Let $X.$ be a space such that its 0-skeleton is reduced to one point. One defines $\mathbb{Z}_\infty X.$ to be the sheaf associated to the presheaf

$$U \mapsto \mathbb{Z}_\infty X.(U),$$

the functor $\mathbb{Z}_\infty$ being the Bousfield-Kan integral completion of [9], § I. It comes equipped with a natural map $X. \to \mathbb{Z}_\infty X.$.

Following [28], §3.1, we consider $(\mathbf{T}, \mathcal{O}_\mathbf{T})$ a ringed topos with $\mathcal{O}_\mathbf{T}$ unitary and commutative. Then, for any integer $N \geq 1$, the linear group of rank N in $\mathbf{T}$, $GL_N = GL_N(\mathcal{O}_\mathbf{T})$, is the sheaf associated to the presheaf

$$U \mapsto GL_N(\Gamma(U, \mathcal{O}_\mathbf{T})).$$

Let $B.GL_N = B.GL_N(\mathcal{O}_\mathbf{T})$ be the classifying space of this sheaf of groups (see example 1.1.11). Observe that for every $N \geq 1$, there is a natural inclusion $B.GL_N \hookrightarrow B.GL_{N+1}$. Consider the space $B.GL = \bigcup_N B.GL_N$ and the following pointed spaces

$$\begin{aligned} \mathbb{K}. &= \mathbb{Z} \times \mathbb{Z}_\infty B.GL, \\ \mathbb{K}^N_. &= \mathbb{Z} \times \mathbb{Z}_\infty B.GL_N. \end{aligned}$$

Here, $\mathbb{Z}$ is the constant simplicial sheaf given by the constant sheaf $\mathbb{Z}$, pointed by zero. For every $N \geq 1$, the direct sum of matrices together with addition over $\mathbb{Z}$ gives a map

$$\mathbb{K}^N_. \wedge \mathbb{K}^N_. \to \mathbb{K}..$$

These maps are compatible with the natural inclusions; thus $\mathbb{K}.$ is equipped with an H-space structure (see [37]).

2.2.1 K-theory

Following [28], for any space $X.$, the *stable K-theory* is defined as

$$H^{-m}(X., \mathbb{K}.) = [S^m \wedge X._+, \mathbb{K}.],$$

and for every $N \geq 1$, the *unstable K-theory* is defined as

$$H^{-m}(X., \mathbb{K}^N_.) = [S^m \wedge X._+, \mathbb{K}^N_.].$$

Since $\mathbb{K}.$ is an H-space, $H^{-m}(X., \mathbb{K}.)$ are abelian groups for all m. However, $H^{-m}(X., \mathbb{K}^N_.)$ are abelian groups for all $m > 0$ and in general only pointed sets for $m = 0$.

Definition 2.2.1. A space $X.$ is *K-coherent* if the natural maps

$$\varinjlim_N H^{-m}(X., \mathbb{K}^N_.) \to H^{-m}(X., \mathbb{K}.)$$

and

$$\varinjlim_N H^{m}(X., \pi_n\mathbb{K}^N_.) \to H^{m}(X., \pi_n\mathbb{K}.)$$

are isomorphisms for all $m, n \geq 0$.

(Here $H^m(X., \pi_n Y)$ are the singular cohomology groups. See [28], §1.2 for a discussion in this language).

The Loday product induces a product structure on $H^{-*}(X., \mathbb{K}.)$ for every K-coherent space $X.$.

2.2.2 Comparison to Quillen's K-theory

Let $(\mathbf{T}, \mathcal{O}_\mathbf{T})$ be a locally ringed topos. For every U in $\mathbf{T}$, let $\mathcal{P}(U)$ be the category of locally free $\mathcal{O}_{\mathbf{T}|U}$-sheaves of finite rank.

Let $B.Q\mathcal{P}$ be the simplicial sheaf obtained by the Quillen construction applied to every $\mathcal{P}(U)$ (see 1.3.1). If $\Omega B.QP$ is the loop space of $B.Q\mathcal{P}$, then, by the results of [28], §3.2.1 and [21], Proposition 2.15, we obtain:

Lemma 2.2.2. *In the homotopy category of simplicial sheaves, there is a natural map of spaces*

$$\mathbb{Z} \times \mathbb{Z}_\infty B.GL \to \Omega B.Q\mathcal{P}$$

which is a weak equivalence.

Observe that this means that $\mathbb{K}.$ has yet another H-space structure, coming from being the loop space of $B.Q\mathcal{P}$. The two structures agree at $m > 0$ and at $m = 0$ at least for schemes and simplicial schemes in ZAR.

It follows from the lemma that for any space $X.$ in $\mathbf{sT}$, there is an isomorphism

$$H^{-m}(X., \mathbb{K}.) \cong H^{-m}(X., \Omega B.Q\mathcal{P}).$$

Hence, the K-groups of a space can be computed using the simplicial sheaf $\Omega B.Q\mathcal{P}$ instead of the simplicial sheaf $\mathbb{K}.$.

Suppose that $\mathbf{T}$ is the category of sheaves over a category of schemes $\mathbf{C}$. Let $\mathbb{K}.^\sim$ be a fibrant resolution of $\Omega B.Q\mathcal{P}$. For every scheme X in $\mathbf{C}$, there is a natural map

$$K_m(X) = \pi_m(\Omega B.Q\mathcal{P}(X)) \to \pi_m(\mathbb{K}.^\sim(X)) \cong H^{-m}(X, \mathbb{K}.). \tag{2.2.3}$$

The next theorem shows that many schemes are K-coherent and that Quillen K-theory agrees with stable K-theory.

Theorem 2.2.4 ([28], Proposition 5)**.** *Suppose that X is a noetherian scheme of finite Krull dimension d and that $\mathbf{T}$ is either*

1. ZAR*, the big Zariski topos of all noetherian schemes of finite Krull dimension,*

2. ZAR(X)*, the big Zariski topos of all schemes of finite type over X,*

3. Zar(X)*, the small Zariski topos of X.*

Then, viewed as a $\mathbf{T}$-space, X is K-coherent with cohomological dimension at most d. Furthermore, the morphisms $K_m(X) \to H^{-m}(X, \mathbb{K}.)$ are isomorphisms for all m.

Remark 2.2.5. Let $\mathcal{C}$ be a small category of schemes over X that contains all open subschemes of its objects. Consider the subsite $Z(X)$ of $\mathrm{ZAR}(X)$ obtained by endowing $\mathcal{C}$ with the Zariski topology. Then, the statement of the theorem will be true with $\mathbf{T} = T(Z(X))$.

For instance, if X is a regular noetherian scheme of finite Krull dimension, we could consider $Z(X)$ to be the site of all regular schemes of finite type over X. Another example would be the site of all quasi-projective schemes of finite type over a noetherian quasi-projective scheme of finite Krull dimension.

K-theory of spaces constructed from schemes. Let $\mathbf{C} = \mathrm{ZAR}$ and let $X.$ be a space constructed from schemes. Then, in the Quillen context, one defines

$$K_m(X.) = \pi_{m+1}(\underset{\overleftarrow{n}}{\operatorname{holim}}\, B.Q\mathcal{P}(X_n)).$$

For a description of the functor holim, see [9], § XI, for the case of simplicial sets or see [36], §19 for a general treatment.

Observe that the construction of the map (2.2.3) can be extended to spaces constructed from schemes. A space $X.$ is said to be *degenerate* (above some simplicial degree) if there exists an $N \geq 0$ such that $X. = \mathrm{sk}_N X.$ (where sk_N means the N-th skeleton of $X.$).

The next proposition is found in [28], §3.2.3.

Proposition 2.2.6. *Let $X.$ be a space constructed from schemes in* ZAR*. Then, the morphism* (2.2.3) *gives an isomorphism $K_m(X.) \cong H^{-m}(X., \mathbb{K}.)$. Moreover, if $X.$ is degenerate, then $X.$ is K-coherent.*

In particular, in the big Zariski site, since for every $N \geq 1$, $B.GL_N$ is a simplicial scheme, we have $K_m(B.GL_N) = H^{-m}(B.GL_N, \mathbb{K}.)$. However, $B.GL_N$ is not degenerate.

In a Zariski site over a base scheme S, the simplicial sheaf $B.GL_N$ is the simplicial scheme given by the fibred product $B.GL_{N/S} = B.GL_N \times_{\mathbb{Z}} S$.

2.3 Characterization of maps from K-theory

Our aim is to characterize functorial maps from K-theory. Since stable K-theory is expressed as a representable functor, a first approximation is obviously given by Yoneda's lemma. That is, given a space $\mathbb{F}.$ and a map of spaces $\mathbb{K}. \xrightarrow{\Phi} \mathbb{F}.$, the induced maps

$$H^{-m}(X., \mathbb{K}.) \xrightarrow{\Phi_*} H^{-m}(X., \mathbb{F}.), \qquad \forall m \geq 0,$$

are determined by the image of $id \in [\mathbb{K}., \mathbb{K}.]$ by the map $\Phi_* : H^0(\mathbb{K}., \mathbb{K}.) \to H^0(\mathbb{K}., \mathbb{F}.)$. Indeed, if $g \in H^{-m}(X., \mathbb{K}.) = [S^m \wedge X., \mathbb{K}]$, there are induced morphisms

$$[\mathbb{K}, \mathbb{K}] \xrightarrow{g^*} [S^m \wedge X., \mathbb{K}] \quad \text{and} \quad [\mathbb{K}, \mathbb{F}] \xrightarrow{g^*} [S^m \wedge X., \mathbb{F}].$$

Then, $g = g^*(id)$ and

$$\Phi_*(g) = \Phi_* g^*(id) = g^*\Phi_*(id).$$

We will see that, under some favorable conditions, the element id can be changed by other universal elements at the level of the simplicial scheme $B.GL_N$, for all $N \geq 1$.

2.3.1 Compatible systems of maps and Yoneda lemma

As in section 2.2, let $(\mathbf{T}, \mathcal{O}_\mathbf{T})$ be a ringed topos and let $\mathbb{F}.$ be a fibrant space in $\mathbf{sT}_*$. A system of maps $\Phi_M \in [\mathbb{K}_.^M, \mathbb{F}.]$, $M \geq 1$, is said to be *compatible* if, for all $M' \geq M$, the diagram

$$\begin{tikzcd} \mathbb{K}_.^M \arrow[d] \arrow[dr, "\Phi_M"] & \\ \mathbb{K}_.^{M'} \arrow[r, "\Phi_{M'}"'] & \mathbb{F}. \end{tikzcd}$$

is commutative in $\mathrm{Ho}(\mathbf{sT}_*)$. We associate to any map $\Phi : \mathbb{K}. \to \mathbb{F}.$ in $\mathrm{Ho}(\mathbf{sT}_*)$, a compatible system of maps $\{\Phi_M\}_{M\geq 1}$, given by the composition of Φ with the natural map from $\mathbb{K}_.^M$ into $\mathbb{K}.$.

Every compatible system of maps $\{\Phi_M\}_{M\geq 1}$ induces a natural transformation of functors

$$\Phi(-) : \varinjlim_M [-, \mathbb{K}_.^M] \to [-, \mathbb{F}.].$$

We state here a variant of Yoneda's lemma for maps induced by a compatible system as above.

Lemma 2.3.1. *Let $\mathbb{F}.$ be a fibrant space in $\mathbf{sT}_*$. The map*

$$\left\{ \begin{array}{c} \textit{compatible systems of maps} \\ \{\Phi_M\}_{M\geq 1},\ \Phi_M \in [\mathbb{K}_.^M, \mathbb{F}.] \end{array} \right\} \xrightarrow{\alpha} \left\{ \begin{array}{c} \textit{natural transformation of functors} \\ \Phi(-) : \varinjlim_M [-, \mathbb{K}_.^M] \to [-, \mathbb{F}.] \end{array} \right\}$$

sending every compatible system of maps to its induced natural transformation, is a bijection.

Proof. We prove the result by giving the explicit inverse arrow β of α.

So, let

$$\Phi(-) : \varinjlim_M [-, \mathbb{K}_.^M] \to [-, \mathbb{F}.]$$

be a natural transformation of functors. For every $N \geq 1$, let

$$e_N \in \varinjlim_M [\mathbb{K}_.^N, \mathbb{K}_.^M]$$

be the image of $id \in [\mathbb{K}_.^N, \mathbb{K}_.^N]$ under the natural morphism

$$[\mathbb{K}_.^N, \mathbb{K}_.^N] \xrightarrow{\sigma_N} \varinjlim_M [\mathbb{K}_.^N, \mathbb{K}_.^M].$$

We define $\Phi_N = \beta(\Phi)_N \in [\mathbb{K}_{\cdot}^N, \mathbb{F}_{\cdot}]$ to be the image of e_N by Φ,

$$\Phi_N := \Phi(\mathbb{K}_{\cdot}^N)(e_N).$$

In general, for every $N' \geq N \geq 1$, consider the map $e_{N,N'} \in [\mathbb{K}_{\cdot}^N, \mathbb{K}_{\cdot}^{N'}]$ induced by the natural inclusion

$$B.GL_N \hookrightarrow B.GL_{N'}.$$

Observe that the image of $e_{N,N'}$ under the map

$$[\mathbb{K}_{\cdot}^N, \mathbb{K}_{\cdot}^{N'}] \xrightarrow{\sigma_{N'}} \varinjlim_M [\mathbb{K}_{\cdot}^N, \mathbb{K}_{\cdot}^M],$$

is exactly e_N. Moreover, by hypothesis, there is a commutative diagram

$$\begin{tikzcd}
\varinjlim_M [\mathbb{K}_{\cdot}^{N'}, \mathbb{K}_{\cdot}^M] \arrow[r, "\Phi(\mathbb{K}_{\cdot}^{N'})"] \arrow[d, "e^*_{N,N'}"'] & {[\mathbb{K}_{\cdot}^{N'}, \mathbb{F}_{\cdot}]} \arrow[d, "e^*_{N,N'}"] \\
\varinjlim_M [\mathbb{K}_{\cdot}^N, \mathbb{K}_{\cdot}^M] \arrow[r, "\Phi(\mathbb{K}_{\cdot}^N)"'] & {[\mathbb{K}_{\cdot}^N, \mathbb{F}_{\cdot}]}
\end{tikzcd}$$

which gives the compatibility of the system $\{\Phi_N\}_{N\geq 1}$. Therefore, the map β is defined.

Now let $X.$ be any space in $\mathbf{sT}_*$. In order to prove that β is a right inverse of α, we have to see that $\Phi(X.)$ is the map induced by the just constructed system $\{\Phi_M\}_{M\geq 1}$.

Let $f \in \varinjlim_M [X., \mathbb{K}_{\cdot}^M]$. Then, there exists an integer $N \geq 1$ and a map $g \in [X., \mathbb{K}_{\cdot}^N]$, such that $\sigma_N(g) = f$. By the commutative diagram

$$\begin{tikzcd}
{[\mathbb{K}_{\cdot}^N, \mathbb{K}_{\cdot}^N]} \arrow[r, "\sigma_N"] \arrow[d, "g^*"'] & \varinjlim_M [\mathbb{K}_{\cdot}^N, \mathbb{K}_{\cdot}^M] \arrow[d, "g^*"] \\
{[X., \mathbb{K}_{\cdot}^N]} \arrow[r, "\sigma_N"'] & \varinjlim_M [X., \mathbb{K}_{\cdot}^M],
\end{tikzcd}$$

we see that in fact, $f = \sigma_N(g) = g^*(e_N)$. Using the fact that Φ is a natural transformation, the diagram

$$\begin{tikzcd}
\varinjlim_M [\mathbb{K}_{\cdot}^N, \mathbb{K}_{\cdot}^M] \arrow[r, "\Phi"] \arrow[d, "g^*"'] & {[\mathbb{K}_{\cdot}^N, \mathbb{F}_{\cdot}]} \arrow[d, "g^*"] \\
\varinjlim_M [X., \mathbb{K}_{\cdot}^M] \arrow[r, "\Phi"'] & {[X., \mathbb{F}_{\cdot}]}
\end{tikzcd} \tag{2.3.2}$$

is commutative. Hence, we obtain

$$\Phi(f) = \Phi(g^*(e_N)) = g^*\Phi(e_N) = \beta(\Phi)_N \circ g = \alpha\beta(\Phi)(f),$$

as desired.

It remains to check that β is a left inverse of α. Let $\{\Phi_N\}_{N\geq 1}$ be a compatible system of maps, let Φ be the associated transformation of functors obtained by α and let $\{\beta(\Phi)_N\}_{N\geq 1}$ be the system $\beta(\Phi)$. From the commutative diagram

$$\begin{tikzcd}
{[\mathbb{K}^N_\cdot, \mathbb{K}^N_\cdot]} \arrow[r, "(\Phi_N)_*"] \arrow[d, "\sigma_N"'] & {[\mathbb{K}^N_\cdot, \mathbb{F}_\cdot]} \\
\lim\limits_{\overrightarrow{M}} [\mathbb{K}^N_\cdot, \mathbb{K}^M_\cdot] \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

we deduce that

$$\Phi_N = (\Phi_N)_*(id) = \Phi\sigma_N(id) = \Phi(e_N) = \beta(\Phi)_N.$$

Therefore, β is the inverse of α and thus α is a bijection. $\square$

Remark 2.3.3. The last lemma is not specific to our category and to our compatible system of maps. It could be directly generalized to any suitable category.

2.3.2 Weakly additive systems of maps

We start by defining the class of weakly additive systems of maps. For this class, we will state results on the comparison of the induced natural transformations. It will be shown below that many usual maps are weakly additive.

Let $\mathbb{K}^N_\cdot, \mathbb{K}_\cdot$ be as in the previous section. Let pr_1 and pr_2 be the projections onto the first and second component respectively

$$\begin{aligned}
\mathrm{pr}_1 &: \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M \to \mathbb{Z}, \\
\mathrm{pr}_2 &: \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M \to \mathbb{Z}_\infty B.GL_M,
\end{aligned}$$

and let j_1, j_2 denote the inclusions obtained using the respective base points

$$\begin{aligned}
\mathbb{Z} &\xrightarrow{j_1} \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M, \\
\mathbb{Z}_\infty B.GL_M &\xrightarrow{j_2} \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M.
\end{aligned}$$

Denote by $\pi_i = j_i \circ \mathrm{pr}_i \in [\mathbb{K}^M_\cdot, \mathbb{K}^M_\cdot]$, $i = 1, 2$, the compositions

$$\begin{aligned}
\pi_1 : \ \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M &\xrightarrow{\mathrm{pr}_1} \mathbb{Z} \xrightarrow{j_1} \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M, \\
\pi_2 : \ \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M &\xrightarrow{\mathrm{pr}_2} \mathbb{Z}_\infty B.GL_M \xrightarrow{j_2} \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M.
\end{aligned}$$

For every space $\mathbb{F}_\cdot$ in $\mathbf{sT}_*$, there are induced maps

$$[\mathbb{K}^M_\cdot, \mathbb{F}_\cdot] \xrightarrow{\pi_i^*} [\mathbb{K}^M_\cdot, \mathbb{F}_\cdot], \qquad i = 1, 2.$$

If $\Phi_M \in [\mathbb{K}^M_\cdot, \mathbb{F}_\cdot]$ we define the maps

$$\Phi^i_M := \pi_i^*(\Phi_M) \in [\mathbb{K}^M_\cdot, \mathbb{F}_\cdot], \qquad i = 1, 2.$$

Remark 2.3.4. Consider a compatible system of maps $\Phi_M \in [\mathbb{K}.^M, \mathbb{F}.]$, $M \geq 1$ and assume that $\mathbb{F}.$ is fibrant (this is no loss of generality). By hypothesis, the diagrams

$$\begin{tikzcd} & \mathbb{K}.^M \arrow[d] \\ \mathbb{Z} \arrow[ur, "j_1"] \arrow[r, "j_1"'] & \mathbb{K}.^{M'} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{K}.^M \arrow[d] \arrow[dr, "\Phi_M"] & \\ \mathbb{K}.^{M'} \arrow[r, "\Phi_{M'}"'] & \mathbb{F}. \end{tikzcd}$$

are commutative in $\mathrm{Ho}(\mathbf{sT}_*)$, for every $M' \geq M$. Therefore, the homotopy class of the map $\Phi_M \circ j_1$ in $[\mathbb{Z}, \mathbb{F}.]$ does not depend on M.

Definition 2.3.5. Let $\mathbb{F}.$ be any space in $\mathbf{sT}_*$. Let $\{\Phi_M\}_{M\geq 1}$, with $\Phi_M \in [\mathbb{K}.^M, \mathbb{F}.]$, be a compatible system of maps. Let

$$\bullet_M : [\mathbb{K}.^M, \mathbb{F}.] \times [\mathbb{K}.^M, \mathbb{F}.] \to [\mathbb{K}.^M, \mathbb{F}.],$$

be an operation on $[\mathbb{K}.^M, \mathbb{F}.]$ (we do not require compatibility for different indexes M). The system $\{\Phi_M\}_{M\geq 1}$ is called *weakly additive with respect to the operation* $\bullet = \{\bullet_M\}_{M\geq 1}$, if for every M,

$$\Phi_M = \Phi_M^1 \bullet_M \Phi_M^2.$$

A map $\Phi \in [\mathbb{K}., \mathbb{F}.]$ is called *weakly additive* if the induced compatible system of maps is weakly additive.

Example 2.3.6 (Φ is trivial over $\mathbb{Z}$)**.** Let $\{\Phi_M\}_{M\geq 1}$ be a compatible system of maps with $\Phi_M \in [\mathbb{K}.^M, \mathbb{F}.]$. Assume that for all $M \geq 1$, $\Phi_M^1 = *$, i.e. the constant map to the base point of $\mathbb{F}.$. Then, $\Phi_M = \Phi_M^2$ for all M. Therefore, with $\bullet_M = \mathrm{pr}_2$, the system $\{\Phi_M\}_{M\geq 1}$ is weakly additive.

Example 2.3.7 ($\mathbb{F}.$ is an H-space)**.** In this case, one can take the $\bullet_N$ operation to be the sum operation in $[\mathbb{K}.^N, \mathbb{F}.]$. Then the condition of weakly additivity means that the maps Φ_N behave additively over the two components of $\mathbb{K}.^N$. Actually, the definition of weakly additive systems of maps was motivated by this example.

The lambda operations on higher algebraic K-theory, defined by Gillet and Soulé in [28], are an example of this type of weakly additive systems of maps.

Remark 2.3.8. If $\mathbb{F}.$ is an H-space and $\{\Phi_M\}_{M\geq 1}$ is a compatible system of maps such that $\Phi_M^1 = *$, then the system is weakly additive with respect to both the H-sum of $\mathbb{F}.$ and the operation $\bullet_M = \mathrm{pr}_2$.

2.3.3 Classifying elements

We now introduce some classifying elements. For every $N \geq 1$, let

$$\sigma_N : [\mathbb{Z}_\infty B.GL_N, \mathbb{K}.^N] \to \varinjlim_M [\mathbb{Z}_\infty B.GL_N, \mathbb{K}.^M]$$

be the natural morphism. Recall that we defined $j_2 \in [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^N_.]$ to be the map induced by the natural inclusion. Then, we define

$$i'_N = \sigma_N(j_2) \in \varinjlim_M [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^M_.].$$

Let $u'_r \in \varinjlim_M [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^M_.]$ be the homotopy class of the constant map

$$\begin{array}{rcl} \mathbb{Z}_\infty B.GL_N & \to & \mathbb{Z} \times \mathbb{Z}_\infty B.GL_N \\ x & \mapsto & (r, *). \end{array}$$

Finally, consider the natural map $B.GL_N \to \mathbb{Z}_\infty B.GL_N$. The images of i'_N and u'_r under the induced map

$$\varinjlim_M [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^M_.] \to \varinjlim_M [B.GL_N, \mathbb{K}^M_.],$$

are denoted by

$$i_N, u_r \in \varinjlim_M [B.GL_N, \mathbb{K}^M_.]$$

respectively.

Proposition 2.3.9. *Let $\mathbb{F}.$ be a space in $\mathbf{sT}_*$ and let $\{\Phi_M\}_{M\geq 1}$, $\{\Phi'_M\}_{M\geq 1}$ be two weakly additive systems of maps with respect to the same operation. Then, the induced maps*

$$\Phi, \Phi' : \varinjlim_M [-, \mathbb{K}^M_.] \to [-, \mathbb{F}.]$$

agree for all spaces, if and only if, in $[\mathbb{Z}_\infty B.GL_N, \mathbb{F}.]$ it holds

$$\begin{array}{rcll} \Phi(i'_N) & = & \Phi'(i'_N), & \textit{for all } N \geq 1, \end{array} \tag{2.3.10}$$
$$\begin{array}{rcll} \Phi(u'_r) & = & \Phi'(u'_r), & \textit{for all } r \in \mathbb{Z},\ N \geq 1. \end{array} \tag{2.3.11}$$

Proof. One implication is obvious. By lemma 2.3.1, it is enough to see that for all N, $\Phi_N = \Phi'_N$. By hypothesis, there is an operation $\bullet_N$ on $[\mathbb{K}^N_., \mathbb{F}.]$ such that

$$\begin{array}{rcl} \Phi_N & = & \Phi^1_N \bullet_N \Phi^2_N, \\ \Phi'_N & = & \Phi'^1_N \bullet_N \Phi'^2_N. \end{array}$$

Therefore, it is enough to see that

$$\Phi^1_N = \Phi'^1_N, \quad \text{and} \quad \Phi^2_N = \Phi'^2_N.$$

The first equality follows from hypothesis (2.3.11). For the second equality, observe that by definition, $\Phi(i'_N) = \Phi_N \circ j_2$. Therefore, by equality (2.3.10),

$$\Phi^2_N = \Phi_N \circ j_2 \circ \mathrm{pr}_2 = \Phi(i'_N) \circ \mathrm{pr}_2 = \Phi'(i'_N) \circ \mathrm{pr}_2 = \Phi'^2_N.$$

□

Corollary 2.3.12. *Let $\mathbb{F}.$ be an H-space in $\mathbf{sT}_*$ and $\{\Phi_M\}_{M\geq 1}$, $\{\Phi'_M\}_{M\geq 1}$ be two weakly additive systems of maps with respect to the same operation. Then, the induced maps*

$$\Phi(-), \Phi'(-) : \varinjlim_M [-, \mathbb{K}.^M] \to [-, \mathbb{F}.]$$

agree if and only if, in $[B.GL_N, \mathbb{F}.]$ it holds

$$\Phi(i_N) = \Phi'(i_N), \quad \textit{for all } N \geq 1, \tag{2.3.13}$$
$$\Phi(u_r) = \Phi'(u_r), \quad \textit{for all } r \in \mathbb{Z},\ N \geq 1. \tag{2.3.14}$$

Proof. It follows from the fact that the natural map

$$[\mathbb{Z}_\infty B.GL_N, \mathbb{F}.] \to [B.GL_N, \mathbb{F}.]$$

is an isomorphism if $\mathbb{F}.$ is an H-space, and under this isomorphism, the elements u_r and i_N correspond to u'_r and i'_N respectively. □

Let $j_N \in H^0(B.GL_N, \mathbb{K}.) = K_0(B.GL_N)$ be the image of i_N under the morphism

$$\varinjlim_M H^0(B.GL_N, \mathbb{K}.^M) \to H^0(B.GL_N, \mathbb{K}.).$$

Denote by u_r the image of $u_r \in \varinjlim_M H^0(B.GL_N, \mathbb{K}.^M)$ in $H^0(B.GL_N, \mathbb{K}.)$.

Corollary 2.3.15. *Let $\mathbb{F}.$ be an H-space in $\mathbf{sT}_*$. Let $\Phi, \Phi' : \mathbb{K}. \to \mathbb{F}.$ be two H-space maps. Then, Φ and Φ' are weakly additive. Moreover,*

$$\Phi, \Phi' : [X., \mathbb{K}.] \to [X., \mathbb{F}.],$$

agree for all K-coherent spaces $X.$, if they agree over j_N and u_r for all $N \geq 1$ and all $r \in \mathbb{Z}$.

Proof. The maps Φ and Φ' are weakly additive with respect to the H-sum of $\mathbb{F}.$, due to example 2.3.7. Therefore, by corollary 2.3.12, the maps

$$\Phi, \Phi' : [X., \mathbb{K}.] \cong \varinjlim_N [X., \mathbb{K}.^N] \to [X., \mathbb{F}.]$$

agree for all K-coherent spaces $X.$, if and only if $\Phi(i_N) = \Phi'(i_N)$ for $N \geq 1$ and $\Phi(u_r) = \Phi'(u_r)$ for all r. Since by construction $\Phi(i_N) = \Phi(j_N)$ (and the same for Φ'), the corollary is proved. □

2.3.4 Application to the Zariski sites

Let S be a finite dimensional noetherian scheme. Fix $\mathbf{C}$ a Zariski subsite of $\mathrm{ZAR}(S)$ containing all open subschemes of its objects and the simplicial scheme $B.GL_{N/S}$. Let $\mathbf{T} = T(\mathbf{C})$.

A direct consequence of corollary 2.3.12 is the following theorem.

Theorem 2.3.16. *Let $\mathbb{F}.$ be an H-space in $\mathbf{sT}_*$ and $\{\Phi_M\}_{M\geq 1}$, $\{\Phi'_M\}_{M\geq 1}$ be two weakly additive systems of maps with respect to the same operation. Then, the induced maps*

$$\Phi, \Phi' : K_m(X.) \cong \varinjlim_{M} H^{-m}(X., \mathbb{K}^M_.) \to H^{-m}(X., \mathbb{F}.)$$

agree for all $m \geq 0$ and all K-coherent spaces $X.$, if and only if it holds in $H^0(B.GL_{N/S}, \mathbb{F}.)$

$$\begin{aligned} \Phi(i_N) &= \Phi'(i_N), \quad \text{for all } N \geq 1, \\ \Phi(u_r) &= \Phi'(u_r), \quad \text{for all } r \in \mathbb{Z},\ N \geq 1. \end{aligned}$$

□

Finally, the next corollary is corollary 2.3.15 applied to the Zariski subsite $\mathbf{C} \subset \mathrm{ZAR}(S)$.

Corollary 2.3.17. *Let $\mathbb{F}.$ be an H-space in $\mathbf{sT}_*$. Let $\chi_1, \chi_2 \in [\mathbb{K}., \mathbb{F}.]$ be two H-space maps in $\mathrm{Ho}(\mathbf{sT}_*)$. Then, the induced maps*

$$\chi_1, \chi_2 : K_m(X.) \to H^{-m}(X., \mathbb{F}.)$$

agree for all degenerate simplicial schemes in $\mathbf{C}$, if for all $N \geq 1$ and $r \in \mathbb{Z}$, they agree at $j_N, u_r \in K_0(B.GL_{N/S})$.

□

The next theorem shows that in fact a weaker condition is needed in order to obtain the uniqueness of maps.

For any scheme X and any simplicial sheaf $\mathbb{F}.$, let $\mathbb{F}_X.$ denote the restriction of $\mathbb{F}.$ to the small Zariski site of X. In the next theorem, we write $[\cdot,\cdot]_{\mathbf{C}}$ for the maps in $\mathrm{Ho}(\mathbf{sT}(\mathbf{C})_*)$, for any site $\mathbf{C}$. If X is a scheme in $\mathbf{C}$, let $\mathbf{C}(X)$ be the subsite of $\mathrm{ZAR}(X)$ whose objects are in $\mathbf{C}$.

Theorem 2.3.18. *Let $\mathbb{F}.$ be a pseudo-flasque sheaf on $\mathbf{sT}_*$, which is an H-space. Assume that*

- *For every scheme X in $\mathbf{C}$, there are two H-space maps*

$$\chi_1(X), \chi_2(X) \in [\mathbb{K}_X., \mathbb{F}_X.]_{\mathrm{Zar}(X)}.$$

- *For any map $X \to Y$ in $\mathbf{C}$, there are commutative diagrams*

$$\begin{tikzcd} \mathbb{K}.(Y) \arrow[r, "\chi_1(Y)\text{,}\chi_2(Y)"] \arrow[d] & \mathbb{F}.(Y) \arrow[d] \\ \mathbb{K}.(X) \arrow[r, "\chi_1(X)\text{,}\chi_2(X)"] & \mathbb{F}.(X) \end{tikzcd} \tag{2.3.19}$$

in $\mathrm{Ho}(\mathbf{SSets}_*)$.

Then, the maps

$$\chi_1, \chi_2 : K_m(X) \to H^{-m}(X, \mathbb{F}.)$$

agree for all schemes in $\mathbf{C}$ *if they agree at* $j_N, u_r \in K_0(B.GL_{N/S})$*, for all* $N \geq 1$ *and* $r \in \mathbb{Z}$ *(see remark below).*

Remark 2.3.20. The condition of $\mathbb{F}.$ being pseudo-flasque means that for any space $X.$ constructed from schemes,

$$H^{-m}(X., \mathbb{F}.) \cong \pi_m(\mathbb{F}.(X.)) := \pi_m(\operatorname{holim}_{\overleftarrow{n}} \mathbb{F}.(X_n)).$$

Now, the commutative diagram (2.3.19), implies that, for any such space, there are induced morphisms

$$\mathbb{K}.(X.) \xrightarrow{\chi_1(X.), \chi_2(X.)} \mathbb{F}.(X.).$$

Hence, the maps χ_1 and χ_2 are defined for $X. = B.GL_{N/S}$.

Proof. For every fixed scheme X, it follows from theorem 2.3.17 that $\chi_1(X) = \chi_2(X)$ if they agree for $j_N, u_r \in [B.GL_{N|X}, \mathbb{K}_X.]_{\mathrm{Zar}(X)}$.

Observe now that by the remarks following proposition 2.2.4,

$$[B.GL_{N|X}, \mathbb{K}_X.]_{\mathrm{Zar}(X)} \cong [B.GL_{N|X}, \mathbb{K}.]_{\mathbf{C}(X)} = K_0(B.GL_{N|X}),$$

and there are commutative diagrams

$$\begin{tikzcd} K_0(B.GL_{N|S}) \arrow[r, "\chi_1,\chi_2"] \arrow[d] & {[B.GL_{N|S}, \mathbb{F}.]_{\mathbf{C}}} \arrow[d] \\ K_0(B.GL_{N|X}) \arrow[r, "\chi_1,\chi_2"'] & {[B.GL_{N|X}, \mathbb{F}.]_{\mathbf{C}(X)}} \end{tikzcd} \quad (2.3.21)$$

Then, the statement follows from the fact that j_N and u_r in $K_0(B.GL_{N|X})$ are the image under the vertical map of j_N and u_r in $K_0(B.GL_{N|S})$. □

2.4 Morphisms between K-groups

2.4.1 Lambda and Adams operations

In this section we focus on the case where $\mathbb{F}. = \mathbb{K}.$. Then, the main application of theorems 2.3.17 and 2.3.18 is to the Adams operations and to the lambda operations on higher algebraic K-theory.

Recall from 1.3.21 that the Grothendieck group of a scheme X, has a λ-ring structure given by $\lambda^k(E) = \bigwedge^k E$, for any vector bundle E over X. In the literature there are several definitions of the extension of the Adams operations of $K_0(X)$ to the higher algebraic K-groups. Our aim in this section is to give a criterion for their comparison.

Soulé, in [53], gives a λ-ring structure to the higher algebraic K-groups of any noetherian regular scheme of finite Krull dimension. Gillet and Soulé then generalize this result in [28], defining lambda operations for all K-coherent spaces in any locally ringed topos. We briefly recall this construction here.

Let $\mathbf{R}_{\mathbb{Z}}(GL_N)$ be the Grothendieck group of representations of the general linear group scheme $GL_{N/\mathbb{Z}}$. The properties of $\mathbf{R}_{\mathbb{Z}}(GL_N)$ that concern us are:

(1) $\mathbf{R}_{\mathbb{Z}}(GL_N)$ has a λ-ring structure.

(2) For any locally ringed topos, there is a ring morphism

$$\varphi : \mathbf{R}_{\mathbb{Z}}(GL_N) \to H^0(B.GL_N, \mathbb{K}.).$$

The operations $\lambda^k_{GS}, \Psi^k_{GS}$ are constructed by transferring the lambda and Adams operations of $\mathbf{R}_{\mathbb{Z}}(GL_N)$ to the K-theory of $B.GL_N$. Namely, consider the representation $id_N - N$ and the maps

$$\varphi(\Psi^k(id_N - N)),\ \varphi(\lambda^k(id_N - N)) : B.GL_N \to \mathbb{K}.$$

in the homotopy category of simplicial sheaves.

Consider the unique λ-ring structure on $\mathbb{Z}$ with trivial involution (see 1.3.20). Then, adding the previous maps with the lambda or Adams operations on the $\mathbb{Z}$-component, we obtain compatible systems of maps

$$\Psi^k_{GS},\ \lambda^k_{GS} : \mathbb{K}^N_. \to \mathbb{K}.,\qquad N \geq 1.$$

Observe that, by example 2.3.7, both systems are weakly additive with respect to the H-sum of $\mathbb{K}.$.

In particular, for any noetherian scheme X of finite Krull dimension, there are induced Adams operations on the higher K-groups

$$\Psi^k_{GS} : K_m(X) \to K_m(X).$$

Gillet and Soulé checked that these maps satisfy the identities of a special lambda ring.

2.4.2 Vector bundles over a simplicial scheme

Let $X.$ be a simplicial scheme, with face maps denoted by d_i and degeneracy maps by s_i. A *vector bundle* $E.$ over $X.$ consists of a collection of vector bundles $E_n \to X_n$, $n \geq 0$, together with isomorphisms $d_i^* E_n \cong E_{n+1}$ and $s_i^* E_{n+1} \cong E_n$ for all face and degeneracy maps. Moreover, these isomorphisms should satisfy the simplicial identities. By a *morphism of vector bundles* we mean a collection of morphisms at each level, compatible with these isomorphisms. An *exact sequence* of vector bundles is an exact sequence at every level.

Let $\mathrm{Vect}(X.)$ be the exact category of vector bundles over $X.$ and consider the algebraic K-groups of $\mathrm{Vect}(X.)$, $K_m(\mathrm{Vect}(X.))$. By the Waldhausen simplicial set, these can be computed as the homotopy groups of the simplicial set $S.(\mathrm{Vect}(X.))$ (see 1.3.2).

For every simplicial scheme $X.$ and every $n \geq 0$, there is a natural simplicial map

$$S.(\mathrm{Vect}(X.)) \to S.(\mathrm{Vect}(X_n)).$$

By the definition of vector bundles over simplicial schemes, it induces a simplicial map

$$S.(\mathrm{Vect}(X.)) \to \underset{\overleftarrow{n}}{\mathrm{holim}}\, S.(\mathrm{Vect}(X_n)),$$

which induces a morphism

$$K_m(\mathrm{Vect}(X.)) \xrightarrow{\psi} K_m(X.), \qquad m \geq 0.$$

At the zero level, $K_0(\mathrm{Vect}(X.))$ is the Grothendieck group of the category of vector bundles over $X.$, and hence it has a λ-ring structure.

In the particular situation where $X.$ is a simplicial object in $\mathrm{ZAR}(S)$, with S a finite dimensional noetherian scheme, the above morphism can be described as follows (see [28]). Recall from 1.1.11, that every covering U of X, has an associated simplicial set, called the nerve of the cover and denoted by $N.U$.

Let $E.^M$ denote the universal vector bundle over $B.GL_{M/S}$ and let $E.$ be a rank N vector bundle over $X.$. Then, there exists a hypercovering $p : N.U \to X.$ and a classifying map $\chi : N.U \to B.GL_{M/S}$, for $M \geq N$, such that $p^*(E.) = \chi^*(E.^M)$. The induced map

$$\chi : N.U \to \{N\} \times \mathbb{Z}_\infty B.GL_M \to \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M \to \mathbb{K}.$$

in $\mathrm{ZAR}(S)$, defines an element χ in $H^0(N.U, \mathbb{K}.) = H^0(X., \mathbb{K}.) = K_0(X.)$, which is $\psi([E.])$. This description also shows that the morphism factorizes through the limit

$$\psi : K_0(\mathrm{Vect}(X.)) \to \varinjlim_{M} H^0(X., \mathbb{K}.^M) \to H^0(X., \mathbb{K}.).$$

When $X. = B.GL_{N/S}$, we obtain that

$$\begin{aligned} \psi(E.^N - N) &= j_N \in K_0(B.GL_{N/S}), \\ \psi(E.^N - N) &= i_N \in \varinjlim_{M} H^0(B.GL_{N/S}, \mathbb{K}.^M). \end{aligned}$$

Here N is the trivial bundle of rank N. Clearly, the trivial bundle of rank $r \geq 0$ in $B.GL_{N/S}$, is mapped to u_r.

Consider the λ-ring structure in $K_0(\mathrm{Vect}(B.GL_{N/S}))$ and denote by Ψ^k the corresponding Adams operations. Gillet and Soulé proved in [28] section 5 that there are equalities

$$\begin{aligned} \varphi(\Psi^k(id_N - N)) &= \psi(\Psi^k(E.^N - N)), \\ \varphi(\lambda^k(id_N - N)) &= \psi(\lambda^k(E.^N - N)). \end{aligned}$$

Moreover, one can easily check that $\varphi(\Psi^k(id_N - N)) = \Psi^k_{GS}(i_N)$, and $\varphi(\lambda^k(id_N - N)) = \lambda^k_{GS}(i_N)$. Therefore,

$$\Psi^k_{GS}(i_N) = \psi(\Psi^k(E.^N - N)), \quad \text{and} \quad \lambda^k_{GS}(i_N) = \psi(\lambda^k(E.^N - N)).$$

Also, it holds by definition that

$$\Psi^k_{GS}(u_r) = \psi(\Psi^k(u_r)) = u_{\psi^k(r)}, \quad \text{and} \quad \lambda^k_{GS}(u_r) = \psi(\lambda^k(u_r)) = u_{\lambda^k(r)}.$$

2.4.3 Uniqueness theorems

Let S be a finite dimensional noetherian scheme. Fix $\mathbf{C}$ a Zariski subsite of ZAR(S) as in section 2.3.4. The following theorems are a consequence of theorems 2.3.17 and 2.3.18 applied to the present situation.

Theorem 2.4.1 (Lambda operations). *Let $\{\rho_N : \mathbb{K}_\cdot^N \to \mathbb{K}_\cdot\}_{N\geq 1}$ be a weakly additive system of maps with respect to the H-sum of $\mathbb{K}_\cdot$. Let ρ be the induced morphism*

$$\rho : \lim_{\overrightarrow{M}} H^*(-, \mathbb{K}_\cdot^M) \to H^*(-, \mathbb{F}_\cdot).$$

If

- $\rho(i_N) = \psi(\lambda^k(E_\cdot^N - N))$, *and,*
- $\rho(u_r) = u_{\lambda^k(r)}$,

then, ρ agrees with $\lambda^k_{GS} : K_m(X_\cdot) \to K_m(X_\cdot)$, for every degenerate simplicial scheme $X_\cdot$ in $\mathbf{C}$.

Theorem 2.4.2 (Adams operations). *Let $\{\rho_N : \mathbb{K}_\cdot^N \to \mathbb{K}_\cdot\}_{N\geq 1}$ be a weakly additive system of maps with respect to the H-sum of $\mathbb{K}_\cdot$. Let ρ be the induced morphism*

$$\lim_{\overrightarrow{M}} H^*(-, \mathbb{K}_\cdot^M) \to H^*(-, \mathbb{F}_\cdot).$$

If

- $\rho(i_N) = \psi(\Psi^k(E_\cdot^N - N))$, *and,*
- $\rho(u_r) = u_{\psi^k(r)}$,

then, ρ agrees with $\Psi^k_{GS} : K_m(X_\cdot) \to K_m(X_\cdot)$, for every degenerate simplicial scheme $X_\cdot$ in $\mathbf{C}$.

Since the Adams operations are group morphisms, it is natural to expect that they will be induced by H-space maps

$$\mathbb{K}_\cdot \to \mathbb{K}_\cdot$$

in Ho($\mathbf{sT}_*$). The next two corollaries follow easily from the last theorem.

Corollary 2.4.3. *Let $\rho : \mathbb{K}_\cdot \to \mathbb{K}_\cdot$ be an H-space map in the homotopy category of simplicial sheaves on* $\mathbf{C}$. *If*

- $\rho(j_N) = \psi(\Psi^k(E_N - N))$, *and,*
- $\rho(u_r) = \psi(\Psi^k(u_r))$,

then ρ agrees with the Adams operation Ψ^k_{GS}, for all degenerate simplicial schemes in $\mathbf{C}$.

Let $S.\mathcal{P}$ denote the Waldhausen simplicial sheaf on $ZAR(S)$ given by

$$X \mapsto S.\mathcal{P}(X) = S.(X).$$

Corollary 2.4.4. *Let $\rho : S.\mathcal{P} \to S.\mathcal{P}$ be an H-space map in $\mathrm{Ho}(\mathbf{sT}_*)$. If for some $k \geq 1$ there is a commutative square*

$$\begin{tikzcd} K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"] \arrow[d, "\Psi^k"'] & K_0(B.GL_{N/S}) \arrow[d, "\rho"] \\ K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"'] & K_0(B.GL_{N/S}), \end{tikzcd}$$

then ρ agrees with the Adams operation Ψ^k_{GS}, for all degenerate simplicial schemes in $\mathbf{C}$.

Therefore, there is a unique way to extend the Adams operations on the Grothendieck group of simplicial schemes by means of a sheaf map $S.\mathcal{P} \to S.\mathcal{P}$.

Grayson, in [31], defines the Adams operations for the K-groups of any exact category with a suitable notion of tensor, symmetric and exterior product. The category of vector bundles over a scheme satisfies the required conditions, as well as the category of vector bundles over a simplicial scheme. For every scheme X, he constructs

- two $(k-1)$-simplicial sets, $S.\tilde{G}^{(k-1)}(X)$ and $\mathrm{Sub}_k(X).$, whose diagonals are weakly equivalent to $S.(X)$, and
- a $(k-1)$-simplicial map $\mathrm{Sub}_k(X). \xrightarrow{\Psi^k} S.\tilde{G}^{(k-1)}(X)$.

His construction is functorial on X and hence induces a map of presheaves.

Grayson has already checked that the operations that he defined induce the usual ones for the Grothendieck group of a suitable category $\mathcal{P}$. Therefore, since the conditions of proposition 2.4.4 are fulfilled, we obtain the following corollary.

Corollary 2.4.5. *Let S be a finite dimensional noetherian scheme. The Adams operations defined by Grayson in [31] agree with the Adams operations defined by Gillet and Soulé in [28], for every scheme in* $\mathrm{ZAR}(S)$. *In particular, they satisfy the usual identities for schemes in* $\mathrm{ZAR}(S)$.

Grayson did not prove that his operations satisfied the identities of a lambda ring. It follows from the previous corollary that they are satisfied for finite dimensional noetherian schemes.

2.5 Morphisms between K-theory and cohomology

2.5.1 Sheaf cohomology as a generalized cohomology theory

Fix $\mathbf{C}$ to be a subsite of the big Zariski site $\mathrm{ZAR}(S)$, as in section 2.3.4.

Consider the Dold-Puppe functor $\mathcal{K}.(\cdot)$ recalled in section 1.2.3, which associates to every cochain complex of abelian groups concentrated in non-positive degrees, G^*, a simplicial abelian group $\mathcal{K}.(G)$, pointed by zero. It satisfies the property that $\pi_i(\mathcal{K}.(G), 0) = H^{-i}(G^*)$.

Now let G^* be an arbitrary cochain complex. Let $(\tau_{\leq n}G)[n]^*$ be the truncation at degree n of G^* followed by the translation by n. That is,

$$(\tau_{\leq n}G)[n]^i = \begin{cases} G^{i+n} & \text{if } i < 0, \\ \ker(d : G^n \to G^{n+1}) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

One defines a simplicial abelian group by

$$\mathcal{K}.(G)_n := \mathcal{K}.((\tau_{\leq n}G)[n]).$$

The simplicial abelian groups $\mathcal{K}.(G)_n$ form an infinite loop spectrum. Moreover, this construction is functorial on G.

Let $\mathcal{F}^*$ be a cochain complex of sheaves of abelian groups in $\mathbf{C}$, and let $\mathcal{K}.(\mathcal{F})$ be the infinite loop spectrum obtained applying section-wise the construction above. For every n, $\mathcal{K}.(\mathcal{F})_n$ is an H-space, since it is a simplicial sheaf of abelian groups.

Lemma 2.5.1 ([37] Prop. B.3.2)**.** *Let $\mathcal{F}^*$ be a bounded below complex of sheaves on $\mathbf{C}$ and let X be a scheme in the underlying category. Then, for all $m \in \mathbb{Z}$,*

$$H^m(X, \mathcal{K}.(\mathcal{F})) \cong H^m_{\mathrm{ZAR}}(X, \mathcal{F}^*).$$

Here, the right hand side is the usual Zariski cohomology and the left hand side is the generalized cohomology of the simplicial sheaf of groups $\mathcal{K}.(\mathcal{F})$. Observe that since $\mathcal{K}.(\mathcal{F})$ is an infinite loop space, we can consider generalized cohomology groups for all $M \in \mathbb{Z}$. We see that the usual Zariski cohomology can be expressed in terms of generalized sheaf cohomology using the Dold-Puppe functor.

2.5.2 Uniqueness of characteristic classes

Now fix a bounded below graded complex of sheaves $\mathcal{F}^*(*)$ of abelian groups, giving a twisted duality cohomology theory in the sense of Gillet, [21]. In loc. cit., Gillet constructed *Chern classes* for higher K-theory. They are given by a map of spaces

$$c_j : \mathbb{K}. \to \mathcal{K}.(\mathcal{F}(j)[2j]), \qquad j \geq 0. \tag{2.5.2}$$

More specifically, they are given by a map

$$\mathbb{Z}_\infty B.GL \to \mathcal{K}.(\mathcal{F}(j)[2j])$$

extended trivially over the $\mathbb{Z}$ component of $\mathbb{K}.$. By example 2.3.6, these maps are weakly additive. In fact, they are weakly additive also with respect to the H-sum of $\mathcal{K}.(\mathcal{F}(j)[2j])$ (see remark 2.3.8).

For any space $X.$, the map induced after taking generalized cohomology,

$$c_j : K_m(X.) \to H^{2j-m}_{\mathrm{ZAR}}(X., \mathcal{F}^*(j)), \qquad j \geq 0,$$

is called the *j-th Chern class.* They are group morphisms for $m > 0$ but only maps for $m = 0$. In this last case, for any vector bundle E over a scheme X, $c_j(E)$ is the standard j-th Chern class taking values in the given cohomology theory.

Using the standard formulas on the Chern classes, one obtains the *Chern character*

$$\mathrm{ch} : K_m(X.) \to \prod_{j\geq 0} H^{2j-m}_{\mathrm{ZAR}}(X., \mathcal{F}^*(j)) \otimes \mathbb{Q},$$

which is now a group morphism for all $m \geq 0$. It is induced by an H-space map

$$\mathrm{ch} : \mathbb{K}. \to \prod_{j\geq 0} \mathcal{K}.(\mathcal{F}(j)[2j]) \otimes \mathbb{Q}.$$

The restriction of ch to $K_0(X)$ is the usual Chern character of a vector bundle.

We will now state the theorems equivalent to theorems 2.4.1 and 2.4.2, for maps from K-theory to cohomology. In order to do this, we should first understand better $c_j(i_N)$ and $\mathrm{ch}(i_N)$ for all j, N. This will be achieved by means of the Grassmanian schemes.

Denote by

$$\begin{aligned} \mathcal{F}^*(*) &= \prod_{i\geq 0,\ j\in\mathbb{Z}} \mathcal{F}^i(j), \\ H^*_{\mathrm{ZAR}}(X., \mathcal{F}^*(*)) &= \prod_{i\geq 0,\ j\in\mathbb{Z}} H^i_{\mathrm{ZAR}}(X., \mathcal{F}^*(j)). \end{aligned}$$

Let $Gr(N,k) = Gr_{\mathbb{Z}}(N,k) \times_{\mathbb{Z}} S$ be the Grassmanian scheme over S. This is a projective scheme over S. Consider $E_{N,k}$ the rank N universal bundle of $Gr(N,k)$ and $\{U_k\} \xrightarrow{p} Gr(N,k)$ its standard trivialization. There is a classifying map of the vector bundle $E_{N,k}$, $\varphi_k : N.U_k \to B.GL_{N/S}$, satisfying $p^*(E_{N,k}) = \varphi_k^*(E^N_.)$. This map induces a map in the Zariski cohomology

$$H^*_{\mathrm{ZAR}}(B.GL_{N/S}, \mathcal{F}^*(*)) \xrightarrow{\varphi_k^*} H^*_{\mathrm{ZAR}}(N.U_k, \mathcal{F}^*(*)) \cong H^*_{\mathrm{ZAR}}(Gr(N,k), \mathcal{F}^*(*)).$$

Moreover, for each m_0, there exists k_0 such that if $m \leq m_0$ and $k \geq k_0$, φ_k^* is an isomorphism on the m-th cohomology group.

Proposition 2.5.3. *Let $\chi_1 = \{\chi_1^N\}$ and $\chi_2 = \{\chi_2^N\}$ be two weakly additive systems of maps*

$$\chi_i^N : \mathbb{K}^N_. \to \mathcal{K}.(\mathcal{F}(*)), \quad i = 1, 2,$$

with respect to the same operation. Then, the induced maps

$$\chi_1, \chi_2 : K_m(X) \to H^*_{\mathrm{ZAR}}(X, \mathcal{F}^*(*))$$

agree on every scheme X in **C**, *if and only if they agree on $X = Gr(N,k)$, for all N and k.*

Proof. One implication is obvious. For the other implication, fix m_0 and let k_0 be an integer such that for every $k \geq k_0$ there is an isomorphism at the m_0 level. Then, there is a commutative diagram

$$\begin{tikzcd}
\lim_{\overrightarrow{M}} H^{-m_0}(B.GL_{N/S}, \mathbb{K}^M_.) \arrow[r, "\chi_1,\chi_2"] \arrow[d, "\varphi_k^*"'] & H^*_{\mathrm{ZAR}}(B.GL_{N/S}, \mathcal{F}^*(*)) \arrow[d, "\cong"', "\varphi_k^*"] \\
\lim_{\overrightarrow{M}} H^{-m_0}(N.U_k, \mathbb{K}^M_.) \arrow[r, "\chi_1,\chi_2"] & H^{-m_0}_{\mathrm{ZAR}}(N.U_k, \mathcal{F}^*(*)) \\
H^{-m_0}(Gr(N,k), \mathbb{K}_.) \arrow[r, "\chi_1,\chi_2"] \arrow[u, "p^*", "\cong"'] & H^{-m_0}_{\mathrm{ZAR}}(Gr(N,k), \mathcal{F}^*(*)). \arrow[u, "\cong", "p^*"']
\end{tikzcd}$$

By theorem 2.3.16, $\chi_1 = \chi_2$ for all schemes X, if they agree on $B.GL_{N/S}$ for all $N \geq 1$. Let $x \in \lim_{\overrightarrow{M}} H^{-m_0}(B.GL_{N/S}, \mathbb{K}^M_.)$. Then,

$$\begin{aligned}
\chi_1(x) = \chi_2(x) \quad &\Leftrightarrow \quad (p^*)^{-1}\varphi_k^*\chi_1(x) = (p^*)^{-1}\varphi_k^*\chi_2(x) \\
&\Leftrightarrow \quad \chi_1(p^*)^{-1}\varphi_k^*(x) = \chi_2(p^*)^{-1}\varphi_k^*(x),
\end{aligned}$$

and since they agree on the Grassmanians, the proposition is proved. □

The following two theorems follow from the results 2.3.16 and 2.3.17, together with the preceding proposition.

Theorem 2.5.4 (Chern classes)**.** *There is a unique way to extend the j-th Chern class of vector bundles over schemes in* $\mathbf{C}$*, by means of a weakly additive system of maps* $\{\rho_N : \mathbb{K}^N_. \to \mathcal{K}.(\mathcal{F}(j)[2j])\}_{N\geq 1}$ *with respect to the H-space operation in* $\mathcal{K}.(\mathcal{F}(*))$*.*

Observe that it follows from the theorem that any weakly additive collection of maps with respect to the H-space operation of $\mathcal{K}.(\mathcal{F}(*))$, inducing the j-th Chern class, is necessarily trivial on the $\mathbb{Z}$-component.

Theorem 2.5.5 (Chern character)**.** *Let*

$$\mathbb{K}_. \longrightarrow \prod_{j\in\mathbb{Z}} \mathcal{K}.(\mathcal{F}(j)[2j])$$

be an H-space map in $\mathrm{Ho}(\mathbf{sT}_*)$*. The induced morphisms*

$$K_m(X) \to \prod_{j\in\mathbb{Z}} H^{2j-m}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j))$$

agree with the Chern character defined by Gillet in [21] for every scheme X, if and only if, the induced map

$$K_0(X) \to \prod_{j\in\mathbb{Z}} H^{2j}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j))$$

is the Chern character for $X = Gr(N,k)$, for all N, k.

Corollary 2.5.6. *There is a unique way to extend the standard Chern character of vector bundles over schemes in* $\mathbf{C}$*, by means of an* H*-space map*

$$\rho : \mathbb{K}. \to \prod_{j \in \mathbb{Z}} \mathcal{K}.(\mathcal{F}(j)[2j]).$$

We deduce from these theorems that any simplicial sheaf map

$$S.\mathcal{P} \to \mathcal{K}.(\mathcal{F}(*))$$

that induces either the Chern character or any Chern class map at the level of $K_0(X)$, induces the Chern character or the Chern class map on the higher K-groups of X. These results hold obviously for the *total Chern class*.

Remark 2.5.7. Let $\mathbf{C}$ be the site of smooth complex varieties and let $\mathcal{D}^*(*)$ be a graded complex computing absolute Hodge cohomology. Burgos and Wang, in [15], constructed a simplicial sheaf map $S.\mathcal{P} \to \mathcal{K}.(\mathcal{D}(*))$ which induces the Chern character on any smooth proper complex variety. A consequence of the last corollary is that their definition agrees with the *Beilinson regulator* (the Chern character for absolute Hodge cohomology).

This is not a new result. Using other methods, Burgos and Wang already proved that the morphism they defined was the same as the Beilinson regulator. The result is proved there by means of the bisimplicial scheme $B.P$ introduced by Schechtman in [51]. This introduced an unnecessary delooping, making the proof generalizable only to sheaf maps inducing group morphisms and introducing irrelevant ingredients to the proof.

Chapter 3

Higher arithmetic Chow groups

Let X be an arithmetic variety over a field, i.e. a regular quasi-projective scheme over an arithmetic field F. In this chapter, we define the *higher arithmetic Chow groups* of X; they are the analog, in the Arakelov context, of the higher Chow groups $CH^p(X,n)$ defined by Bloch in [7].

In the first two sections we review the theory of diagrams of chain complexes due to Beilinson in [5], introduce the refined normalized complex of a cubical abelian group, and give the definition and main properties of the higher algebraic Chow groups given by Bloch.

In the next two sections, we construct a representative of the *Beilinson regulator* by means of the Deligne complex of differential forms and the cubical scheme of affine lines. The regulator that we obtained turns out to be a minor modification of the regulator described by Bloch in [8].

We then develop an analogous construction of the regulator using projective lines instead of affine lines, valid only for proper varieties. This description will be useful in the last section of this chapter.

In the remaining sections we establish the theory of *higher arithmetic Chow groups* $\widehat{CH}^p(X,n)$. We will obtain the following results:

- Let $\widehat{CH}^p(X)$ denote the arithmetic Chow group defined by Burgos. Then, there is an isomorphism
$$\widehat{CH}^p(X) \to \widehat{CH}^p(X,0).$$
- There is a long exact sequence
$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$
$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0,$$
with ρ the Beilinson regulator.
- *Pull-back.* Let $f : X \to Y$ be a morphism between two arithmetic varieties over a field. Then, there is a pull-back morphism
$$\widehat{CH}^p(Y,n) \xrightarrow{f^*} \widehat{CH}^p(X,n),$$

for every p and n, compatible with the pull-back maps on the groups $CH^p(X,n)$ and $H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$.

- ► *Homotopy invariance.* Let $\pi : X \times \mathbb{A}^m \to X$ be the projection on X. Then, the pull-back map

$$\pi^* : \widehat{CH}^p(X,n) \to \widehat{CH}^p(X \times \mathbb{A}^m, n), \quad n \geq 1$$

is an isomorphism.

- ► *Product.* There exists a product on

$$\widehat{CH}^*(X,*) := \bigoplus_{p\geq 0, n\geq 0} \widehat{CH}^p(X,n).$$

It is associative, graded commutative with respect to the degree n and commutative with respect to the degree p.

3.1 Preliminaries

In this section we describe the ideas of Beilinson, in [5], on the simple complexes associated to a diagram of complexes. Next, we introduce a refined version of the normalized complex associated to a cubical abelian group. It will play a key role in the proof of the commutativity of the higher arithmetic Chow groups product.

3.1.1 The simple of a diagram of chain complexes

Following the work of Beilinson, in [5], we study the simple complex associated to a diagram of chain complexes. In this section, all complexes are complexes of abelian groups.

Consider a diagram of morphisms of chain complexes of the form

$$\mathcal{D}_* = \left(\begin{array}{ccccccccc} & B^1_* & & B^2_* & & \cdots & & B^n_* & \\ \overset{f_1}{\nearrow} & & \overset{g_1}{\nwarrow} \quad \overset{f_2}{\nearrow} & & & & & & \overset{g_n}{\nwarrow} \\ A^1_* & & A^2_* & & \cdots & \cdots & \cdots & & A^{n+1}_* \end{array} \right). \qquad (3.1.1)$$

Then, the simple complex associated to $\mathcal{D}_*$ is defined as follows. Let

$$A_* = \bigoplus_{i=1}^{n+1} A^i_*, \qquad B_* = \bigoplus_{i=1}^{n} B^i_*,$$

and consider the morphisms

$$\begin{array}{ccc} A_* & \xrightarrow{\varphi_1} & B_* \\ (a_1, \cdots, a_{n+1}) & \mapsto & (f_1(a_1), \ldots, f_n(a_n)), \end{array}$$

and

$$\begin{array}{ccc} A_* & \xrightarrow{\varphi_2} & B_* \\ (a_1, \cdots, a_{n+1}) & \mapsto & (g_1(a_2), \ldots, g_n(a_{n+1})). \end{array}$$

Definition 3.1.2. Let $\mathcal{D}_*$ be a diagram as (3.1.1). The *simple of* $\mathcal{D}_*$ is the chain complex given by the simple of the difference map $\varphi_1 - \varphi_2$,

$$s(\mathcal{D})_* := s(\varphi_1 - \varphi_2)_*.$$

The long exact sequence associated to the simple of $\varphi_1 - \varphi_2$, gives a long exact sequence

$$\cdots \to H_n(s(\mathcal{D})_*) \to H_n(A_*) \xrightarrow{\varphi_1 - \varphi_2} H_n(B_*) \to H_{n-1}(s(\mathcal{D})_*) \to \cdots \tag{3.1.3}$$

Functoriality.

Definition 3.1.4. Let $\mathcal{D}_*$ and $\mathcal{D}'_*$ be two diagrams as (3.1.1). A morphism of diagrams

$$\mathcal{D}_* \xrightarrow{h} \mathcal{D}'_*$$

is a collection of morphisms

$$A^i_* \xrightarrow{h^A_i} A'^i_*, \qquad B^i_* \xrightarrow{h^B_i} B'^i_*,$$

commuting with the morphisms f_i and g_i, for all i.

Any morphism of diagrams

$$\mathcal{D}_* \xrightarrow{h} \mathcal{D}'_*,$$

induces a morphism on the associated simple complexes

$$s(\mathcal{D})_* \xrightarrow{s(h)} s(\mathcal{D}')_*,$$

by

$$(a_1, \ldots, a_{n+1}, b_1, \ldots, b_n) \mapsto (h^A_1(a_1), \ldots, h^A_{n+1}(a_{n+1}), h^B_1(b_1), \ldots, h^B_n(b_n)).$$

Lemma 3.1.5. *Let h be a morphism of diagrams. If for every i, h^A_i and h^B_i are quasi-isomorphisms, then $s(h)$ is also a quasi-isomorphism.*

Proof. It follows from the five lemma, considering the long exact sequences associated to every diagram. □

We are interested in a particular type of such diagrams, namely, diagrams of the form

$$\mathcal{D}_* = \left(\begin{tikzcd} & B^1_* & & B^2_* \\ A^1_* \arrow[ur, "f_1"] & & A^2_* \arrow[ul, "g_1", "\sim"'] \arrow[ur, "f_2"] & \end{tikzcd} \right), \tag{3.1.6}$$

with g_1 a quasi-isomorphism.

Lemma 3.1.7. *Let $\mathcal{D}_*$ be a diagram like* (3.1.6)*. Then there is a well-defined morphism*

$$\begin{array}{rcl} H_n(A^1_*) & \xrightarrow{\rho} & H_n(B^2_*) \\ [a_1] & \mapsto & f_2 g_1^{-1} f_1[a_1]. \end{array}$$

Moreover, there is a long exact sequence

$$\cdots \to H_n(s(\mathcal{D})_*) \to H_n(A^1_*) \xrightarrow{\rho} H_n(B^2_*) \to H_{n-1}(s(\mathcal{D})_*) \to \cdots \tag{3.1.8}$$

Proof. Let $\varphi = \varphi_1 - \varphi_2$. Consider the following diagram of chain complexes

$$\begin{tikzcd} & H_n(A^1_* \oplus A^2_*) \arrow[r, "\varphi"] \arrow[d, "\chi"] & H_n(B^1_* \oplus B^2_*) \arrow[d, "\chi'"] \arrow[dr, "\psi_2"] & \\ \cdots \arrow[r] & H_n(s(\mathcal{D})_*) \arrow[u, "\psi_1"] \arrow[d, "\psi'_1"'] & & H_{n-1}(s(\mathcal{D})_*) \arrow[r] & \cdots \\ & H_n(A^1_*) \arrow[r, "\rho"] & H_n(B^2_*) \arrow[ur, "\psi'_2"'] & \end{tikzcd}$$

The maps χ and χ' are defined by

$$\chi(a_1, a_2) = a_1, \qquad \text{and} \qquad \chi'(b_1, b_2) = f_2 g_1^{-1}(b_1) + b_2.$$

Observe that χ admits a section

$$\begin{array}{rcl} \sigma : H_n(A^1_*) & \to & H_n(A^1_* \oplus A^2_*) \\ a & \mapsto & (a, g_1^{-1} f_1(a)), \end{array}$$

satisfying $\rho = \chi' \varphi \sigma$. Moreover, χ' admits a section:

$$\begin{array}{rcl} \sigma' : H_n(B^1_*) & \to & H_n(B^1_* \oplus B^2_*) \\ b & \mapsto & (0, b), \end{array}$$

satisfying $\sigma' \rho = \varphi \sigma$.

We have:

- ▷ $\ker \psi_1 = \ker \psi'_1$, since $\chi \psi_1 = \psi'_1$ and $\psi_1 = \sigma \psi'_1$.
- ▷ $\operatorname{im} \psi_2 = \operatorname{im} \psi'_2$, since $\psi'_2 \chi' = \psi_2$ and $\psi'_2 = \psi_2 \sigma'$.

Since the upper row is exact, this gives exactness at $H_*(s(\mathcal{D}))$. Finally, we have

$$\chi \ker \varphi = \ker \rho, \quad \chi \operatorname{im} \psi_1 = \operatorname{im} \psi'_1, \quad \chi' \ker \psi_2 = \ker \psi'_2, \quad \text{and} \quad \chi' \operatorname{im} \varphi = \operatorname{im} \rho,$$

and hence, by the exactness of the first row,

$$\ker \rho = \chi \ker \varphi = \chi \operatorname{im} \psi_1 = \operatorname{im} \psi'_1, \quad \text{and} \quad \operatorname{im} \rho = \chi' \operatorname{im} \varphi = \chi' \ker \psi_2 = \ker \psi'_2.$$

□

Product structure on the simple of a diagram. In [5], Beilinson introduces a product structure on the simple of a diagram of the form (3.1.1). We recall the construction here, but restrict ourselves to diagrams of the form (3.1.6).

Let $\mathcal{D}_*$ and $\mathcal{D}'_*$ be two diagrams as (3.1.6). Consider the diagram obtained by the tensor product of complexes (see example 1.2.6):

$$(\mathcal{D}\otimes\mathcal{D}')_* = \left(\begin{tikzcd}[column sep=small] & B^1_*\otimes B'^1_* & & B^2_*\otimes B'^2_* \\ A^1_*\otimes A'^1_* \arrow[ur, "f_1\otimes f'_1"] & & A^2_*\otimes A'^2_* \arrow[ul, "g_1\otimes g'_1", "\sim"'] \arrow[ur, "f_2\otimes f'_2"] & \end{tikzcd}\right). \tag{3.1.9}$$

For every $\beta\in\mathbb{Z}$, a morphism

$$s(\mathcal{D})_*\otimes s(\mathcal{D}')_* \xrightarrow{\star_\beta} s(\mathcal{D}\otimes\mathcal{D}')_*$$

is defined as follows. For $a\in A, a'\in A', b\in B$ and $b'\in B'$, set:

$$\begin{aligned} a\star_\beta a' &= a\otimes a', \\ b\star_\beta a' &= b\otimes((1-\beta)\varphi_1(a')+\beta\varphi_2(a')), \\ a\star_\beta b' &= (-1)^{\deg a}(\beta\varphi_1(a)+(1-\beta)\varphi_2(a))\otimes b', \\ b\star_\beta b' &= 0. \end{aligned}$$

Lemma 3.1.10 (Beilinson). *(i) The map $\star_\beta$ is a morphism of complexes.*

(ii) For every $\beta,\beta'\in\mathbb{Z}$, $\star_\beta$ is homotopic to $\star_{\beta'}$.

(iii) There is a commutative diagram

$$\begin{tikzcd} s(\mathcal{D})_*\otimes s(\mathcal{D}') \arrow[r, "\star_\beta"] \arrow[d, "\sigma"'] & s(\mathcal{D}\otimes\mathcal{D}')_* \arrow[d, "\sigma"] \\ s(\mathcal{D}')_*\otimes s(\mathcal{D}) \arrow[r, "\star_{1-\beta}"] & s(\mathcal{D}'\otimes\mathcal{D})_*. \end{tikzcd}$$

(iv) The multiplications $\star_0$ and $\star_1$ are associative.

Remark 3.1.11. Let $\mathcal{D}_*$ be a diagram like (3.1.6). Assume that each of the chain complexes A^i_* and B^i_* has a product structure which commutes with the maps f_* and g_*. Denote by $\bullet$ the product structure on A^i_* and on B^i_* for all i. Then, there is a morphism

$$\begin{aligned} s(\mathcal{D}\otimes\mathcal{D})_* &\to s(\mathcal{D})_* \\ c\otimes d &\mapsto c\bullet d, \end{aligned}$$

which is the product $\bullet$ at each component. Composing with $\star_\beta$, we obtain a pairing for $s(\mathcal{D})_*$

$$\bullet_\beta : s(\mathcal{D})_*\otimes s(\mathcal{D})_* \xrightarrow{\star_\beta} s(\mathcal{D}\otimes\mathcal{D})_* \xrightarrow{\bullet} s(\mathcal{D})_*.$$

If the product structures in A^i_* and B^i_* are both associative and commutative, the product $\bullet_\beta$ will be associative for $\beta = 0, 1$. By 3.1.10, (iii), under the transformation σ, the multiplication $\bullet_\beta$ is transformed to $\bullet_{1-\beta}$. Since all products $\bullet_\beta$ are homotopic, $\bigoplus_{n\in\mathbb{Z}} H_n(s(\mathcal{D})_*)$ becomes a graded commutative and associative ring.

3.1.2 Refined normalized complex of a cubical abelian group

For certain cubical abelian groups, the normalized chain complex constructed in 1.2.34 can be simplified, up to homotopy equivalence, by considering the elements which belong to the kernel of all faces but δ_1^0.

Definition 3.1.12. Let $C.$ be a cubical abelian group. Let N_0C_* be the complex defined by

$$N_0C_n = \bigcap_{i=1}^{n} \ker \delta_i^1 \cap \bigcap_{i=2}^{n} \ker \delta_i^0, \quad \text{and differential} \quad \delta = -\delta_1^0. \tag{3.1.13}$$

The proof of the next proposition is analogous to the proof of theorem 4.4.2 in [6]. The result is proved there only for the cubical abelian group defining the higher Chow complex (see section 3.2). We give here a general statement, valid for a certain type of cubical abelian groups.

Proposition 3.1.14. *Let $C.$ be a cubical abelian group. Assume that it comes equipped with a collection of maps*

$$h_j : C_n \to C_{n+1}, \qquad j = 1, \dots, n,$$

such that, for any $k = 0, 1$,

$$\begin{aligned} \delta_j^1 h_j &= \delta_{j+1}^1 h_j = s_j \delta_j^1, \\ \delta_j^0 h_j &= \delta_{j+1}^0 h_j = id, \\ \delta_i^k h_j &= \begin{cases} h_{j-1}\delta_i^k & i < j, \\ h_j \delta_{i-1}^k & i > j+1. \end{cases} \end{aligned}$$

Then, the inclusion of complexes

$$i : N_0C_* \hookrightarrow NC_*$$

is a homotopy equivalence.

Proof. Let $g_j : NC_* \to NC_{*+1}$ be defined as $g_j = (-1)^{j+1} h_j$ if $j \leq n$ and $g_j = 0$ if $j > n$. Then there is a morphism of chain complexes

$$H_j = (id - \delta g_{j+1} - g_j \delta) : NC_* \to NC_*.$$

Observe that by hypothesis, for every j and k, there exists a map f and an index k' such that $\delta_k^1 h_j = f \delta_{k'}^1$. Therefore, for every $x \in NC_n$, we have $\delta_k^1 H_j(x) = 0$ and thus $H_j(x) \in NC_*$.

Let $\varphi : NC_n \to NC_n$ be the chain morphism given by

$$\varphi := H_1 \circ \cdots \circ H_{n-1}.$$

By definition, it is homotopy equivalent to the identity.

Let $x \in NC_n$. Then,

$$\delta h_{j+1}(x) \quad = \quad \sum_{i=1}^{n+1}(-1)^i \delta_i^0 h_{j+1}(x) = \sum_{i=1}^{j}(-1)^i h_j \delta_i^0(x) + \sum_{i=j+3}^{n+1}(-1)^i h_{j+1}\delta_{i-1}^0(x),$$

and

$$h_j\delta(x) \quad = \quad \sum_{i=1}^{n}(-1)^i h_j \delta_i^0(x).$$

Hence,

$$\delta g_{j+1}(x) + g_j\delta(x) = \sum_{i=j+2}^{n}(-1)^i g_{j+1}\delta_i^0(x) + \sum_{i=j+1}^{n}(-1)^{i+1} g_j \delta_i^0(x).$$

Therefore, if x is such that $\delta_i^0(x) = 0$ for $i > j$, then $\delta g_{j+1}(x) + g_j\delta(x) = 0$ and thus, $H_j(x) = x$. Moreover, in this case

$$\delta_j^0(x - \delta g_j(x) - g_{j-1}\delta(x)) \quad = \quad \delta_j^0(x) - \delta_j^0(x) = 0.$$

It follows that φ is the identity on N_0C_* and that its image lies in N_0C_*. Hence, $\varphi \circ i$ is 4he identity while $i \circ \varphi$ is homotopy equivalent to the identity. $\square$

Remark 3.1.15. The maps h_j of proposition 3.1.14 behave almost like the degeneracy maps of a cubical abelian group, except that they vanish after composition with δ_j^1 and δ_{j+1}^1. Consider the morphisms given by

$$\begin{array}{ccc} \{0,1\}^n & \xrightarrow{h_k} & \{0,1\}^{n-1} \\ (j_1, \ldots, j_n) & \mapsto & (j_1, \ldots, j_k \text{ or } j_{k+1}, \ldots, j_n), \end{array}$$

where by "or" we mean the boolean operator or. Recall that the category $\langle \mathbf{1} \rangle$ was defined in section 1.2.4. It is the category whose objects are the sets $\{0,1\}^n$ and the morphisms are generated by the face and degeneracy maps. Let $\langle \mathbf{1} \rangle'$ be the category whose objects are the sets $\{0,1\}^n$ and the morphisms are generated by the face and degeneracy maps together with the morphisms $\{h_k\}_k$. Then, the objects in proposition 3.1.14 are functors from the category $\langle \mathbf{1} \rangle'$ to $\mathbf{Ab}$.

Remark 3.1.16. Recall that to every cubical abelian group $C.$ we have associated four chain complexes: C_*, NC_*, N_0C_* and $\widetilde{C}_*$. In some situations it will be necessary to consider the cochain complexes associated to these chain complexes. In this case we will write, respectively,

$$C^*, NC^*, N_0C^* \quad \text{and} \quad \widetilde{C}^*.$$

Cubical cochain complexes. We finish this section with a couple of results that will be needed in the forthcoming sections. A *cubical cochain complex* $X^*_\cdot$ is a collection of cochain complexes (X^*_n, d), together with face and degeneracy cochain maps satisfying the cubical identities. Observe that, for every m, the cochain complexes $NX^*_m, N_0X^*_m$ and $\widetilde{X}^*_m$ are defined.

If $X^*_\cdot$ is a cubical cochain complex, then there is an associated 2-iterated cochain complex, X^{**}, whose (n,m)-th graded piece is

$$X^{nm} = X^n_{-m}$$

and whose differentials are (d,δ). The 2-iterated cochain complexes

$$NX^{**}, N_0X^{**} \quad \text{and} \quad \widetilde{X}^{**}$$

are defined analogously.

Proposition 3.1.17. *Let $X^*_\cdot, Y^*_\cdot$ be two cubical cochain complexes and let $f : X^*_\cdot \to Y^*_\cdot$ be a morphism. Assume that for every m, the cochain morphism*

$$X^*_m \xrightarrow{f_m} Y^*_m$$

is a quasi-isomorphism. Then, the induced morphisms

$$\begin{array}{ccc} NX^*_m & \xrightarrow{f_m} & NY^*_m, \\ \widetilde{X}^*_m & \xrightarrow{f_m} & \widetilde{Y}^*_m, \end{array}$$

are quasi-isomorphisms.

Proof. Since the maps f_m commute with the face and degeneracy maps, there are induced cochain maps

$$NX^*_m \xrightarrow{f_m} NY^*_m, \qquad DX^*_m \xrightarrow{f_m} DY^*_m,$$

and decompositions $X^*_m = NX^*_m \oplus DX^*_m$ and $Y^*_m = NY^*_m \oplus DY^*_m$. Then, the proposition follows from the decompositions

$$\begin{array}{rcl} H^r(X^*_m) & = & H^r(NX^*_m) \oplus H^r(DX^*_m), \\ H^r(Y^*_m) & = & H^r(NY^*_m) \oplus H^r(DY^*_m). \end{array}$$

□

Lemma 3.1.18. *Let $X^*_\cdot$ be a cubical cochain complex. Then the natural morphism*

$$H^r(NX^*_n) \xrightarrow{f} NH^r(X^*_n)$$

is an isomorphism for all $n \geq 0$.

Proof. The cohomology groups $H^r(X^*_\cdot)$ have a cubical abelian group structure. Hence, by lemma 1.2.35, there is a decomposition

$$H^r(X^*_\cdot) = NH^r(X^*_\cdot) \oplus DH^r(X^*_\cdot).$$

In addition, there is a decomposition $X^*_n = NX^*_n \oplus DX^*_n$. Therefore

$$H^r(X^*_\cdot) = H^r(NX^*_\cdot) \oplus H^r(DX^*_\cdot).$$

The lemma follows from the fact that the identity morphism in $H^r(X^*_\cdot)$ maps $NH^r(X^*_\cdot)$ to $H^r(NX^*_\cdot)$ and $DH^r(X^*_\cdot)$ to $H^r(DX^*_\cdot)$.

□

3.2 The cubical Bloch complex

We recall here the definition and main properties of the *higher Chow groups* defined by Bloch in [7]. Initially, they were defined using the chain complex associated to a simplicial abelian group. However, for reasons that will become apparent in the later sections, it is more convenient to use the cubical presentation, as given by Levine in [41]. The main reason is that the cubical setting is more suitable for defining products.

3.2.1 Higher Chow groups

Fix a base field k and let $\mathbb{P}^1$ be the projective line over k. Let

$$\square = \mathbb{P}^1 \setminus \{1\} \cong \mathbb{A}^1.$$

The cartesian product $(\mathbb{P}^1)^\cdot$ has a cocubical scheme structure. Specifically, the face and degeneracy maps

$$\begin{aligned} \delta^i_j : (\mathbb{P}^1)^n &\to (\mathbb{P}^1)^{n+1}, \quad i = 1, \dots, n, \ j = 0, 1, \\ \sigma^i : (\mathbb{P}^1)^n &\to (\mathbb{P}^1)^{n-1}, \quad i = 1, \dots, n, \end{aligned}$$

are defined as

$$\begin{aligned} \delta^i_0(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, (0:1), x_i, \dots, x_n), \\ \delta^i_1(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, (1:0), x_i, \dots, x_n), \\ \sigma^i(x_1, \dots, x_n) &= (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n). \end{aligned}$$

These maps satisfy the usual identities for a cocubical object in a category, and leave invariant $\square$. Hence, both $(\mathbb{P}^1)^\cdot$ and $\square^\cdot$ are cocubical schemes. An r-dimensional *face* in $\square^n$ is any subscheme of the form $\delta^{i_1}_{j_1} \cdots \delta^{i_r}_{j_r}(\square^{n-r})$.

Let X be an equidimensional quasi-projective algebraic scheme of dimension d over the field k. Let $Z^p(X, n)$ be the free abelian group generated by the codimension p closed irreducible subvarieties of $X \times \square^n$, which intersect properly all the faces of $\square^n$. Then the pull-backs by δ_0^i and δ_1^i are well defined and thus there are induced maps

$$\delta_i^j : Z^p(X,n) \to Z^p(X,n-1), \quad i = 1,\dots,n, \quad j = 0,1.$$

Since for all $i = 1,\dots,n$ the map $\sigma^i : \square^n \to \square^{n-1}$ is a flat map, there are pull-back morphisms

$$\sigma_i : Z^p(X,n) \to Z^p(X,n+1).$$

From the cocubical identities in $\square^{\cdot}$, it follows that $Z^p(X,\cdot)$ is a cubical abelian group. Let $(Z^p(X,*),\delta)$ be the associated chain complex (see section 1.2.4). Let $D^p(X,*)$ be the subcomplex of *degenerate elements*, and let

$$\widetilde{Z}^p(X,*) := Z^p(X,*)/D^p(X,*).$$

Alternatively, we consider the *normalized chain complex* associated to $Z^p(X,*)$,

$$Z^p(X,*)_0 := NZ^p(X,*) = \bigcap_{i=1}^{n} \ker \delta_i^1.$$

Definition 3.2.1. Let X be a quasi-projective equidimensional algebraic scheme over a field k. The *higher Chow groups* defined by Bloch are

$$CH^p(X,n) := H_n(\widetilde{Z}^p(X,*)) = H_n(Z^p(X,*)_0).$$

Remark 3.2.2. For $n = 0$, $CH^p(X,0) = CH^p(X)$ is the classical Chow group of X.

Let N_0 be the refined normalized complex of definition (3.1.12) and let $Z^p(X,*)_{00}$ be the complex

$$Z^p(X,*)_{00} := N_0 Z^p(X,n).$$

Definition 3.2.3. Let $n \geq 0$. For every $j = 1,\dots,n+1$, we define the map

$$\begin{array}{rcl} \square^{n+1} & \xrightarrow{h^j} & \square^n \\ (x_1,\dots,x_{n+1}) & \mapsto & (x_1,\dots,(x_j-1)\cdot(x_{j+1}-1),\dots,x_{n+1}). \end{array} \tag{3.2.4}$$

Lemma 3.2.5. *The inclusion*

$$Z^p(X,*)_{00} := N_0 Z^p(X,n) \to Z^p(X,*)_0$$

is a homotopy equivalence.

Proof. It follows from proposition 3.1.14 with the maps 3.2.3. $\square$

3.2.2 Functoriality and product structure

Functoriality. It follows easily from the definition that the complex $Z^p(X,*)_0$ is:

- ▷ covariant with respect to proper maps (with a shift in the grading), and,
- ▷ contravariant for flat maps.

Let $f : X \to Y$ be an arbitrary map between two smooth schemes X, Y. Let $Z^p_f(Y,n)_0 \subset Z^p(Y,n)_0$ be the subgroup generated by the codimension p irreducible subvarieties $Z \subset Y \times \square^n$, intersecting properly the faces of $\square^n$ and such that the pull-back $X \times Z$ intersects properly the graph of f, Γ_f. Then, with the differential induced by the differential of $Z^p(Y,*)_0$, $Z^p_f(Y,*)_0$ is a chain complex and the inclusion of complexes $Z^p_f(Y,*)_0 \subseteq Z^p(Y,*)_0$ is a quasi-isomorphism. Moreover, the pull-back by f is defined for algebraic cycles in $Z^p_f(Y,*)_0$ and hence there is a well-defined pull-back morphism

$$CH^p(Y,n) \xrightarrow{f^*} CH^p(X,n).$$

A proof of this fact can be found in [42], § 3.5.

Product structure. Let X and Y be quasi-projective algebraic schemes over k. Then, there is a chain morphism

$$s(Z^p(X,*)_0 \otimes Z^q(Y,*)_0) \xrightarrow{\cup} Z^{p+q}(X \times Y,*)_0$$

inducing exterior products

$$CH^p(X,n) \otimes CH^q(Y,m) \xrightarrow{\cup} CH^{p+q}(X \times Y, n+m).$$

Specifically, let Z be a codimension p irreducible subvariety of $X \times \square^n$, intersecting properly the faces of $\square^n$ and let W be a codimension q irreducible subvariety of $Y \times \square^m$, intersecting properly the faces of $\square^m$. Then, the codimension $p+q$ subvariety

$$Z \times W \subseteq X \times \square^n \times Y \times \square^m \cong X \times Y \times \square^n \times \square^m \cong X \times Y \times \square^{n+m},$$

intersects properly the faces of $\square^{n+m}$. By linearity, we obtain a morphism

$$Z^p(X,n) \otimes Z^q(Y,m) \xrightarrow{\cup} Z^{p+q}(X \times Y, n+m).$$

Observe that if $Z \in Z^p(X,n)_0$ and $W \in Z^q(Y,m)_0$, then $Z \times W \in Z^{p+q}(X \times Y, n+m)_0$. Hence there is a chain morphism

$$s(Z^p(X,*)_0 \otimes Z^q(Y,*)_0) \xrightarrow{\cup} Z^{p+q}(X \times Y,*)_0,$$

inducing an external product

$$\cup : CH^p(X,n) \otimes CH^q(Y,m) \to CH^{p+q}(X \times Y, n+m), \tag{3.2.6}$$

for all p, q, n, m.

If X is smooth, then the pull-back by the diagonal map $\Delta : X \to X \times X$ is defined on the higher Chow groups

$$CH^p(X \times X, *) \xrightarrow{\Delta^*} CH^p(X, *).$$

Therefore, for all p, q, n, m, we obtain an internal product

$$\cup : CH^p(X, n) \otimes CH^q(X, m) \to CH^{p+q}(X \times X, n+m) \xrightarrow{\Delta^*} CH^{p+q}(X, n+m). \quad (3.2.7)$$

In the derived category of chain complexes, the internal product is given by the morphism

$$\begin{tikzcd} s(Z^p(X,*)_0 \otimes Z^q(X,*)_0) \arrow[r, "\cup"] & Z^{p+q}(X \times X, *)_0 & \\ & Z^{p+q}_{\Delta}(X \times X, *)_0 \arrow[u, "\sim"'] \arrow[r, "\Delta^*"] & Z^{p+q}(X, *)_0. \end{tikzcd}$$

Proposition 3.2.8. *Let X be a quasi-projective algebraic scheme over k. The pairing (3.2.7) defines an associative product on $CH^*(X, *) = \bigoplus_{p,n} CH^p(X, n)$. This product is commutative with respect to the codimension degree p and graded commutative with respect to the degree given by n.*

Proof. See [41], Theorem 5.2. □

3.2.3 Other properties of the higher Chow groups

Codimension 1 cycles. The following theorem is proven by Bloch, in [7].

Theorem 3.2.9. *Let X be a regular noetherian scheme over a field k. Then*

$$CH^1(X, m) = \begin{cases} \mathrm{Pic}(X) & m = 0, \\ \Gamma(X, \mathbb{G}_m) & m = 1, \\ 0 & m \neq 0, 1. \end{cases}$$

The isomorphism $CH^1(X, 0) \cong \mathrm{Pic}(X)$ is the standard correspondence between codimension 1 algebraic cycles and invertible sheaves. The isomorphism $CH^1(X, 1) \cong \Gamma(X, \mathbb{G}_m)$ is defined as follows. By the proof of the theorem in loc. cit., every class $[Z] \in CH^1(X, 1)$ is represented by an element of the form $Z = \mathrm{div}(f)$, with f a function whose restriction to $X \times \{0\}$ and to $X \times \{1\}$ is invertible. Then, the isomorphism maps $[Z]$ to the invertible function $f_{|X \times \{0\}} f^{-1}_{|X \times \{1\}}$.

Homotopy invariance. Let $\pi : X \times \mathbb{A}^n \to X$ be the projection onto X. Since this map is flat, the pull-back by π is well defined.

As proved by Bloch in [7], for all n, the morphism

$$\pi^* : Z^p(X, *)_0 \to Z^p(X \times \mathbb{A}^n, *)_0$$

is a quasi-isomorphism.

Projective Dold-Thom isomorphism. Let X be a quasi-projective scheme over a field k and let E be a rank n vector bundle over X. Let $\mathbb{P}(E)$ be the associated projective bundle and let $\pi : \mathbb{P}(E) \to X$ be the structural morphism. Let $\xi \in \mathrm{Pic}(X) \cong CH^1(X,0)$ be the class of $\mathcal{O}_{\mathbb{P}(E)}(1)$.

As proved by Bloch in [7], for every $m \geq 0$, there is an isomorphism

$$\bigoplus_{i=0}^{n-1} \xi^i \cdot \pi^* : \bigoplus_{i=0}^{n-1} \bigoplus_{p\geq 0} CH^p(X,m) \longrightarrow \bigoplus_{q\geq 0} CH^q(\mathbb{P}(E),m).$$

3.3 Differential forms and higher Chow groups

In this section we construct a complex of differential forms, which is quasi-isomorphic to the complex $Z^p(X,*)_0 \otimes \mathbb{R}$. The key point is the isomorphism (1.4.7).

This complex is very similar to the complex introduced by Bloch in [8] in order to construct the cycle map for the higher Chow groups. In both constructions one considers a 2-iterated complex of differential forms on a cubical or simplicial scheme. The main difference is the direction of the truncation. We truncate the 2-iterated complex at the degree given by the differential forms, while he truncated the complex at the degree given by the simplicial scheme.

Assume from now on that all schemes are equidimensional complex algebraic manifolds.

3.3.1 Differential forms and affine lines

For every $n, p \geq 0$, let $\mathcal{D}^*_{\log}(X \times \square^n, p)$ be the Deligne complex of differential forms in $X \times \square^n$, with logarithmic singularities at infinity (see section 1.4.1). For every $i = 1, \ldots, n$ and $l = 0, 1$, the structural maps δ^i_l and σ^i of the cocubical scheme $\square^{\cdot}$ induce cochain morphisms

$$\begin{aligned} \delta^l_i : \mathcal{D}^*_{\log}(X \times \square^n, p) &\rightarrow \mathcal{D}^*_{\log}(X \times \square^{n-1}, p), \\ \sigma_i : \mathcal{D}^*_{\log}(X \times \square^{n-1}, p) &\rightarrow \mathcal{D}^*_{\log}(X \times \square^n, p), \end{aligned}$$

which define a cubical structure on $\mathcal{D}^r_{\log}(X \times \square^*, p)$ for every r and p.

Consider the 2-iterated cochain complex given by

$$\mathcal{D}^{r,-n}_{\mathbb{A}}(X,p) = \mathcal{D}^r_{\log}(X \times \square^n, p)$$

and with differential $(d_{\mathcal{D}}, \delta = \sum_{i=1}^n (-1)^i(\delta^0_i - \delta^1_i))$. Recall that, as fixed in subsection 1.4.1, we defined the complex $\mathcal{D}^*_{\log}(Y,p)$ as the Deligne complex of differential forms truncated at the degree $2p$. Let

$$\mathcal{D}^*_{\mathbb{A}}(X,p) = s(\mathcal{D}^{*,*}_{\mathbb{A}}(X,p))$$

be the simple complex associated to the 2-iterated complex $\mathcal{D}^{*,*}_{\mathbb{A}}(X,p)$. Then the differential d_s in $\mathcal{D}^*_{\mathbb{A}}(X,p)$ is given by

$$d_s(\alpha) = d_{\mathcal{D}}(\alpha) + (-1)^r \delta(\alpha),$$

for every $\alpha \in \mathcal{D}_{\mathbb{A}}^{r,-n}(X,p)$.

For every r,n, let

$$\mathcal{D}_{\log}^{r}(X \times \square^n, p)_0 = N\mathcal{D}_{\log}^{r}(X \times \square^n, p),$$

be the normalized complex of definition (1.2.37), and let

$$\mathcal{D}_{\mathbb{A}}^{r,-n}(X,p)_0 = \mathcal{D}_{\log}^{r}(X \times \square^n, p)_0.$$

Denote by $\mathcal{D}_{\mathbb{A}}^{*}(X,p)_0$ the associated simple complex.

Proposition 3.3.1. *The natural morphism of complexes*

$$\mathcal{D}_{\log}^{*}(X,p) = \mathcal{D}_{\mathbb{A}}^{*,0}(X,p)_0 \to \mathcal{D}_{\mathbb{A}}^{*}(X,p)_0$$

is a quasi-isomorphism.

Proof. Consider the second quadrant spectral sequence with E_1 term given by

$$E_1^{r,-n} = H^r(\mathcal{D}_{\mathbb{A}}^{*,-n}(X,p)_0).$$

When it converges, it converges to the cohomology groups $H^*(\mathcal{D}_{\mathbb{A}}^{*}(X,p)_0)$. Since it is a second quadrant spectral sequence, to ensure convergence, we need to truncate the complexes at degree $2p$.

If we see that, for all $n>0$, the cohomology of the complex $\mathcal{D}_{\mathbb{A}}^{*,-n}(X,p)_0$ is zero, the spectral sequence converges and the proposition is proven. By the homotopy invariance of Deligne-Beilinson cohomology, there is an isomorphism

$$\delta_1^1 \circ \cdots \circ \delta_1^1 : H^*(\mathcal{D}_{\log}^{*}(X \times \square^n, p)) \to H^*(\mathcal{D}_{\log}^{*}(X,p)).$$

By definition, the image of $H^*(\mathcal{D}_{\log}^{*}(X \times \square^n, p)_0)$ under the isomorphism is zero. Therefore, since $H^*(\mathcal{D}_{\log}^{*}(X \times \square^n, p)_0)$ is a direct summand of $H^*(\mathcal{D}_{\log}^{*}(X \times \square^n, p))$ (by 1.2.35), it has zero cohomology for all $n>0$. $\square$

Remark 3.3.2. Our construction differs from the construction given by Bloch, in [8], in two points:

- ▷ He considered the 2-iterated complex of differential forms on the simplicial scheme $\mathbb{A}^n$, instead of the differential forms on the cubical scheme $\square^n$.
- ▷ In order to ensure the convergence of the spectral sequence in the proof of last proposition, he truncated the 2-iterated complex in the direction given by the affine schemes.

We define the complex $\mathcal{D}_{\mathbb{A}}^{*}(X,p)_{00}$ to be the simple complex associated to the 2-iterated complex with

$$\mathcal{D}_{\mathbb{A}}^{r,-n}(X,p)_{00} = N_0\mathcal{D}_{\log}^{r}(X \times \square^n, p).$$

Corollary 3.3.3. *The natural morphism of complexes*

$$\mathcal{D}^*_{\log}(X,p) = \mathcal{D}^{*,0}_{\mathbb{A}}(X,p)_{00} \to \mathcal{D}^*_{\mathbb{A}}(X,p)_{00}$$

is a quasi-isomorphism.

Proof. It follows from proposition 3.3.1, proposition 3.1.14 (with maps $\{h_j\}$ induced by the maps defined in 3.2.3) and proposition 3.1.17. □

3.3.2 A complex with differential forms for the higher Chow groups

Let $\mathcal{Z}^p_{n,X}$ be the set of all codimension p closed subvarieties of $X \times \square^n$ intersecting properly the faces of $\square^n$. When there is no source of confusion, we simply write $\mathcal{Z}^p_n$ or even $\mathcal{Z}^p$. Consider the cubical abelian group

$$\mathcal{H}^p(X,*) := H^{2p}_{\mathcal{D},\mathcal{Z}^p_*}(X \times \square^*, \mathbb{R}(p)), \tag{3.3.4}$$

with faces and degeneracies induced by δ^l_i and σ_i. Let $\mathcal{H}^p(X,*)_0$ be the associated normalized complex.

Lemma 3.3.5. *Let X be a complex algebraic manifold. There is an isomorphism of chain complexes*

$$f_1 : Z^p(X,*)_0 \otimes \mathbb{R} \xrightarrow{\cong} \mathcal{H}^p(X,*)_0,$$

sending z to $cl(z)$.

Proof. It follows from the isomorphism (1.4.7). □

Remark 3.3.6. Observe that the complex $\mathcal{H}^p(X,*)_0$ has the same functorial properties as $Z^p(X,*)_0 \otimes \mathbb{R}$.

Let $\mathcal{D}^{*,*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ be the 2-iterated cochain complex, whose component of bidegree $(r,-n)$ is

$$\mathcal{D}^r_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0,$$

and whose differentials are $(d_{\mathcal{D}}, \delta)$. As usual, we denote by $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ the associated simple complex and by d_s its differential.

Let $\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ be the chain complex whose n-graded piece is $\mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$.

Proposition 3.3.7. *The morphism*

$$\begin{array}{ccc} \mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 & \xrightarrow{g_1} & \mathcal{H}^p(X,n)_0 \\ ((\omega_n,g_n),\dots,(\omega_0,g_0)) & \mapsto & [(\omega_n,g_n)] \end{array}$$

is a quasi-isomorphism of chain complexes.

Proof. Consider the second quadrant spectral sequence with E_1-term

$$E_1^{r,-n} = H^r(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0).$$

Observe that, by construction, $E_1^{r,-n} = 0$ for all $r > 2p$. Moreover, for all $r < 2p$ and for all n, the semipurity property of Deligne-Beilinson cohomology implies that

$$H^r(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)) = 0. \tag{3.3.8}$$

Hence, by proposition 3.1.17,

$$H^r(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0) = 0, \qquad r < 2p.$$

Therefore, the E_1-term of the spectral sequence is

$$E_1^{r,-n} = \begin{cases} 0 & \text{if } r \neq 2p, \\ H^{2p}(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0) & \text{if } r = 2p. \end{cases}$$

Finally, from proposition 3.1.18, it follows that the natural map

$$H^{2p}(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0) \to \mathcal{H}^p(X, n)_0$$

is an isomorphism, and the proposition is proved. $\square$

We denote

$$CH^p(X, n)_{\mathbb{R}} = CH^p(X, n) \otimes \mathbb{R}.$$

Corollary 3.3.9. *Let $z \in CH^p(X, n)_{\mathbb{R}}$ be the class of an algebraic cycle z in $X \times \square^n$. By the isomorphisms of lemma 3.3.5 and proposition 3.3.7, the algebraic cycle z is represented, in $H^{2p-n}(\mathcal{D}_{\mathbb{A},\mathcal{Z}^p}(X, p)_0)$, by any cycle*

$$((\alpha_n, g_n), \dots, (\alpha_0, g_0)) \in \mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X, p)_0$$

such that

$$cl(z) = [(\alpha_n, g_n)].$$

Remark 3.3.10. Let $D_n^r = \sum_{i=1}^n \sigma_i(\mathcal{D}^r_{\log}(X \times \square^n, p)) \subset \mathcal{D}^r_{\log}(X \times \square^n, p)$ be the subcomplex of degenerate elements. Then, we denote

$$\widetilde{\mathcal{D}}^{r,n}_{\mathbb{A}}(X, p) = \mathcal{D}^{r,n}_{\mathbb{A}}(X, p)/D_n^r,$$

and let $\widetilde{\mathcal{D}}^*_{\mathbb{A}}(X, p)$ denote the associated simple complex. By lemma 1.2.36, there is an isomorphism of complexes

$$\mathcal{D}^*_{\mathbb{A}}(X, p)_0 \cong \widetilde{\mathcal{D}}^*_{\mathbb{A}}(X, p).$$

Hence, all the results could be stated using the complex $\widetilde{\mathcal{D}}^*_{\mathbb{A}}(X, p)$ instead of $\mathcal{D}^*_{\mathbb{A}}(X, p)_0$.

One defines analogously the complex $\widetilde{\mathcal{D}}^*_{\mathbb{A},\mathcal{Z}^p}(X, p)$, which is isomorphic to $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X, p)_0$. In general, the notation $\widetilde{\mathcal{D}}^*_*(X, p)$ refers to the construction obtained taking the quotient by the degenerate elements.

3.3.3 Functoriality of $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$

In many aspects, the complex $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ behaves like the complex $Z^*(X,*)_0$.

Lemma 3.3.11. *Let $f : X \to Y$ be a flat map between two equidimensional complex algebraic manifolds. Then there is a pull-back map*

$$f^* : \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(Y,p)_0 \to \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0.$$

Proof. We will see that in fact there is a map of iterated complexes

$$f^* : \mathcal{D}^{r,-n}_{\mathbb{A},\mathcal{Z}^p}(Y,p) \to \mathcal{D}^{r,-n}_{\mathbb{A},\mathcal{Z}^p}(X,p).$$

Let Z be a codimension p subvariety of $Y \times \square^n$ intersecting properly the faces of $\square^n$. Since f is flat, the $f^*(Z)$ is defined. It is a codimension p subvariety of $X \times \square^n$ intersecting properly the faces of $\square^n$, and whose support is $f^{-1}(Z)$. Then, by [24] 1.3.3, the pull-back of differential forms gives a morphism

$$\mathcal{D}^*_{\log}(Y \times \square^n \setminus Z, p) \xrightarrow{f^*} \mathcal{D}^*_{\log}(X \times \square^n \setminus f^{-1}(Z), p).$$

Hence, there is an induced morphism

$$\mathcal{D}^*_{\log}(Y \times \square^n \setminus \mathcal{Z}^p_Y, p) \xrightarrow{f^*} \varinjlim_{Z \in \mathcal{Z}^p_Y} \mathcal{D}^*_{\log}(X \times \square^n \setminus f^{-1}(Z), p) \to \mathcal{D}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p_X, p),$$

and thus, there is a pull-back morphism

$$f^* : \mathcal{D}^{*,-n}_{\mathbb{A},\mathcal{Z}^p}(Y,p) \to \mathcal{D}^{*,-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)$$

compatible with the differential δ. □

Remark 3.3.12. The pull-back defined here agrees with the pull-back defined by Bloch under the isomorphisms of lemma 3.3.5 and proposition 3.3.7. Indeed, let $f : X \to Y$ be a flat map. Then, if Z is an irreducible subvariety of Y and (ω, g) a couple representing the class of $[Z]$ in the Deligne-Beilinson cohomology with support, then the couple $(f^*\omega, f^*g)$ represents the class of $[f^*(Z)]$ (see [24], theorem 3.6.1).

Proposition 3.3.13. *Let $f : X \to Y$ be a morphism of equidimensional complex algebraic manifolds. Let $\mathcal{Z}^p_f$ be the subset consisting of the subvarieties Z of $Y \times \square^n$ intersecting properly the faces of $\square^n$ and such that $X \times Z \times \square^n$ intersects properly the graph of f, Γ_f. Then,*

(i) The complex $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p)_0$ is quasi-isomorphic to $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(Y,p)_0$.

(ii) There is a well-defined pull-back

$$f^* : \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p)_0 \to \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0.$$

Proof. Arguing as in the proof of the previous proposition, there is a pull-back map

$$f^* : \mathcal{D}^*_{\log}(Y \times \square^n \setminus \mathcal{Z}^p_f, p) \xrightarrow{f^*} \mathcal{D}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p),$$

inducing a morphism

$$f^* : \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p) \to \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p),$$

and hence a morphism

$$f^* : \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p)_0 \to \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0.$$

All that remains to be shown is that the inclusion

$$\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p)_0 \xrightarrow{i} \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(Y,p)_0$$

is a quasi-isomorphism. By the quasi-isomorphisms of 3.2.2 and 3.3.7, there is a commutative diagram

$$\begin{tikzcd}
Z^p_f(Y,*) \otimes \mathbb{R} \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p)_0 \arrow[d, "i"] \\
Z^p(Y,*) \otimes \mathbb{R} \arrow[r, "\sim"] & \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(Y,p)_0.
\end{tikzcd}$$

The proof that the upper horizontal arrow is a quasi-isomorphism is analogous to the proof of proposition 3.3.7. Thus, we deduce that i is a quasi-isomorphism. $\square$

3.4 Algebraic cycles and the Beilinson regulator

In this section we define a chain morphism $Z^p(X,*) \to \mathcal{D}^{2p-*}_{\log}(X,p)$, in the derived category of chain complexes, that, after composition with the isomorphism $K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}$ induces in homology the Beilinson regulator.

The construction is analogous to the definition of the cycle class map given by Bloch in [8], with the minor modifications mentioned in 3.3.2. However, in loc. cit. there is no proof of the fact that the cycle class map agrees with the Beilinson regulator.

3.4.1 Definition of the regulator

Consider the map of iterated cochain complexes defined by the projection onto the first factor

$$\begin{array}{rcl}
\mathcal{D}^{r,-n}_{\mathbb{A},\mathcal{Z}^p}(X,p) = \mathcal{D}^r_{\log}(X \times \square^n, p) \oplus \mathcal{D}^{r-1}_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p) & \xrightarrow{\rho} & \mathcal{D}^r_{\log}(X \times \square^n, p) \\
(\alpha, g) & \mapsto & \alpha.
\end{array}$$

It induces a cochain morphism

$$\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \xrightarrow{\rho} \mathcal{D}^*_{\mathbb{A}}(X,p)_0,$$

and hence a chain morphism

$$\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0. \tag{3.4.1}$$

The morphism induced by ρ in homology, together with the isomorphisms of propositions 3.3.1, 3.3.5 and 3.3.7, induce a morphism

$$\rho : CH^p(X,n) \to CH^p(X,n)_{\mathbb{R}} \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)). \tag{3.4.2}$$

By abuse of notation, all these morphisms are denoted by ρ.

By corollary 3.3.9, we deduce that, if $z \in Z^p(X,n)_0$, then

$$\rho(z) = (\alpha_n,\dots,\alpha_0),$$

for any cycle $((\alpha_n,g_n),\dots,(\alpha_0,g_0)) \in Z\mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ such that $[(\alpha_n,g_n)] = cl(z)$.

3.4.2 Properties of the regulator

Proposition 3.4.3. *(i) The morphism $\rho : \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \to \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0$ is contravariant for flat maps.*

(ii) The induced morphism $\rho : CH^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$ is contravariant for arbitrary maps.

Proof. Both assertions are obvious. Let $((\alpha_n,g_n),\dots,(\alpha_0,g_0)) \in \mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ be a cycle such that f^* is defined. Then, since

$$f^*((\alpha_n,g_n),\dots,(\alpha_0,g_0)) = ((f^*\alpha_n,f^*g_n),\dots,(f^*\alpha_0,f^*g_0))$$

the claim follows. □

Remark 3.4.4. Let X be an equidimensional compact complex algebraic manifold. Observe that, by definition, the morphism

$$\rho : CH^p(X,0) = CH^p(X) \to H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p))$$

agrees with the cycle class map cl introduced in 1.4.6.

Now let E be a vector bundle of rank n over X. For every $p = 1,\dots,n$, there exists a characteristic class $C^{CH}_p(E) \in CH^p(X)$ (see [32]) and a characteristic class $C^{\mathcal{D}}_p(E) \in H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p))$, called the p-th Chern class of the vector bundle E. By definition, $cl(C^{CH}_p(E)) = C^{\mathcal{D}}_p(E)$. Hence,

$$\rho(C^{CH}_p(E)) = C^{\mathcal{D}}_p(E),$$

for all $p = 1,\dots,n$.

3.4.3 Comparison with the Beilinson regulator

In this section we prove that the regulator defined in (3.4.2) agrees, after composition with the isomorphism $K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}$ with the Beilinson regulator. The pattern of the proof is very similar to the comparison theorems developed in chapter 2. However, due to the failure of the strict functoriality of the Chow complex, we cannot apply directly the results proved in that chapter.

The comparison is based on the following facts:

▷ The morphism ρ has good contravariant properties.

▷ The morphism ρ is defined for quasi-projective schemes.

In view of these properties, it is enough to prove that the two regulators agree when X is a Grassmanian manifold, which in turn follows from remark 3.4.4.

Theorem 3.4.5. *Let X be an equidimensional complex algebraic scheme. Let ρ' be the composition of ρ with the isomorphism given by the Chern character*

$$\rho' : K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}} \xrightarrow{\rho} \bigoplus_{p\geq 0} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Then, the morphism ρ' agrees with the Beilinson regulator.

Proof. The notations introduced in chapter 2 are kept throughout the proof of this theorem.

Definition of the Beilinson regulator. The Beilinson regulator is the Chern character taking values in Deligne-Beilinson cohomology. The regulator can be described in terms of homotopy theory of sheaves as in [28].

Recall that we denoted

$$\mathbb{K}^N_{\cdot} = \mathbb{Z} \times \mathbb{Z}_{\infty} B.GL_N, \qquad \mathbb{K}. = \mathbb{Z} \times \mathbb{Z}_{\infty} B.GL.$$

Then, there exist Chern classes

$$C_p^{\mathcal{D}} \in [\mathbb{K}^N_{\cdot}, \mathcal{K}.(\mathcal{D}_{\log}(X,p))], \qquad N \gg 0,$$

compatible with the morphisms $\mathbb{K}^N_{\cdot} \to \mathbb{K}^{N+1}_{\cdot}$. Let $\mathbb{K}^N_{X,\cdot}$ and $\mathcal{K}_{X,\cdot}(\mathcal{D}_{\log}(X,p))$ denote the restrictions of $\mathbb{K}^N_{\cdot}$ and $\mathcal{K}.(\mathcal{D}_{\log}(X,p))$ to the small Zariski site of X. Then, we obtain Chern classes

$$C_p^{\mathcal{D}} \in [\mathbb{K}^N_{X,\cdot}, \mathcal{K}_{X,\cdot}(\mathcal{D}_{\log}(X,p))],$$

compatible with the morphisms $\mathbb{K}^N_{X,\cdot} \to \mathbb{K}^{N+1}_{X,\cdot}$. Therefore, we obtain a morphism

$$K_m(X) = \varinjlim_{N} H^{-m}(X, \mathbb{K}^N_X) \xrightarrow{C^{\mathcal{D}}_{p,X}} H^{2p-m}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Using the standard formula for the Chern character in terms of the Chern classes, we obtain a morphism

$$K_m(X) \xrightarrow{ch^{\mathcal{D}}} H_{\mathcal{D}}^{2p-m}(X, \mathbb{R}(p)),$$

which is the Beilinson regulator.

Let $E_\cdot^N$ be the universal rank N vector bundle over $B.GL_N$ and let $\mathbb{P}.(E_\cdot^N)$ be the simplicial associated projective bundle over $B.GL_N$. In [21], Gillet constructs a tautological class $\xi \in \operatorname{Pic}(\mathbb{P}(E_\cdot^N)) = H^1(\mathbb{P}(E_\cdot^N), \mathbb{G}_m)$.

Construction of the Chern character for higher Chow groups. The description of the isomorphism $K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}$ given by Bloch follows the same pattern as the description of the Beilinson regulator. However, since the complexes that define the higher Chow groups are not sheaves on the big Zariski site, a few modifications are necessary. We give here a sketch of the construction. For details see [7].

If $Y.$ is a simplicial scheme whose face maps are flat, then there is a well-defined 2-iterated cochain complex $Z^p(Y., *)_0$, whose (n,m)-bigraded group is

$$Z^p(Y_{-n}, m)_0,$$

and induced differentials. The higher algebraic Chow groups of $Y.$ are then defined as

$$CH^p(Y., n) = H^n(Z^p(Y., *)_0).$$

Since the face maps of $B.GL_N$ are flat, the group $CH^p(B.GL_N, n)$ is well defined for every p and n. Analogously, the groups $CH^p(\mathbb{P}.(E_\cdot^N), n)$ are defined.

The first step in the definition of the Chern character for higher Chow groups, is to construct universal Chern classes

$$C_p^{CH} \in CH^p(B.GL_N, 0).$$

There is a spectral sequence

$$E_1^{nm} = CH^p(\mathbb{P}_m(E_\cdot^N), -n) \Rightarrow CH^p(\mathbb{P}.(E_\cdot^N), -n-m),$$

which converges for $p = 1$.

This spectral sequence together with the comparison of the Chow groups in codimension 1 and the cohomology of $\mathbb{G}_m$, implies that there is an isomorphism

$$CH^1(\mathbb{P}.(E_\cdot^N), 0) \cong \operatorname{Pic}(\mathbb{P}.(E_\cdot^N)).$$

Let $C_1^{CH}(\xi) \in CH^1(\mathbb{P}.(E_\cdot^N), 0)$ be the class corresponding to the tautological class $\xi \in \operatorname{Pic}(\mathbb{P}.(E_\cdot^N))$. Then, the above spectral sequences and the projective bundle isomorphism (see section 3.2.3), imply that, for $1 \leq p \leq n$, the classes

$$C_p^{CH} \in CH^p(B.GL_N, 0)$$

are defined. These classes are represented by elements $C_p^{CH,i} \in Z^p(B_iGL_N, i)_0$.

Because at the level of complexes the pull-back morphism is not defined for arbitrary maps, one cannot consider the pull-back of these classes $C_p^{CH,i}$ to X, as was the case for the Beilinson regulator. However, by [7] § 6, there exists a purely transcendental extension L of $\mathbb{C}$, and classes $C_p^{CH,i}$ defined over L, such that the pull-back $f^*C_p^{CH,i}$ is defined for every $\mathbb{C}$-morphism $f : V \to B_iGL_N$.

Then, there is a map of simplicial Zariski sheaves on X

$$B.GL_{N,X} \to \mathcal{K}_X(p_* Z^p_{X_L}(-,*)_0),$$

where $p : X_L \to X$ is the natural map obtained by extension to L.

There is a specialization process described in [7], which, in the homotopy category of sheaves over X, gives a well-defined map

$$\mathcal{K}_X(p_* Z^p_{X_L}(-,*)_0) \to \mathcal{K}_X(Z^p_X(-,*)_0).$$

Therefore, there are maps $C^{CH}_{p,X} \in [B.GL_{N,X}, \mathcal{K}_X(Z^*_X(\cdot,p))]$. Proceeding as above, we obtain the Chern character isomorphism

$$K_m(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}.$$

End of the proof. Since, at the level of complexes, ρ is functorial for flat maps, there is a sheaf map

$$\rho : \mathcal{K}_X(Z^*_X(\cdot,p)) \to \mathcal{K}.(\mathcal{D}_{\log}(X,p))$$

in the small Zariski site of X.

It follows that the composition $\rho \circ C_p^{CH}$ is obtained by the same procedure as the Beilinson regulator, but with starting characteristic classes $\rho(C_p^{CH}) \in H^{2p}_{\mathcal{D}}(X,\mathbb{R}(p))$ instead of $C_p^{\mathcal{D}}$. Therefore, it remains to see that

$$\rho(C_p^{CH}) = C_p^{\mathcal{D}}. \tag{3.4.6}$$

Let $Gr(N,k)$ be the complex Grassmanian scheme of k-planes in $\mathbb{C}^N$. It is a smooth complex projective scheme. Let $E_{N,k}$ be the rank N universal bundle of $Gr(N,k)$ and $U_k = (U_{k,\alpha})_\alpha$ its standard trivialization. Let $N.U_k$ denote the nerve of this cover. It is a hypercover of $Gr(N,k)$, $N.U_k \xrightarrow{\pi} Gr(N,k)$. Consider the classifying map of the vector bundle $E_{N,k}$, $\varphi_k : N.U_k \to B.GL_N$, which satisfies $\pi^*(E_{N,k}) = \varphi_k^*(E^N_.)$. Observe that all the faces and degeneracy maps of the simplicial scheme $N.U_k$ are flat, as well as the inclusion maps $N_lU_k \to Gr(N,k)$. Therefore, $CH^p(N.U_k,m)$ is defined and there is a pull-back map $CH^p(Gr(N,k),m) \xrightarrow{\pi^*} CH^p(N.U_k,m)$.

Since ρ is defined on $N.U_k$ and is a functorial map, we obtain the following commutative diagram

$$\begin{tikzcd}
 & CH^p(B.GL_N,0) \arrow[r, "\rho"] \arrow[d, "\varphi_k^*"'] & H^{2p}_{\mathcal{D}}(B.GL_N,\mathbb{R}(p)) \arrow[d, "\varphi_k^*"] \\
 & CH^p(N.U_k,0) \arrow[r, "\rho"] & H^{2p}_{\mathcal{D}}(N.U_k,\mathbb{R}(p)) \\
K_0(Gr(N,k)) \arrow[r, "C_p^{CH}"] \arrow[rr, bend right, "C_p^{\mathcal{D}}"'] & CH^p(Gr(N,k),0) \arrow[r, "\rho"] \arrow[u, "\pi^*"] & H^{2p}_{\mathcal{D}}(Gr(N,k),\mathbb{R}(p)) \arrow[u, "\pi^*"']
\end{tikzcd}$$

By construction, $C_p^{CH}(E_{N,k})$ is the standard p-th Chern class in the classical Chow group of $Gr(N,k)$, and $C_p^{\mathcal{D}}(E_{N,k})$ is the p-th Chern class in Deligne-Beilinson cohomology. It then follows from proposition 3.4.4 that

$$\rho(C_p^{CH}(E_{N,k})) = C_p^{\mathcal{D}}(E_{N,k}). \tag{3.4.7}$$

The vector bundle $E_{N,k} \in K_0(Gr(N,k)) = \lim_{\overrightarrow{M}}[Gr(N,k),\mathbb{K}^M_{.}]$ is represented by the map of sheaves on $Gr(N,k)$ induced by

$$Gr(N,k) \xleftarrow{\pi} N.U_k \xrightarrow{\varphi_k} B.GL_N.$$

Here, since $N.U_k$ is a hypercover of $Gr(N,k)$, the map π is a weak equivalence of sheaves. This means that

$$\varphi_k^*(C_p^{CH}(E^N_{.})) = \pi^*(C_p^{CH}(E_{N,k})). \tag{3.4.8}$$

Finally, we proceed as in section 2.5.2. For each m_0, there exists k_0 such that if $m \leq m_0$ and $k \geq k_0$, φ_k^* is an isomorphism on the m-th cohomology group. Moreover, π also induces an isomorphism in Deligne-Belinson cohomology. Under these isomorphisms, we obtain the equality

$$C_p^{\mathcal{D}}(E_{N,k}) = (\pi^*)^{-1}\varphi_k^*(C_p^{\mathcal{D}}(E^N_{.})). \tag{3.4.9}$$

Hence,

$$\begin{aligned}
\rho(C_p^{CH}(E^N_{.})) = C_p^{\mathcal{D}}(E^N_{.}) \quad &\Leftrightarrow \quad \varphi_k^*\rho(C_p^{CH}(E^N_{.})) = \varphi_k^* C_p^{\mathcal{D}}(E^N_{.}) \\
&\Leftrightarrow \quad \rho\varphi_k^*(C_p^{CH}(E^N_{.})) = \varphi_k^* C_p^{\mathcal{D}}(E^N_{.}).
\end{aligned}$$

The last equality follows directly from (3.4.7), (3.4.8) and (3.4.9). Therefore, the theorem is proved. $\square$

3.5 The regulator for compact complex algebraic manifolds

In the last two sections we constructed a chain morphism

$$\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p) \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p),$$

for every complex algebraic manifold X. There, it is proved that the composition

$$Z^p(X,*)_0 \to \mathcal{H}^p(X,*)_0 \xleftarrow{\sim} \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0 \xleftarrow{\sim} \mathcal{D}^{2p-*}_{\log}(X,p), \tag{3.5.1}$$

represents, in the derived category of complexes, the Beilinson regulator.

Assume now that X is compact. Then, by considering differential forms over $X \times (\mathbb{P}^1)^n$ instead of over $X \times \square^n$, one can obtain a representative of the Beilinson regulator as a morphism in the derived category

$$\widetilde{Z}^p(X,*) \to \widetilde{\mathcal{H}}^p_{\mathbb{P}}(X,*) \xleftarrow{\sim} \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \xrightarrow{\rho} \mathcal{D}^{2p-*}(X,p). \tag{3.5.2}$$

Note that this construction differs from (3.5.1) in the fact that the target complex is exactly the Deligne complex of differential forms on X and not a complex of differential forms on $X \times \square^{\cdot}$ as defined in the previous sections. The key point is proposition 3.5.7 below, from Burgos and Wang in [15].

The particularity of this representative is that then a definition of higher arithmetic Chow groups, analogous to the definition of higher arithmetic K-theory given by Takeda in [57], can be given. This alternative approach to higher arithmetic Chow groups is sketched in the last section.

In this section, we describe the morphism (3.5.2) and compare it to the regulator given in section 3.4. The equidimensional compact complex algebraic manifold X is fixed throughout this section.

3.5.1 Differential forms over the projective space

In this section we introduce the analog of the complex $\mathcal{D}^*_{\mathbb{A}}(X,p)_0$, obtained by means of the cocubical scheme $(\mathbb{P}^1)^{\cdot}$ instead of the cocubical scheme $\square^{\cdot}$. In order to prove that the morphism

$$\mathcal{D}^*(X,p) \xrightarrow{\sim} \mathcal{D}^*_{\mathbb{A}}(X,p)_0$$

is a quasi-isomorphism, the homotopy invariance of Deligne-Beilinson cohomology is needed (see proposition 3.3.1). Using projective lines, the equivalent statement is no longer true. This forces us to consider the quotient of the complex of differential forms on $X \times (\mathbb{P}^1)^n$ by the complex of differential forms on the projective lines. Moreover, in this situation, instead of using the normalized complex associated to a cubical abelian group, it is more convenient to work with the complex given by the quotient by the degenerate elements.

Consider the cocubical scheme $(\mathbb{P}^1)^{\cdot}$ as described in section 3.2.1. The coface and codegeneracy maps induce, for every $i = 1, \dots, n$ and $l = 0, 1$, morphisms

$$\begin{aligned} \delta_i^l : \mathcal{D}^*(X \times (\mathbb{P}^1)^n, p) &\to \mathcal{D}^*(X \times (\mathbb{P}^1)^{n-1}, p), \\ \sigma_i : \mathcal{D}^*(X \times (\mathbb{P}^1)^{n-1}, p) &\to \mathcal{D}^*(X \times (\mathbb{P}^1)^n, p). \end{aligned}$$

Let $\mathcal{D}_{\mathbb{P}}^{*,*}(X, p)$ be the 2-iterated cochain complex given by

$$\mathcal{D}_{\mathbb{P}}^{r,-n}(X, p) = \mathcal{D}^r(X \times (\mathbb{P}^1)^n, p)$$

and differentials $(d_{\mathcal{D}}, \delta = \sum_{i=1}^n (-1)^i (\delta_i^0 - \delta_i^1))$ and denote by $\mathcal{D}_{\mathbb{P}}^*(X, p)$ the associated simple complex.

Let $(x : y)$ be homogeneous coordinates in $\mathbb{P}^1$ and consider

$$h = -\frac{1}{2} \log \frac{(x-y)\overline{(x-y)}}{x\bar{x} + y\bar{y}}.$$

It defines a function on the open set $\mathbb{P}^1 \setminus \{1\}$, with logarithmic singularities along 1. Therefore, it belongs to $\mathcal{D}_{\log}^1(\square^1, 1)$. Consider the differential $(1,1)$-form

$$\omega = d_{\mathcal{D}} h \in \mathcal{D}^2(\mathbb{P}^1, 1).$$

This is a smooth form all over $\mathbb{P}^1$ representing the class of the first Chern class of the canonical bundle of $\mathbb{P}^1$, $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$. Observe that,

$$\delta_1^0(h) = \delta_1^1(h) = 0, \qquad \text{and} \qquad \delta_1^0(\omega) = \delta_1^1(\omega) = 0.$$

For every n, denote by $\pi : X \times (\mathbb{P}^1)^n \to X$ the projection onto X and let $p_i : X \times (\mathbb{P}^1)^n \to \mathbb{P}^1$ be the projection onto the i-th projective line. Denote, for $i = 1, \dots, n$,

$$\begin{aligned} \omega_i &= p_i^*(\omega) \in \mathcal{D}^2(X \times (\mathbb{P}^1)^n, 1) \\ h_i &= p_i^*(h) \in \mathcal{D}_{\log}^1(X \times \square^n, 1). \end{aligned}$$

Let

$$D_n^r = \sum_{i=1}^n \sigma_i(\mathcal{D}^r(X \times (\mathbb{P}^1)^{n-1}, p)), \tag{3.5.3}$$

be the complex of degenerate elements and let $\mathcal{W}_n^*$ be the subcomplex of $\mathcal{D}^*(X \times (\mathbb{P}^1)^n, p)$ given by

$$\mathcal{W}_n^r = \sum_{i=1}^n \omega_i \wedge \sigma_i(\mathcal{D}^{r-2}(X \times (\mathbb{P}^1)^{n-1}, p-1)). \tag{3.5.4}$$

This complex is meant to kill the cohomology classes coming from the projective lines. We define the 2-iterated complex

$$\widetilde{\mathcal{D}}_{\mathbb{P}}^{r,-n}(X, p) := \widetilde{\mathcal{D}}^r(X \times (\mathbb{P}^1)^n, p) := \frac{\mathcal{D}^r(X \times (\mathbb{P}^1)^n, p)}{D_n^r + \mathcal{W}_n^r}$$

and denote by $\widetilde{\mathcal{D}}_{\mathbb{P}}^*(X, p)$ the associated simple complex.

Proposition 3.5.5. *The natural map*

$$\mathcal{D}^*(X,p) = \widetilde{\mathcal{D}}^{*,0}_{\mathbb{P}}(X,p) \xrightarrow{i} \widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p)$$

is a quasi-isomorphism.

Proof. The proof is analogous to the proof of lemma 1.3 in [15]. It follows from the fact that, by the Dold-Thom isomorphism in Deligne-Beilinson cohomology,

$$H^r(\widetilde{\mathcal{D}}^*(X \times (\mathbb{P}^1)^n, p)) = 0 \quad \forall n > 0. \tag{3.5.6}$$

□

In [15], Burgos and Wang gave an explicit quasi-inverse of the morphism i. Here we recall the construction.

Let $\mathbb{C}^* = \mathbb{P}^1 \setminus \{0,\infty\}$. If $(x_i : y_i)$ are projective coordinates on the i-th projective line in $(\mathbb{P}^1)^n$, let $z_i = x_i/y_i$ be the Euclidian coordinates. For any $0 \leq i \leq n$, consider the differential form on $(\mathbb{P}^1)^n$

$$S^i_n := \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \log|z_{\sigma(1)}|^2 \frac{dz_{\sigma(2)}}{z_{\sigma(2)}} \wedge \cdots \wedge \frac{dz_{\sigma(i)}}{z_{\sigma(i)}} \wedge \frac{d\bar{z}_{\sigma(i+1)}}{\bar{z}_{\sigma(i+1)}} \wedge \cdots \wedge \frac{d\bar{z}_{\sigma(n)}}{\bar{z}_{\sigma(n)}}.$$

This is a differential form with logarithmic singularities along the hyperplanes $x_i = 0$ and $y_i = 0$. Therefore, $S^i_n \in \mathcal{D}^n_{\log}((\mathbb{C}^*)^n, n)$.

Consider the differential form

$$T_n = \frac{1}{2n!} \sum_{i=1}^{n} (-1)^i S^i_n \in \mathcal{D}^n_{\log}((\mathbb{C}^*)^n, n).$$

Proposition 3.5.7. *There is a morphism of complexes*

$$\mathcal{D}^*_{\mathbb{P}}(X,p) \xrightarrow{\varphi} \mathcal{D}^*(X,p),$$

given by

$$\alpha \in \mathcal{D}^r(X \times (\mathbb{P}^1)^n, p) \mapsto \pi_*(\alpha \bullet T_n) = \begin{cases} \frac{1}{(2\pi i)^n} \int_{(\mathbb{P}^1)^n} \alpha \bullet T_n & n > 0, \\ \alpha & n = 0. \end{cases}$$

Moreover, the morphism φ factorizes through $\widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p)$ inducing a cochain morphism

$$\widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p) \xrightarrow{\varphi} \mathcal{D}^*(X,p).$$

The morphism φ is a quasi-inverse of the quasi-isomorphism i of proposition 3.5.5.

Proof. See [15], § 6. □

3.5.2 Differential forms over the projective space and higher Chow groups

In this section we define a cochain complex, $\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X,p)$, which satisfies the same properties as $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$. Recall that the complex $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ is obtained from the complexes

$$s(\mathcal{D}^*_{\log}(X\times\square^n,p)\to\mathcal{D}^*_{\log}(X\times\square^n\setminus\mathcal{Z}^p_n,p))$$

that compute cohomology with support. In order to define $\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X,p)$, the complex $\widetilde{\mathcal{D}}^*(X\times(\mathbb{P}^1)^*,p)$ will play the role of the complex $\mathcal{D}^*_{\log}(X\times\square^n,p)$. The role of the complex $\mathcal{D}^*_{\log}(X\times\square^n\setminus\mathcal{Z}^p_n,p)$ will be played by a complex consisting of differential forms on $X\times\square^n\setminus\mathcal{Z}^p_n$ and not over $X\times(\mathbb{P}^1)^n\setminus\mathcal{Z}^p_n$, as one might think.

Denote by

$$j: X\times\square^n\to X\times(\mathbb{P}^1)^n$$

the inclusion induced by $\square=\mathbb{P}^1\setminus\{1\}\subset\mathbb{P}^1$ and let $j^*:\mathcal{D}^*(X\times(\mathbb{P}^1)^n,p)\to\mathcal{D}^*_{\log}(X\times\square^n,p)$ be the induced inclusion of complexes. Consider the complex

$$\mathcal{G}^r_n=\sum_{i=1}^n h_i\bullet\sigma_i(j^*\mathcal{D}^{r-1}(X\times(\mathbb{P}^1)^{n-1},p-1))\subset\mathcal{D}^r_{\log}(X\times\square^n,p).$$

Observe that if $h_i\bullet\alpha\in\mathcal{G}^r_n$, then

$$d_{\mathcal{D}}(h_i\bullet\alpha)=\omega_i\wedge\alpha-h_i\bullet d_{\mathcal{D}}(\alpha)\in j^*\mathcal{W}^{r+1}_n+\mathcal{G}^{r+1}_n.$$

Therefore, $j^*\mathcal{W}^*_n+\mathcal{G}^*_n$ is a subcomplex of $\mathcal{D}^*_{\log}(X\times\square^n,p)$.

When there is no source of confusion, we will identify $j^*\mathcal{W}^*_n$ with $\mathcal{W}^*_n$.

Lemma 3.5.8. *The cochain complex $j^*\mathcal{W}^*_n+\mathcal{G}^*_n$ is acyclic.*

Proof. We have to see that, for every $\mu+\gamma\in j^*\mathcal{W}^r_n+\mathcal{G}^r_n$, such that $d_{\mathcal{D}}(\mu+\gamma)=0$, there exists $\eta\in j^*\mathcal{W}^{r-1}_n+\mathcal{G}^{r-1}_n$ with $d_{\mathcal{D}}(\eta)=\mu+\gamma$.

We write $\mu=\sum_{i=1}^n\omega_i\wedge\sigma_i(\alpha_i)$ with $\alpha_i\in\mathcal{D}^{r-2}(X\times(\mathbb{P}^1)^{n-1},p-1)$ and $\gamma=\sum_{i=1}^n h_i\bullet\sigma_i(\beta_i)$ with $\beta_i\in j^*\mathcal{D}^{r-1}(X\times(\mathbb{P}^1)^{n-1},p-1)$. Then,

$$\begin{aligned}d_{\mathcal{D}}\mu &= \sum_{i=1}^n\omega_i\wedge\sigma_i(d_{\mathcal{D}}(\alpha_i)),\\ d_{\mathcal{D}}\gamma &= \sum_{i=1}^n\omega_i\wedge\sigma_i(\beta_i)-\sum_{i=1}^n h_i\bullet\sigma_i d_{\mathcal{D}}(\beta_i).\end{aligned}$$

Therefore, the hypothesis $d_{\mathcal{D}}(\mu+\gamma)=0$ means that

$$\sum_{i=1}^n\omega_i\wedge\sigma_i(d_{\mathcal{D}}(\alpha_i)+\beta_i)=\sum_{i=1}^n h_i\bullet\sigma_i d_{\mathcal{D}}(\beta_i).$$

The left hand side is smooth all over $X \times (\mathbb{P}^1)^n$, hence so is $\sum_{i=1}^n h_i \bullet \sigma_i d_{\mathcal{D}}(\beta_i)$. Since for every i, h_i has logarithmic singularities along $D_i = X \times (\mathbb{P}^1)^{i-1} \times \{1\} \times (\mathbb{P}^1)^{n-i}$, this implies that $\sigma_i d_{\mathcal{D}}(\beta_i)$ is flat along D_i. In particular $\sigma_i d_{\mathcal{D}}(\beta_i)$ is zero along D_i and hence $d_{\mathcal{D}}(\beta_i) = 0$ (see [12] for details on flat differential forms and logarithmic singularities).

Therefore both sides of the equality are zero. This means that $d_{\mathcal{D}}(\beta_i) = 0$ and $d_{\mathcal{D}}(\alpha_i) = -\beta_i$. Hence, $\eta = \sum_{i=1}^n h_i \wedge \sigma_i(\alpha_i)$, satisfies

$$d_{\mathcal{D}}(\eta) = \sum_{i=1}^n \omega_i \wedge \sigma_i(\alpha_i) + \sum_{i=1}^n h_i \bullet \sigma_i(\beta_i) = \mu + \gamma.$$

□

Corollary 3.5.9. *The natural morphism*

$$\frac{\mathcal{D}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p)}{D^*_n} \to \frac{\mathcal{D}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p)}{D^*_n + \mathcal{W}^*_n + \mathcal{G}^*_n}$$

is a quasi-isomorphism.

□

Let

$$i : \mathcal{D}^r(X \times (\mathbb{P}^1)^n, p) \to \mathcal{D}^r_{\log}(X \times \square^n \setminus \mathcal{Z}^p_n, p)$$

be the inclusion morphism obtained by the restriction of differential forms on $X \times (\mathbb{P}^1)^n$ to $X \times \square^n \setminus Y$, for every $Y \in \mathcal{Z}^p_n$. Observe that there is a well-defined map

$$\frac{\mathcal{D}^r(X \times (\mathbb{P}^1)^n, p)}{D^r_n + \mathcal{W}^r_n} \xrightarrow{\ J_n\ } \frac{\mathcal{D}^r_{\log}(X \times \square^n \setminus \mathcal{Z}^p_n, p)}{D^r_n + \mathcal{W}^r_n + \mathcal{G}^r_n}.$$

Let

$$\widetilde{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p) := \tau_{\leq 2p} s(J_n),$$

be the truncation at degree $2p$ of the associated simple complex.

Proposition 3.5.10. *If $r \leq 2p$, then*

$$H^r(\widetilde{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)) \cong H^r(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0).$$

Proof. By remark 3.3.10,

$$\begin{aligned} H^r(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0) &\cong H^r(\widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)), \\ H^r(\mathcal{D}^*_{\log}(X \times \square^n, p)_0) &\cong H^r(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p)). \end{aligned}$$

If $n \geq 1$, it follows from the proof of proposition 3.3.1 that

$$H^r(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p)) = 0$$

for all r.

From the exact sequence

$$H^{r-1}(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p)) \to H^{r-1}(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p)) \to H^r(\widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p))$$
$$\to H^r(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p)) \to H^r(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p)),$$

we deduce that

$$H^{r-1}(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p)) \cong H^r(\widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)),$$

for all r.

By (3.5.6),
$$H^r\big(\widetilde{\mathcal{D}}^*(X \times (\mathbb{P}^1)^n, p)\big) = 0, \qquad \text{for all } r.$$

Therefore, by corollary 3.5.9,

$$H^{r-1}(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p)) \cong H^r\big(\widetilde{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)\big).$$

Hence
$$H^r(\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0) \cong H^r\big(\widetilde{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)\big).$$

If $n = 0$, the result is obvious. $\square$

The semipurity property of Deligne-Beilinson cohomology together with the last proposition give the next corollary.

Corollary 3.5.11. *For all n and $r < 2p$,*

$$H^r(\widetilde{\mathcal{D}}_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)) = 0.$$

$\square$

Since the differential δ is compatible with the quotient $\widetilde{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)$, there is a 2-iterated cochain complex $\widetilde{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^*, p)$, whose $(r, -n)$-graded piece is $\widetilde{\mathcal{D}}^r_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)$. Denote by $\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X, p)$ the associated simple complex. Let

$$\widetilde{\mathcal{H}}^p_{\mathbb{P}}(X, *) = H^{2p}(\widetilde{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^*, p)).$$

Corollary 3.5.12. *The morphism*

$$\begin{array}{rcl} \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X, p) & \xrightarrow{g_1} & \widetilde{\mathcal{H}}^p_{\mathbb{P}}(X, *) \\ ((\alpha_n, g_n), \ldots, (\alpha_0, g_0)) & \mapsto & [(\alpha_n, g_n)] \end{array}$$

is a quasi-isomorphism.

Proof. It follows from a spectral sequence argument analogous to the proof of proposition 3.3.7, together with corollary 3.5.11. $\square$

The following corollary is a consequence of proposition 3.5.10 and the semipurity property of Deligne-Beilinson cohomology.

Let $Y \subseteq X \times \square^n$ be a codimension p irreducible subvariety and let $\bar{Y}$ be its closure in $X \times (\mathbb{P}^1)^n$. Then, there exists a representative

$$[(\omega, g)] \in H^{2p}(s(\mathcal{D}^*(X \times (\mathbb{P}^1)^n, p) \to \mathcal{D}^*(X \times (\mathbb{P}^1)^n \setminus \bar{Y}, p))),$$

of the class of Y. Denote by $f_1(Y)$ the image of this element in the cohomology group $\widetilde{\mathcal{H}}^p_{\mathbb{P}}(X,*)$. Extending f_1 by linearity, we obtain a morphism

$$\widetilde{Z}^p(X,n)_{\mathbb{R}} \xrightarrow{f_1} \widetilde{\mathcal{H}}^p_{\mathbb{P}}(X,*).$$

Corollary 3.5.13. *The morphism f_1 is an isomorphism.*

□

3.5.3 Definition of the regulator

The projection on the first component induces a chain morphism

$$\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P}}(X,p),$$

analogous to the definition of the morphism ρ in section 3.4.1.

Let $\bar{\rho}$ be the composition of φ with ρ:

$$\bar{\rho} : \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P}}(X,p) \xrightarrow{\varphi} \mathcal{D}^{2p-*}(X,p).$$

Then, there is a morphism in the derived category of chain complexes

$$\widetilde{Z}^p(X,*) \xrightarrow{f_1} \widetilde{\mathcal{H}}^p_{\mathbb{P}}(X,*) \xleftarrow[\sim]{g_1} \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \xrightarrow{\bar{\rho}} \mathcal{D}^{2p-*}(X,p) \tag{3.5.14}$$

that we call the regulator.

Let

$$\bar{\rho} : CH^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$$

be the induced morphism.

We introduce now some notation for the next proposition. Let

$$h_1 : H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)) \to H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X,p))$$

be the composition of the isomorphism

$$f_1^{-1} \circ g_1 : H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)) \xrightarrow{\cong} \widetilde{Z}^p(X,n)_{\mathbb{R}}$$

of propositions 3.3.5 and 3.3.7, with the isomorphism

$$g_1^{-1} \circ f_1 : \widetilde{Z}^p(X,n)_{\mathbb{R}} \xrightarrow{\cong} H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X,p))$$

of propositions 3.5.12 and 3.5.13. Let h_2 be the composition of the isomorphisms

$$H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A}}(X,p)) \cong H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) \cong H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p))$$

of propositions 3.3.1 and 3.5.5.

Proposition 3.5.15. *Let X be an equidimensional compact complex algebraic manifold. Then, the following diagram*

$$\begin{tikzcd}
 & H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)) \arrow[r, "\rho"] \arrow[d, leftrightarrow, "h_1"', "\cong"] & H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A}}(X,p)) \arrow[d, leftrightarrow, "h_2"', "\cong"] \arrow[dr, "\cong"] & \\
CH^p(X,n) \arrow[ur] \arrow[dr] & & & H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) \\
 & H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X,p)) \arrow[r, "\bar{\rho}"'] & H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p)) \arrow[ur, "\cong", "\varphi"'] &
\end{tikzcd}$$

is commutative.

Proof. It is a consequence of propositions 3.5.21 and 3.5.23 below. □

Corollary 3.5.16. *Let X be an equidimensional compact complex algebraic manifold. The morphism*

$$K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}} \xrightarrow{\bar{\rho}} \bigoplus_{p\geq 0} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)),$$

is the Beilinson regulator.

Proof. It follows from proposition 3.5.15 and theorem 3.4.5. □

In order to prove proposition 3.5.15, it is necessary to understand the isomorphism

$$H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)) \cong H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X,p))$$

in more detail.

Consider the following commutative diagram

$$\begin{tikzcd}
N\mathcal{D}^*(X\times(\mathbb{P}^1)^n,p) \arrow[r, "j^*"] \arrow[d, "\cong"'] & N\mathcal{D}^*_{\log}(X\times\square^n,p) \arrow[d, "\cong"] \\
\mathcal{D}^*(X\times(\mathbb{P}^1)^n,p)/D^*_n \arrow[r, "j^*"'] & \widetilde{\mathcal{D}}^*_{\log}(X\times\square^n,p).
\end{tikzcd}$$

Since the upper row is an inclusion of complexes, there is an inclusion of complexes

$$\mathcal{D}^*(X\times(\mathbb{P}^1)^n,p)/D^*_n \xrightarrow{j^*} \widetilde{\mathcal{D}}^*_{\log}(X\times\square^n,p).$$

Let $\mathcal{Q}(n)^*$ be the cochain complex given by the quotient

$$\mathcal{Q}(n)^* = \frac{\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p)}{\mathcal{D}^*(X \times (\mathbb{P}^1)^n, p)/D^*_n}.$$

Then, there is a short exact sequence

$$0 \to \mathcal{D}^*(X \times (\mathbb{P}^1)^n, p)/D^*_n \xrightarrow{i^*} \widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p) \xrightarrow{p} \mathcal{Q}(n)^* \to 0. \tag{3.5.17}$$

Let I_n be the composition morphism

$$I_n : \mathcal{D}^*(X \times (\mathbb{P}^1)^n, p)/D^*_n \xrightarrow{j^*} \widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p) \to \widetilde{\mathcal{D}}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p),$$

and let

$$\widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p) := \tau_{\leq 2p} s(I_n).$$

The notation $\widehat{\mathcal{D}}^*(\cdot, *)$ is used here temporarily. In the definition of higher arithmetic Chow groups, this notation will refer to another complex.

There is a natural injective morphism

$$\begin{array}{rcl} \widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p) & \xrightarrow{j^*} & \widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p) \\ (\alpha, g) & \mapsto & (j^*(\alpha), g). \end{array}$$

Lemma 3.5.18. *For every $n \geq 0$, there is a short exact sequence*

$$0 \to \widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p) \xrightarrow{j^*} \widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p) \xrightarrow{p} \mathcal{Q}(n)^* \to 0.$$

Proof. Let $\mathcal{F}(n)^*$ the quotient complex $\frac{\widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X\times\square^n,p)}{\widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X\times(\mathbb{P}^1)^n,p)}$. The claim follows from the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p)[-1] \arrow[r, "="] \arrow[d] & \widetilde{\mathcal{D}}^*_{\log}(X \times \square^n \setminus \mathcal{Z}^p, p)[-1] \arrow[r] \arrow[d] & 0 \arrow[d] \\
\widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p) \arrow[r, "j^*"] \arrow[d] & \widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p) \arrow[r, "p"] \arrow[d] & \mathcal{F}(n)^* \arrow[d] \\
\mathcal{D}^*(X \times (\mathbb{P}^1)^n, p)/D^*_n \arrow[r, "j^*"] & \widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p) \arrow[r, "p"] & \mathcal{Q}(n)^*,
\end{tikzcd}$$

where all the rows and the first and second columns are short exact sequences. $\square$

Therefore, for every $n \geq 0$ there is an exact sequence

$$\begin{array}{c} \cdots \to H^i(\widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)) \xrightarrow{j^*} H^i(\widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)) \xrightarrow{p} H^i(\mathcal{Q}(n)^*) \xrightarrow{\partial} \\ \xrightarrow{\partial} H^{i+1}(\widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)) \xrightarrow{j^*} H^{i+1}(\widetilde{\mathcal{D}}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)) \to \dots \end{array} \tag{3.5.19}$$

By definition of the connection morphism, the image by ∂ of $[a] \in H^i(\mathcal{Q}(n)^*)$ is $[(d_{\mathcal{D}}a, a)]$, for any representative a of $[a]$.

Lemma 3.5.20. *For every $n \geq 1$, the morphism*

$$\begin{aligned}
\bigoplus_{i=1}^{n} \mathcal{D}^{r-1}(X \times (\mathbb{P}^1)^{n-1}, p-1) &\xrightarrow{\Gamma} \mathcal{Q}(n)^r \\
(\eta_1, \dots, \eta_n) &\mapsto \sum_{i=1}^{n} h_i \bullet \sigma_i(\eta_i),
\end{aligned}$$

is a quasi-isomorphism.

Proof. We write

$$\begin{aligned}
\widetilde{H}^*_{(\mathbb{P}^1)^n,p} &:= H^*(\mathcal{D}^*(X \times (\mathbb{P}^1)^n, p)/D^*_n), \\
H^*_{\square^n} &:= H^*(\widetilde{\mathcal{D}}^*_{\log}(X \times \square^n, p)), \\
H^*_{(\mathbb{P}^1)^n,p} &:= H^*(\mathcal{D}^*(X \times (\mathbb{P}^1)^n, p)).
\end{aligned}$$

Consider the following commutative diagram

$$\begin{tikzcd}
\widetilde{H}^r_{(\mathbb{P}^1)^n,p} \arrow[r] & H^r_{\square^n} \arrow[r] & H^r(\mathcal{Q}(n)^*) \arrow[r] & \widetilde{H}^{r+1}_{(\mathbb{P}^1)^n,p} \arrow[r] & H^{r+1}_{\square^n} \\
\bigoplus_{i=1}^n H^{*-2}_{(\mathbb{P}^1)^{n-1},p-1} \arrow[r] \arrow[u, "\varphi_r"', "\cong"] & 0 \arrow[r] \arrow[u, "="'] & \bigoplus_{i=1}^n H^{*-1}_{(\mathbb{P}^1)^{n-1},p-1} \arrow[r] \arrow[u, "\Gamma"] & \bigoplus_{i=1}^n H^{*-1}_{(\mathbb{P}^1)^{n-1},p-1} \arrow[r] \arrow[u, "\varphi_{r+1}"', "\cong"] & 0. \arrow[u, "="']
\end{tikzcd}$$

Observe that:

- ▷ The first row is the exact sequence associated to the short exact sequence (3.5.17). The connecting morphism $H^r(\mathcal{Q}(n)^*) \to \widetilde{H}^{r+1}_{(\mathbb{P}^1)^n,p}$ is given by $[a] \mapsto [da]$. The second row is obviously exact.

- ▷ By the proof of proposition 3.3.1, $H^r_{\square^n} = 0$.

- ▷ The morphisms φ_r are induced by

$$\begin{aligned}
\bigoplus_{i=1}^{n} \mathcal{D}^{r-2}(X \times (\mathbb{P}^1)^{n-1}, p-1) &\to \mathcal{D}^r(X \times (\mathbb{P}^1)^n, p)/D^r_n \\
(\eta_1, \dots, \eta_n) &\mapsto \sum_{i=1}^{n} \omega_i \wedge \sigma_i(\eta_i).
\end{aligned}$$

  This morphism is injective and the quotient has zero cohomology. Therefore, it is a quasi-isomorphism.

Then, the lemma follows from the five lemma. □

Proposition 3.5.21. *Let z be a codimension p algebraic cycle in $X \times \square^n$ intersecting properly all the faces of $\square^n$ and such that $\delta(z) = 0$ in $\widetilde{Z}^p(X, n-1)$. Let $Z \subset X \times \square^n$ be the support of z. Then, for every $m = 0, \dots, n$, there exists a pair of differential forms*

$$(\alpha'_m, g_m) \in \mathcal{D}^{2p-n+m}(X \times (\mathbb{P}^1)^m, p) \oplus \mathcal{D}^{2p-n+m-1}_{\log}(X \times \square^m \setminus Z, p),$$

and a differential form

$$a_m \in j^*\mathcal{W}^{2p-n+m}_m + \mathcal{G}^{2p-n+m}_m, \quad \text{with } a_0 = a_n = 0,$$

such that if we set $\alpha_m = j^*(\alpha'_m) - a_m$, then

$$\begin{aligned} ((\alpha_n, g_n), \dots, (\alpha_0, g_0)) &\in \widetilde{\mathcal{D}}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X, p), \\ ((\alpha'_n, g_n), \dots, (\alpha'_0, g_0)) &\in \widetilde{\mathcal{D}}^{2p-n}_{\mathbb{P},\mathcal{Z}^p}(X, p), \end{aligned}$$

represent $cl(z)$ in $H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A},\mathcal{Z}^p}(X, p))$ and $H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X, p))$ respectively.

Proof. Set $b_n = a_n = 0$. Let $\bar{Z}$ be the closure of Z in $X \times (\mathbb{P}^1)^n$ and let $\bar{z}$ be the closure of the algebraic cycle z. Then, by definition $\delta_z := j^*\delta_{\bar{z}}$. Therefore, there exists a pair $(\alpha'_n, g_n) \in \mathcal{D}^{2p}(X \times (\mathbb{P}^1)^n, p) \oplus \mathcal{D}^{2p-1}_{\log}(X \times (\mathbb{P}^1)^n \setminus \bar{Z}, p)$ satisfying the equality

$$d_{\mathcal{D}}[g_n] = \alpha'_n - \delta_{\bar{z}}.$$

Since $\delta(z) = 0$, the class of $\delta(\alpha'_n, g_n) \in H^{2p}(\widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^{n-1}, p))$ maps to zero in $H^{2p}(\widetilde{\mathcal{D}}_{\log,\mathcal{Z}^p}(X \times \square^{n-1}, p))$. By (3.5.19) and lemma 3.5.20, there exists a collection of closed differential forms $\eta_i \in \mathcal{D}^{2p-1}(X \times (\mathbb{P}^1)^{n-1}, p)$, $i = 1, \dots, n$, such that

$$[\delta(\alpha'_n, g_n)] = [\sum_{i=1}^n \omega_i \wedge \sigma_i(\eta_i), \sum_{i=1}^n h_i \bullet \sigma_i(\eta_i)] \in H^{2p}(\widehat{\mathcal{D}}^*_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^n, p)). \tag{3.5.22}$$

We define the differential forms a_{n-1}, b_{n-1} as follows:

$$a_{n-1} = \sum_{i=1}^n h_i \bullet \sigma_i(\eta_i) \in \mathcal{G}^{2p-1}_{n-1}, \quad \text{and} \quad b_{n-1} = \sum_{i=1}^n \omega_i \wedge \sigma_i(\eta_i) \in \mathcal{W}^{2p}_{n-1}.$$

Observe that there is an equality $d_{\mathcal{D}}(a_{n-1}) = j^*(b_{n-1})$ in $j^*\mathcal{W}^{2p}_{n-1}$. By (3.5.22), there exists a pair $(\alpha'_{n-1}, g_{n-1}) \in \widehat{\mathcal{D}}^{2p-1}_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^{n-1}, p)$ such that

$$d_s(\alpha'_{n-1}, g_{n-1}) + \delta(\alpha'_n, g_n) = (b_{n-1}, a_{n-1}).$$

Assume that for every $m = n, \dots, k$, we have already obtained pairs of differential forms

$$\begin{aligned} (\alpha'_m, g_m) &\in \widehat{\mathcal{D}}^{2p-n+m}_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^m, p), \\ (b_m, a_m) &\in \mathcal{W}^{2p-n+m}_m \oplus \mathcal{G}^{2p-n+m-1}_m, \end{aligned}$$

such that

$$\begin{aligned} d_s(\alpha'_{m-1}, g_{m-1}) + (-1)^{n-m}\delta(\alpha'_m, g_m) &= (b_{m-1}, a_{m-1}), \\ j^*(b_{m-1}) - d_{\mathcal{D}}(a_{m-1}) &= (-1)^{n-m}\delta(a_m). \end{aligned}$$

for all $m = n, \dots, k+1$. Let us construct (α'_{k-1}, g_{k-1}) and (b_{k-1}, a_{k-1}) satisfying the equalities above. Applying δ to the first equality corresponding to $m = k+1$, we obtain

$$d_s\delta(\alpha'_k, g_k) = \delta(b_k, a_k),$$

and hence, $d_s\delta(\alpha'_k, g_k) = 0$ in $\widetilde{\mathcal{D}}^{2p-n+k+1}_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^{k-1}, p)$. By corollary 3.5.11, there exist pairs of differential forms $(\alpha'_{k-1}, g_{k-1}) \in \widehat{\mathcal{D}}^{2p-n+k+1}_{\mathcal{Z}^p}(X \times (\mathbb{P}^1)^{k-1}, p)$ and $(b_{k-1}, a_{k-1}) \in \mathcal{W}^{2p-n+k-1}_{k-1} \oplus \mathcal{G}^{2p-n+k-2}_{k-1}$ such that

$$d_s(\alpha'_{k-1}, g_{k-1}) + (-1)^{n-k}\delta(\alpha'_k, g_k) = (b_{k-1}, a_{k-1}).$$

Applying d_s to this equality, we obtain

$$d_s(b_{k-1}, a_{k-1}) = (-1)^{n-k}\delta d_s(\alpha'_k, g_k) = (-1)^{n-k}\delta(b_k, a_k).$$

Hence,

$$j^*(b_{k-1}) - d_{\mathcal{D}}(a_{k-1}) = (-1)^{n-k}\delta(a_k).$$

Observe that the process ends with $a_0 = 0$.

By construction,

$$((\alpha'_n, g_n), \dots, (\alpha'_0, g_0)) \in \widetilde{\mathcal{D}}^{2p-n}_{\mathbb{P},\mathcal{Z}^p}(X, p),$$

is a cycle representing the class of z. The fact that

$$((\alpha_n, g_n), \dots, (\alpha_0, g_0)) \in \widetilde{\mathcal{D}}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X, p)$$

is a cycle follows from the equality

$$\begin{aligned} d_s(a_{m-1}, 0) + (-1)^{m-n}\delta(a_m, 0) &= (j^*(b_{m-1}), a_{m-1}) \\ &= d_s(j^*(\alpha'_{m-1}), g_{m-1}) + (-1)^{n-m}\delta(j^*(\alpha'_m), g_m), \end{aligned}$$

for all m. □

Recall that we want to prove that the following diagram is commutative

$$\begin{tikzcd} & H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)) \arrow[r, "\rho"] \arrow[d, leftrightarrow, "h_1"', "\cong"] & H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A}}(X,p)) \arrow[dr, "\cong"] \arrow[d, leftrightarrow, "h_2"', "\cong"] & \\ CH^p(X,n) \arrow[ur] \arrow[dr] & & & H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) \\ & H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X,p)) \arrow[r, "\bar\rho"'] & H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p)) \arrow[ur, "\cong", "\varphi"'] & \end{tikzcd}$$

It follows from the last proposition that we can choose representatives of the image of a cycle $Z \in CH^p(X,n)$ in $H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A},\mathcal{Z}^p}(X,p))$ and in $H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P},\mathcal{Z}^p}(X,p))$, satisfying a certain relation. Hence, considering these representatives, we know how to relate the image of Z by ρ in $H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A}}(X,p))$ with the image of Z by $\bar{\rho}$ in $H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p))$. In the next proposition we show that any two cycles $\alpha' \in \bigoplus_{m=0}^n \widetilde{\mathcal{D}}^{2p-n+m,-m}_{\mathbb{P}}(X,p)$ and $\alpha \in \bigoplus_{m=0}^n \widetilde{\mathcal{D}}^{2p-n+m,-m}_{\mathbb{A}}(X,p)$ satisfying the same relation agree in the cohomology group $H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$. It follows then that $\rho(Z) = \bar{\rho}(Z)$ in $H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$.

Proposition 3.5.23. *Let*

$$(\alpha'_0,\dots,\alpha'_n) \in \bigoplus_{m=0}^n \widetilde{\mathcal{D}}^{2p-n+m,-m}_{\mathbb{P}}(X,p), \text{ and } (\alpha_0,\dots,\alpha_n) \in \bigoplus_{m=0}^n \widetilde{\mathcal{D}}^{2p-n+m,-m}_{\mathbb{A}}(X,p)$$

be cycles satisfying:

(i) $\alpha_n = j^*(\alpha'_n), \quad \alpha_0 = \alpha'_0.$

(ii) For $m = 1,\dots,n-1$, *there exists* $a_m \in j^*\mathcal{W}^{2p-n+m}_m + \mathcal{G}^{2p-n+m}_m$ *such that*

$$\alpha_m = j^*(\alpha'_m) + a_m.$$

Then, under the isomorphisms

$$\begin{aligned} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) &\xrightarrow{\cong} H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p)),\\ H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) &\xrightarrow{\cong} H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A}}(X,\mathbb{R}(p))),\end{aligned}$$

of propositions 3.3.1 and 3.5.5, there is an equality

$$[(\alpha'_0,\dots,\alpha'_n)] = [(\alpha_0,\dots,\alpha_n)] \in H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Proof. Since $(\alpha'_0,\dots,\alpha'_n)$ and $(\alpha_0,\dots,\alpha_n)$ are cycles by hypothesis, the relations

$$\begin{aligned} d_{\mathcal{D}}(\alpha_m) + (-1)^{n-m-1}\delta(\alpha_{m+1}) &= 0,\\ d_{\mathcal{D}}(\alpha'_m) + (-1)^{n-m-1}\delta(\alpha'_{m+1}) &= v_m,\end{aligned}$$

hold for some $v_m \in \mathcal{W}^{2p-n+m}_m$. Let $x, x' \in \mathcal{D}^{2p-n}(X,p)$ such that

$$[(\alpha_n,\dots,\alpha_0)] = [(0,\dots,0,x)] \in H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{A}}(X,p))$$

and

$$[(\alpha'_n,\dots,\alpha'_0)] = [(0,\dots,0,x')] \in H^{2p-n}(\widetilde{\mathcal{D}}^*_{\mathbb{P}}(X,p)).$$

We have to see that $[x] = [x'] \in H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$. For $m = 1,\dots,n+1$, let $\gamma_m \in \widetilde{\mathcal{D}}^{2p-n+m-1}_{\log}(X \times \square^m, p)$, and $\gamma'_m \in \widetilde{\mathcal{D}}^{2p-n+m-1}(X \times (\mathbb{P}^1)^m, p)$, such that

$$\begin{aligned} \alpha'_m &= d_{\mathcal{D}}(\gamma'_m) + (-1)^{n-m}\delta(\gamma'_{m+1}) + u_m,\\ \alpha_m &= d_{\mathcal{D}}(\gamma_m) + (-1)^{n-m}\delta(\gamma_{m+1}),\end{aligned}$$

for some $u_m \in \mathcal{W}_m^{2p-n+m}$ and

$$x' = \delta(\gamma_1') - \alpha_0, \qquad x = \delta(\gamma_1) - \alpha_0.$$

We want to find $\xi \in \mathcal{D}^{2p-n-1}(X,p)$ such that $x'-x = d_{\mathcal{D}}(\xi)$. We will find the differential form ξ by proving inductively that, for all m, there exists $\xi_m \in \mathcal{D}_{\log}^{2p-n+m}(X \times \square^m, p)$ such that

$$j^*\delta(\gamma_m') - \delta(\gamma_m) = d_{\mathcal{D}}\delta(\xi_m). \tag{3.5.24}$$

Observe that we can assume that $\gamma_{n+1} = \gamma_{n+1}' = 0$. Therefore, the equality is trivially satisfied for $m = n+1$. Let us proceed by induction on m. Assume that equation (3.5.24) is satisfied for $m+1$. Then,

$$d_{\mathcal{D}}(j^*\gamma_m' - \gamma_m) = -a_m - u_m - (-1)^{n-m} d_{\mathcal{D}}\delta(\xi_{m+1}). \tag{3.5.25}$$

Therefore,

$$d_{\mathcal{D}}(j^*\gamma_m' - \gamma_m + (-1)^{n-m}\delta(\xi_{m+1})) = -a_m - u_m.$$

By hypothesis, $a_m + u_m \in \mathcal{W}_m^{2p-n+m} + \mathcal{G}_m^{2p-n+m}$. Then, it follows from lemma 3.5.8 that there exists $\eta \in \mathcal{W}_m^{2p-n+m-1} + \mathcal{G}_m^{2p-n+m-1}$ such that

$$d_{\mathcal{D}}(\eta) = a_m + u_m.$$

Therefore,

$$d_{\mathcal{D}}(j^*\gamma_m' - \gamma_m + (-1)^{n-m}\delta(\xi_{m+1}) + \eta) = 0.$$

Since

$$j^*(\gamma_m') - \gamma_m + (-1)^{n-m}\delta(\xi_{m+1}) + \eta \in \widetilde{\mathcal{D}}_{\log}^{2p-n-m+1}(X \times \square^m, p),$$

which is an acyclic complex, there exists $\xi_m \in \widetilde{\mathcal{D}}_{\log}^{2p-n-m}(X \times \square^m, p)$ such that

$$d_{\mathcal{D}}(\xi_m) = j^*(\gamma_m') - \gamma_m + (-1)^{n-m}\delta(\xi_{m+1}) + \eta.$$

Since $\delta(\eta) = 0$, applying δ to this equation, we obtain

$$d_{\mathcal{D}}\delta(\xi_m) = j^*\delta(\gamma_m') - \delta(\gamma_m).$$

Then, for $m = 1$,

$$x' - x = \delta(j^*\gamma_1') - \delta(\gamma_1) = d_{\mathcal{D}}\delta(\xi_1).$$

Therefore,

$$[x'] = [x] \in H_{\mathcal{D}}^{2p-n}(X, \mathbb{R}(p)).$$

$\square$

3.6 Higher arithmetic Chow groups

Let X be an arithmetic variety over a field. Using the description of the Beilinson regulator given in the previous sections, we define the *higher arithmetic Chow groups*, $\widehat{CH}^n(X,p)$. The definition is analogous to the definition given by Goncharov, in [30]. The following properties will be shown:

- *Functoriality:* For every map of arithmetic varieties $f : X \to Y$, there is a pull-back morphism
$$f^* : \widehat{CH}^p(Y,n) \to \widehat{CH}^p(X,n).$$

- *Product:* There is a pairing
$$\widehat{CH}^p(X,n) \otimes \widehat{CH}^q(X,m) \to \widehat{CH}^{p+q}(X,n+m),$$
which turns $\bigoplus_{n,p} \widehat{CH}^p(X,n)$ into an associative ring, commutative with respect to the degree given by p, and graded commutative with respect to the degree given by n.

- For $n = 0$, $\widehat{CH}^p(X,0)$ agrees with the arithmetic Chow group defined by Burgos in [13] and hence, if X is proper, it agrees with the arithmetic Chow group defined by Gillet and Soulé in [24].

- *Long exact sequence:* There is a long exact sequence
$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$
$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \mathcal{D}_{\log}^{2p-1}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0,$$
where $\rho : CH^p(X,1) \to \mathcal{D}_{\log}^{2p-1}(X,p)/\operatorname{im} d_{\mathcal{D}}$ is the composition of the Beilinson regulator $CH^p(X,1) \to H_{\mathcal{D}}^{2p-1}(X,\mathbb{R}(p))$ with the natural map $H_{\mathcal{D}}^{2p-1}(X,\mathbb{R}(p)) \to \mathcal{D}_{\log}^{2p-1}(X,p)/\operatorname{im} d_{\mathcal{D}}$.

Observe that we need to restrict ourselves to arithmetic varieties over a field, because the theory of higher algebraic Chow groups by Bloch is only established for schemes over a field.

3.6.1 Arithmetic varieties

The following definitions are taken from [24].

Definition 3.6.1. An *arithmetic ring* is a triple (A, Σ, F_∞) consisting of:

(i) An excellent regular noetherian integral domain A.

(ii) A finite non-empty set of monomorphisms $\sigma : A \to \mathbb{C}$.

(iii) Write $\mathbb{C}^\Sigma = \coprod_{\sigma\in\Sigma}\mathbb{C}$. Then, F_∞ is an antilinear involution of $\mathbb{C}$-algebras, $F_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$, such that the diagram

$$\begin{tikzcd} & \mathbb{C}^\Sigma \arrow[d, "F_\infty"] \\ A \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & \\ & \mathbb{C}^\Sigma \end{tikzcd}$$

commutes.

An *arithmetic field* is an arithmetic ring which is also a field.

For simplicity, when Σ and F_∞ are clear from the context, we say that A is an arithmetic ring. The main examples of arithmetic fields are the number fields, $\mathbb{R}$ and $\mathbb{C}$.

Definition 3.6.2. An *arithmetic variety* X over a field F is a regular scheme X which is quasi-projective over an arithmetic field F.

Let X be an arithmetic variety over a field F. If $\sigma \in \Sigma$, we write

$$X_\sigma := X \otimes_\sigma \mathbb{C}, \quad \text{and} \quad X_\Sigma = \coprod_{\sigma\in\Sigma} X_\sigma.$$

We denote by X_∞ the complex algebraic manifold

$$X_\infty = X_\Sigma(\mathbb{C}).$$

Recall that a real variety X consists of a couple $(X_\mathbb{C}, F)$, where $X_\mathbb{C}$ is a complex algebraic manifold and F an antilinear involution on $X_\mathbb{C}$. Since F_∞ induces an antilinear involution on X_Σ, $X_\mathbb{R} = (X_\Sigma, F_\infty)$ is a real variety.

3.6.2 Higher arithmetic Chow groups

We start by adapting the results of the previous sections from complex varieties to real varieties. Let $X = (X_\mathbb{C}, F_\infty)$ be a real variety. Recall from subsection 1.4.7, that the Deligne complex of differential forms of X is defined as

$$\mathcal{D}^n_{\log}(X,p) := \mathcal{D}^n_{\log}(X_\mathbb{C},p)^{\overline{F}^*_\infty = id}.$$

We define analogously the complexes

$$\mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0, \quad \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_{00}, \quad \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0, \quad \text{and} \quad \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_{00}.$$

Then, the results of the previous sections remain true for real varieties. In particular:

(i) The chain complexes $\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{A}}(X,p)$ and $\mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0$ are both quasi-isomorphic to $\mathcal{D}^n_{\log}(X,p)$.

(ii) The Beilinson regulator for real varieties is represented, in the derived category of chain complexes, by

$$Z^p(X,*)_0 \to \mathcal{H}^p(X,*)_0 \xleftarrow{\sim} \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0.$$

Fix an arithmetic field F and let X be an arithmetic variety over F. Then, the Deligne cohomology groups of X and the Deligne complexes of X considered here are defined as the Deligne cohomology groups and complexes of the associated real variety $X_{\mathbb{R}}$.

Definition 3.6.3. Let $\widehat{\mathcal{D}}^{*,*}_{\mathbb{A}}(X,p)_0$ be the 2-iterated cochain complex given by the quotient $\mathcal{D}^{*,*}_{\mathbb{A}}(X,p)_0/\mathcal{D}^{2p,0}(X,p)$. That is, for all r,n,

$$\widehat{\mathcal{D}}^{r,-n}_{\mathbb{A}}(X,p)_0 = \begin{cases} 0 & \text{if } r=2p \text{ and } n=0, \\ \mathcal{D}^{r,-n}_{\mathbb{A}}(X,p)_0 & \text{otherwise.} \end{cases}$$

Let $\widehat{\mathcal{D}}^{*}_{\mathbb{A}}(X,p)_0$ denote the simple complex associated to $\widehat{\mathcal{D}}^{*,*}_{\mathbb{A}}(X,p)_0$.

Lemma 3.6.4. *The cohomology groups of the complex* $\widehat{\mathcal{D}}^{*}_{\mathbb{A}}(X,p)$ *are*

$$H^{2p-n}(\widehat{\mathcal{D}}^{*}_{\mathbb{A}}(X,p)_0) = \begin{cases} 0 & \textit{if } n\leq 0, \\ \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} & \textit{if } n=1, \\ H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) & \textit{if } n>1. \end{cases}$$

Proof. It follows from a spectral sequence argument as in proposition 3.3.1. □

Consider the composition of ρ with the projection map

$$\rho : \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0 \to \widehat{\mathcal{D}}^{2p-*}_{\mathbb{A}}(X,p)_0.$$

This morphism is also denoted by ρ. Let f_1 be the composition

$$f_1 : Z^p(X,n)_0 \xrightarrow{\otimes\mathbb{R}} Z^p(X,n)_0\otimes\mathbb{R} \xrightarrow{\times_F\mathbb{R}} Z^p(X_{\mathbb{R}},n)_0\otimes\mathbb{R} \cong \mathcal{H}^p(X,n)_0.$$

Then, there is a diagram of chain complexes of the type of (3.1.6)

$$\widehat{\mathcal{Z}}^p(X,*)_0 = \left(\begin{tikzcd} & \mathcal{H}^p(X,*)_0 & & \widehat{\mathcal{D}}^{2p-*}_{\mathbb{A}}(X,p)_0 \\ Z^p(X,*)_0 \arrow[ur, "f_1"] & & \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\rho"] & \end{tikzcd} \right).$$

Definition 3.6.5. The *higher arithmetic Chow complex* is the simple complex of the diagram $\widehat{\mathcal{Z}}^p(X,*)_0$, as defined in 3.1.2:

$$\widehat{Z}^p(X,*)_0 := s(\widehat{\mathcal{Z}}^p(X,*)_0).$$

Recall that, by definition, $\widehat{Z}^p(X,n)_0$ consists of 4-tuples

$$(Z,\alpha_0,\alpha_1,\alpha_2) \in Z^p(X,n)_0 \oplus \mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \oplus \mathcal{H}^p(X,n+1)_0 \oplus \widehat{\mathcal{D}}^{2p-n-1}_{\mathbb{A}}(X,p)_0,$$

and the differential is given by

$$\begin{array}{rcl} \widehat{Z}^p(X,n)_0 & \xrightarrow{D} & \widehat{Z}^p(X,n-1)_0 \\ (Z,\alpha_0,\alpha_1,\alpha_2) & \mapsto & (\delta(Z), d_s(\alpha_0), f_1(Z)-g_1(\alpha_0)-\delta(\alpha_1), \rho(\alpha_0)-d_s(\alpha_2)). \end{array}$$

Definition 3.6.6. Let X be an arithmetic variety over a field. The (p,n)-th *higher arithmetic Chow group* of X is defined by

$$\widehat{CH}^p(X,n) := H_n(\widehat{Z}^p(X,*)_0), \qquad p,n \geq 0.$$

Proposition 3.6.7. *There is a long exact sequence*

$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$

with ρ the Beilinson regulator. The end of this long exact sequence is

$$CH^p(X,1) \xrightarrow{\rho} \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X,0) \xrightarrow{\zeta} CH^p(X,0) \to 0. \tag{3.6.8}$$

Proof. It follows from lemma 3.1.7, theorem 3.4.5 and 3.6.4. □

Remark 3.6.9. If in the diagram $\widehat{\mathcal{Z}}^p(X,*)_0$ we consider the target complex $\mathcal{D}^*_{\mathbb{A}}(X,p)_0$ instead of $\widehat{\mathcal{D}}^*_{\mathbb{A}}(X,p)_0$, the homology groups of the diagram will agree, for $n \geq 1$, with the higher arithmetic Chow groups just defined. The difference is only in degree $n=0$. The purpose of the modification of the target complex is to obtain, for $n=0$, the arithmetic Chow group defined by Burgos.

3.6.3 Agreement with the arithmetic Chow group

Let X be an arithmetic variety and let $\widehat{CH}^p(X)$ denote the p-th arithmetic Chow group of X as defined by Burgos in [13]. We recall here its definition.

For every p, let $Z^p(X) = Z^p(X,0)$ and let $Z\mathcal{D}^{2p}_{\log}(X,p)$ denote the subgroup of cycles of $\mathcal{D}^{2p}_{\log}(X,p)$.

Given a couple

$$(\omega,\tilde{g}) \in Z\mathcal{D}^{2p}_{\log}(X,p) \oplus \frac{\mathcal{D}^{2p-1}_{\log}(X\setminus Z,p)}{\operatorname{im} d_{\mathcal{D}}},$$

one defines $cl(\omega,\tilde{g}) = cl(\omega,g)$, for any representative g of $\tilde{g}$. If $Z \in Z^p(X)$, a *Green form* for Z is a couple $(\omega,\tilde{g})$ such that

$$cl(Z) = cl(\omega,\tilde{g}).$$

Let

$$\widehat{Z}^p(X) = \left\{ (Z,(\omega,\tilde{g})) \in Z^p(X) \oplus Z\mathcal{D}^{2p}_{\log}(X,p) \oplus \frac{\mathcal{D}^{2p-1}_{\log}(X\setminus Z,p)}{\operatorname{im} d_{\mathcal{D}}} \;\middle|\; \begin{array}{c} cl(Z)=cl(\omega,\tilde{g}), \\ \omega = d_{\mathcal{D}}\tilde{g} \end{array} \right\}.$$

Let Y be a codimension $p-1$ subvariety of X and let $f \in k^*(Y)$. As shown in [13], §7, there is a canonical Green form attached to $\operatorname{div} f$. It is denoted by $\mathfrak{g}(f)$ and it is of the form $(0,\widetilde{cl}(f))$.

One defines the following subgroup of $\widehat{Z}^p(X)$:

$$\widehat{\operatorname{Rat}}^p(X) = \{(\operatorname{div} f, \mathfrak{g}(f)) \mid f \in k^*(Y),\ Y \subset X \text{ a codimension } p-1 \text{ subvariety}\}.$$

Definition 3.6.10. Let X be an arithmetic variety. For every $p \geq 0$, the *arithmetic Chow group* of X is

$$\widehat{CH}^p(X) = \widehat{Z}^p(X)/\widehat{\operatorname{Rat}}^p(X).$$

It is proved in [24], Theorem 3.3.5 and [13], Theorem 7.3, that these groups fit into exact sequences

$$CH^{p-1,p}(X) \xrightarrow{\rho} \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0$$

where:

- ▷ $CH^{p-1,p}(X)$ is the term $E_2^{p-1,-p}$ in the Quillen spectral sequence (see [48], § 7).
- ▷ ρ is the Beilinson regulator.
- ▷ The map ζ is the projection on the first component.
- ▷ The map a sends α to $(0,(0,\alpha))$.

Theorem 3.6.11. *The morphism*

$$\begin{array}{rcl} \widehat{CH}^p(X) & \xrightarrow{\Phi} & \widehat{CH}^p(X,0) \\ [(Z,(\omega,\tilde{g}))] & \mapsto & [(Z,(\omega,g),0,0)], \end{array}$$

where g is any representative of $\tilde{g} \in \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}}$, is an isomorphism.

Proof. We first prove that Φ is well defined. Afterwards, we will prove that the diagram

$$\begin{tikzcd} CH^{p-1,p}(X) \arrow[r, "\rho"] \arrow[d, "\cong"] & \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \arrow[r, "a"] \arrow[d, "="] & \widehat{CH}^p(X) \arrow[r, "\zeta"] \arrow[d, "\Phi"] & CH^p(X) \arrow[r] \arrow[d, "\cong"] & 0 \\ CH^p(X,1) \arrow[r, "\rho"] & \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \arrow[r, "a"] & \widehat{CH}^p(X,0) \arrow[r, "\zeta"] & CH^p(X,0) \arrow[r] & 0 \end{tikzcd}$$

is commutative. The statement then follows from the five lemma.

The proof is a consequence of lemmas 3.6.12, 3.6.13 and 3.6.14 below.

Lemma 3.6.12. *The map Φ is well defined.*

Proof. We have to prove that:

(i) The elements in the image of Φ are indeed cycles in $\widehat{Z}^p(X,0)_0$.

(ii) The map Φ does not depend on the choice of a representative of g.

(iii) The map Φ is zero on $\widehat{\mathrm{Rat}}^p(X)$.

Let $[(Z,(\omega,\tilde{g}))] \in \widehat{CH}^p(X)$. The claim (i) follows from the equality $cl(Z) = cl(\omega,\tilde{g}) = cl(\omega,g)$. Indeed, since $d_s(\omega,g) = 0$,

$$D(Z,(\omega,g),0,0) = (0,0,cl(Z) - cl(\omega,g),0) = 0.$$

To see (ii), assume that $g_1, g_2 \in \mathcal{D}^{2p-1}_{\log}(X,p)$ are representatives of $\tilde{g}$, i.e. there exists $h \in \mathcal{D}^{2p-2}_{\log}(X,p)$ such that $d_{\mathcal{D}}h = g_1 - g_2$. Then

$$D(0,(0,h),0,0) = (0,(0,g_1-g_2),0,0) = (Z,(\omega,g_1),0,0) - (Z,(\omega,g_2),0,0)$$

and therefore,

$$[(Z,(\omega,g_1),0,0)] = [(Z,(\omega,g_2),0,0)].$$

Finally, to prove (iii), we have to see that

$$\Phi(\operatorname{div} f, \mathfrak{g}(f)) = 0 \in \widehat{CH}^p(X,0),$$

i.e. that

$$[(\operatorname{div} f,(0,cl(f)),0,0)] = 0,$$

for any fixed representative $cl(f)$ of $\widetilde{cl}(f)$.

Let $\hat{f}$ be the function in $Y \times \mathbb{A}^1$ given by

$$(y,(t_1,t_2)) \mapsto t_1 - t_2 f(y).$$

Its divisor defines a codimension p subvariety of $X \times \mathbb{A}^1$. Moreover, it intersects properly $(0,1)$ and $(1,0)$. Fix $cl(\hat{f})$ to be any representative of $\widetilde{cl}(\hat{f})$. Since

$$\delta(\widetilde{cl}(\hat{f})) = \widetilde{cl}(f),$$

there exists $h \in \mathcal{D}^{2p-1}_{\log}(X \times \square \setminus \operatorname{div}\hat{f}, p)$ with

$$d_{\mathcal{D}}h = \delta(cl(\hat{f})) - cl(f).$$

Then,

$$D(\operatorname{div}\hat{f},(0,cl(\hat{f}),(0,h)),0,0) \;\;=\;\; (\operatorname{div} f,(0,cl(f)),0,0)$$

as desired. □

Lemma 3.6.13. *There are isomorphisms*

$$\begin{aligned} CH^p(X) &\xrightarrow{\pi_1} CH^p(X,0), \\ CH^{p-1,p}(X) &\xrightarrow{\pi_2} CH^p(X,1), \end{aligned}$$

making the two following diagrams commute

$$\begin{tikzcd} CH^{p-1,p}(X) \arrow[r, "\rho"] \arrow[d, "\pi_2"'] & \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \arrow[d, "="] \\ CH^p(X,1) \arrow[r, "\rho"] & \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \end{tikzcd} \qquad \begin{tikzcd} \widehat{CH}^p(X) \arrow[r, "\zeta"] \arrow[d, "\Phi"'] & CH^p(X) \arrow[d, "\pi_1"] \\ \widehat{CH}^p(X,0) \arrow[r, "\zeta"] & CH^p(X,0). \end{tikzcd}$$

Proof. Both isomorphisms are well known. The morphism π_1 is the isomorphism between the classical Chow group $CH^p(X)$ and the Bloch Chow group $CH^p(X,0)$. The diagram is obviously commutative, since $\pi_1([Z]) = [Z]$.

Consider the canonical isomorphism $CH^{p-1,p}(X) \cong K_1(X)_{\mathbb{Q}}$ given in [53]. Then, theorem 3.4.5 together with [24] §3.5 give a commutative diagram

$$\begin{tikzcd} CH^{p-1,p}(X) \arrow[d, "\cong"'] \arrow[dr, "\rho"] & \\ K_1(X)_{\mathbb{Q}} \arrow[r, "\rho'"] & \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \\ CH^p(X,1) \arrow[u, "\cong"] \arrow[ur, "\rho"'] & \end{tikzcd}$$

with ρ' the Beilinson regulator. Therefore, π_2 is the composition of the isomorphisms

$$CH^{p-1,p}(X) \xrightarrow{\cong} K_1(X)_{\mathbb{Q}} \xrightarrow{\cong} CH^p(X,1).$$

$\square$

Lemma 3.6.14. *The following diagram is commutative:*

$$\begin{tikzcd} & \widehat{CH}^p(X) \arrow[d, "\Phi"] \\ \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \arrow[ur, "a"] \arrow[r, "a"'] & \widehat{CH}^p(X,0) \end{tikzcd}$$

Proof. Let $\tilde{\alpha} \in \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}}$. Then, the lemma follows from the equality

$$D(0,(-\alpha,0),0,0) = (0,(-d\alpha,-\alpha),0,0) - (0,0,0,\alpha)$$

in $\widehat{CH}^p(X,0)$. $\square$

This finishes the proof of theorem 3.6.11. $\square$

3.6.4 Functoriality of the higher arithmetic Chow groups

Proposition 3.6.15 (Pull-back)**.** *Let $f : X \to Y$ be a morphism between two arithmetic varieties. Then, for all $p \geq 0$, there exists a chain complex, $\widehat{Z}^p_f(Y,*)_0$ such that*

(i) There is a quasi-isomorphism

$$\widehat{Z}^p_f(Y,*)_0 \xrightarrow{\sim} \widehat{Z}^p(Y,*)_0.$$

(ii) There is a pull-back morphism

$$f^* : \widehat{Z}^p_f(Y,*)_0 \to \widehat{Z}^p(X,*)_0,$$

inducing a pull-back morphism of higher arithmetic Chow groups

$$\widehat{CH}^p(Y,n) \xrightarrow{f^*} \widehat{CH}^p(X,n),$$

for every $p, n \geq 0$.

(iii) The pull-back is compatible with the morphisms a and ζ. That is, there are commutative diagrams

$$\begin{tikzcd} \cdots \arrow[r] & H^{2p-n-1}_{\mathcal{D}}(Y,\mathbb{R}(p)) \arrow[r, "a"] \arrow[d, "f^*"'] & \widehat{CH}^p(Y,n) \arrow[r, "\zeta"] \arrow[d, "f^*"'] & CH^p(Y,n) \arrow[r] \arrow[d, "f^*"'] & \cdots \\ \cdots \arrow[r] & H^{2p-n-1}_{\mathcal{D}}(X,\mathbb{R}(p)) \arrow[r, "a"'] & \widehat{CH}^p(X,n) \arrow[r, "\zeta"'] & CH^p(X,n) \arrow[r] & \cdots \end{tikzcd} \qquad (3.6.16)$$

Proof. Recall that there are inclusions of complexes

$$\begin{aligned} Z^p_f(Y,*)_0 &\subseteq Z^p(Y,*)_0, \\ \mathcal{H}^p_f(Y,*)_0 &\subseteq \mathcal{H}^p(Y,*)_0, \\ \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p)_0 &\subseteq \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(Y,p)_0, \end{aligned}$$

which are quasi-isomorphisms. The pull-back by f is defined for any α in $Z^p_f(Y,*)_0$, in $\mathcal{H}^p_f(Y,*)_0$ or in $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p)_0$. Moreover, by construction, there is a commutative diagram

$$\begin{tikzcd} Z^p_f(Y,*)_0 \arrow[r, "f_1"] \arrow[d, "f^*"'] & \mathcal{H}^p_f(Y,*)_0 \arrow[d, "f^*"'] & \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p_f}(Y,p)_0 \arrow[l, "g_1"', "\sim"] \arrow[r, "\rho"] \arrow[d, "f^*"'] & \mathcal{D}^*_{\mathbb{A}}(Y,p)_0 \arrow[d, "f^*"'] \\ Z^p(X,*)_0 \arrow[r, "f_1"'] & \mathcal{H}^p(X,*)_0 & \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \arrow[l, "\sim", "g_1"'] \arrow[r, "\rho"'] & \mathcal{D}^*_{\mathbb{A}}(X,p)_0. \end{tikzcd}$$

Let $\widehat{Z}^p_f(Y,*)_0$ denote the simple associated to the first row diagram. Then, there is a pull-back morphism

$$f^* : \widehat{Z}^p_f(Y,*)_0 \to \widehat{Z}^p(X,*)_0.$$

Moreover, it follows from lemma 3.1.5 that the natural map

$$\widehat{Z}^p_f(Y,*)_0 \to \widehat{Z}^p(Y,*)_0$$

is a quasi-isomorphism. Therefore, (i) and (ii) are proved. Statement (iii) follows from the construction. □

Remark 3.6.17. If the map is flat, then the pull-back is already defined at the level of the chain complexes $\widehat{Z}^p(Y,*)_0$ and $\widehat{Z}^p(X,*)_0$.

Proposition 3.6.18 (Functoriality of pull-back)**.** *Let $f: X \to Y$ and $g: Y \to Z$ be two morphisms of arithmetic varieties. Then,*

$$f^* \circ g^* = (g\circ f)^* : \widehat{CH}^p(Z,n) \to \widehat{CH}^p(X,n).$$

Proof. Let $\widehat{Z}^p_{gf\cup g}(Z,n)_0$ be the subgroup of $\widehat{Z}^p(Z,n)_0$ obtained considering, at each of the complexes of the diagram $\widehat{\mathcal{Z}}^p(Z,*)_0$, the subvarieties W of $Z\times\square^n$ intersecting properly the faces of $\square^n$ and such that

- ▷ $X \times W \times \square^n$ intersects properly the graph of $g\circ f$,
- ▷ $Y \times W \times \square^n$ intersects properly the graph of g.

That is,

$$\widehat{Z}^p_{gf\cup g}(Z,n)_0 = \widehat{Z}^p_{gf}(Z,n)_0 \cap \widehat{Z}^p_{g}(Z,n)_0.$$

Then, the proposition follows from the commutative diagram

$$\begin{tikzcd} & \widehat{Z}^p(X,*)_0 \\ \widehat{Z}^p_{gf\cup g}(Z,*)_0 \arrow[ur, "(g\circ f)^*"] \arrow[dr, "g^*"'] & \\ & \widehat{Z}^p_f(Y,*)_0. \arrow[uu, "f^*"'] \end{tikzcd}$$

□

Corollary 3.6.19 (Homotopy invariance)**.** *Let $\pi: X\times\mathbb{A}^m \to X$ be the projection on X. Then, the pull-back map*

$$\pi^* : \widehat{CH}^p(X,n) \to \widehat{CH}^p(X\times\mathbb{A}^m,n)$$

is an isomorphism for all $n\geq 1$.

Proof. It follows from the five lemma in the diagram (3.6.16), using the fact that both the higher Chow groups and the Deligne-Beilinson cohomology groups are homotopy invariant. Observe that to prove the homotopy invariance for $n=1$, we need to consider the non-truncated complex $\mathcal{D}^*_{\mathbb{A}}(X,p)$ instead of $\widehat{\mathcal{D}}^*_{\mathbb{A}}(X,p)$ (see remark 3.6.9). □

3.7 Product structure

Let X, Y be arithmetic varieties over a field F. In this section, we define an external product,

$$\widehat{CH}^*(X,*) \otimes \widehat{CH}^*(Y,*) \to \widehat{CH}^*(X \times Y,*),$$

and an internal product

$$\widehat{CH}^*(X,*) \otimes \widehat{CH}^*(X,*) \to \widehat{CH}^*(X,*),$$

for the higher arithmetic Chow groups. The internal product endows $\widehat{CH}^*(X,*)$ with a ring structure. The commutativity and associativity of the product will be checked in the next two sections.

Recall that the higher arithmetic Chow groups are the homology groups of the simple complex associated to a diagram of complexes. Therefore, in order to define a product, we use the general procedure developed by Beilinson, as recalled in section 3.1.1. Hence, we need to define a product for each of the complexes in the diagram $\widehat{\mathcal{Z}}^p(X,*)_0$, commuting with the morphisms f_1, g_1 and ρ.

The pattern for the construction of the product is analogous to the pattern followed to define the product for the cubical higher Chow groups, described in section 3.2.2. That is, the external product for the higher Chow groups is given by the cartesian product

$$s(Z^p(X,n)_0 \otimes Z^q(Y,m)_0) \xrightarrow{\cup} Z^{p+q}(X \times Y, n+m)_0.$$

Then, the internal product is obtained by pull-back along the diagonal map on X (observe that X is smooth)

$$\begin{tikzcd} s(Z^p(X,n)_0 \otimes Z^q(X,m)_0) \arrow[r, "\cup"] & Z^{p+q}(X \times X, n+m)_0 & \\ & Z^{p+q}_{\Delta}(X \times X, n+m)_0 \arrow[u, "\sim"'] \arrow[r, "\Delta^*"] & Z^{p+q}(X, n+m)_0. \end{tikzcd}$$

Since the complex $\mathcal{H}^p(X,*)_0$ is isomorphic to $Z^p_{\mathbb{R}}(X_{\mathbb{R}},*)_0$, the external and internal products on the complex $\mathcal{H}^*(X,*)_0$ can be defined by means of this isomorphism.

3.7.1 Product structure for $\mathcal{D}^*_{\mathbb{A}}(X,p)$

We start by defining a product structure on $\mathcal{D}^*_{\mathbb{A}}(X,p)$ which will induce a product structure on $\widehat{\mathcal{D}}^{2p-*}_{\mathbb{A}}(X,p)$.

Let

$$X \times Y \times \square^n \times \square^m \xrightarrow{p_{13}} X \times \square^n$$
$$X \times Y \times \square^n \times \square^m \xrightarrow{p_{24}} Y \times \square^m$$

be the projections indicated by the subindices. For every $\omega_1 \in \mathcal{D}^r_{\log}(X \times \square^n, p)$ and $\omega_2 \in \mathcal{D}^s_{\log}(Y \times \square^m, q)$, we define

$$\omega_1 \bullet_{\mathbb{A}} \omega_2 := (-1)^{ns} p_{13}^* \omega_1 \bullet p_{24}^* \omega_2 \in \mathcal{D}^{r+s}_{\log}(X \times Y \times \square^{n+m}, p+q).$$

This gives a map

$$\begin{array}{rcl} \mathcal{D}^{r_1}_{\mathbb{A}}(X,p)\otimes \mathcal{D}^{r_2}_{\mathbb{A}}(Y,q) & \xrightarrow{\bullet_{\mathbb{A}}} & \mathcal{D}^{r_1+r_2}_{\mathbb{A}}(X\times Y,p+q) \\ (\omega_1,\omega_2) & \mapsto & \omega_1\bullet_{\mathbb{A}}\omega_2. \end{array}$$

Lemma 3.7.1. *The map $\bullet_{\mathbb{A}}$ satisfies the Leibniz rule. Therefore, there is a cochain morphism*

$$s(\mathcal{D}^*_{\mathbb{A}}(X,p)\otimes \mathcal{D}^*_{\mathbb{A}}(Y,q)) \xrightarrow{\bullet_{\mathbb{A}}} \mathcal{D}^*_{\mathbb{A}}(X\times Y,p+q).$$

Proof. Let $\omega_1 \in \mathcal{D}^r_{\log}(X,n)$ and $\omega_2 \in \mathcal{D}^s_{\log}(Y,m)$. By definition of δ, the following equality holds

$$\delta(p_{13}^*\omega_1\bullet p_{24}^*\omega_2) = \delta(p_{13}^*\omega_1)\bullet p_{24}^*\omega_2 + (-1)^n p_{13}^*\omega_1\bullet \delta(p_{24}^*\omega_2).$$

Then,

$$\begin{array}{rcl} d_s(\omega_1\bullet_{\mathbb{A}}\omega_2) & = & (-1)^{ns} d_s(p_{13}^*\omega_1\bullet p_{24}^*\omega_2) \\ & = & (-1)^{ns} d_{\mathcal{D}}(p_{13}^*\omega_1\bullet p_{24}^*\omega_2) + (-1)^{r+s+ns}\delta(p_{13}^*\omega_1\bullet p_{24}^*\omega_2) \\ & = & (-1)^{ns} d_{\mathcal{D}}(p_{13}^*\omega_1)\bullet p_{24}^*\omega_2 + (-1)^{r+ns} p_{13}^*\omega_1\bullet d_{\mathcal{D}}(p_{24}^*\omega_2) + \\ & & +(-1)^{r+s+ns}\delta(p_{13}^*\omega_1)\bullet p_{24}^*\omega_2 + (-1)^{r+s+n+ns} p_{13}^*\omega_1\bullet \delta(p_{24}^*\omega_2) \\ & = & d_s(\omega_1)\bullet_{\mathbb{A}}\omega_2 + (-1)^{r+n}\omega_1\bullet_{\mathbb{A}} d_s(\omega_2), \end{array}$$

as desired. □

If $X = Y$, composing with the pull-back along the diagonal, we obtain an internal product

$$\mathcal{D}^r_{\mathbb{A}}(X,p)\otimes \mathcal{D}^s_{\mathbb{A}}(X,q) \xrightarrow{\bullet_{\mathbb{A}}} \mathcal{D}^{r+s}_{\mathbb{A}}(X,p+q).$$

Remark 3.7.2. Let p_1 and p_2 denote the projections

$$X\times \square^{n+m} \xrightarrow{p_1} X\times\square^n, \qquad X\times\square^{n+m}\xrightarrow{p_2} X\times\square^m,$$

onto the first n coordinates and the last m coordinates, respectively. Observe that the internal product then agrees with the morphism

$$\begin{array}{rcl} \mathcal{D}^r_{\mathbb{A}}(X,p)\otimes \mathcal{D}^s_{\mathbb{A}}(X,q) & \xrightarrow{\bullet_{\mathbb{A}}} & \mathcal{D}^{r+s}_{\mathbb{A}}(X,p+q) \\ (\omega_1,\omega_2) & \mapsto & (-1)^{ns'} p_1^*\omega_1\bullet p_2^*\omega_2. \end{array}$$

if $\omega_1\in \mathcal{D}^{r'}_{\log}(X\times\square^n,p)$ and $\omega_2\in\mathcal{D}^{s'}_{\log}(X\times\square^m,q)$.

Proposition 3.7.3. *The product $\bullet_{\mathbb{A}}$ induces the Beilinson product in cohomology:*

$$H^*_{\mathcal{D}}(X,\mathbb{R}(p))\otimes H^*_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{\bullet} H^*_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Proof. The result follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{D}^r_{\log}(X,p) \otimes \mathcal{D}^s_{\log}(X,p) \arrow[r, "\bullet"] \arrow[d, "\sim"'] & \mathcal{D}^{r+s}_{\log}(X,p) \arrow[d, "\sim"] \\
\mathcal{D}^r_{\mathbb{A}}(X,p)_0 \otimes \mathcal{D}^s_{\mathbb{A}}(X,p)_0 \arrow[r, "\bullet_{\mathbb{A}}"'] & \mathcal{D}^{r+s}_{\mathbb{A}}(X,p)_0
\end{tikzcd}$$

where the vertical arrows are the quasi-isomorphisms of corollary 3.3.1. □

Observe that the product $\bullet_{\mathbb{A}}$ induces a product

$$\begin{array}{ccc} s(\widehat{\mathcal{D}}^*_{\mathbb{A}}(X,p)_0 \otimes \widehat{\mathcal{D}}^*_{\mathbb{A}}(X,q)_0) & \xrightarrow{\bullet_{\mathbb{A}}} & \widehat{\mathcal{D}}^*_{\mathbb{A}}(X,p+q)_0 \\ (\omega_1,\omega_2) & \mapsto & \omega_1 \bullet_{\mathbb{A}} \omega_2. \end{array} \tag{3.7.4}$$

The following remark will be important in order to discuss the commutativity of the product on the higher arithmetic Chow groups.

Remark 3.7.5. Let $\mathcal{D}^*_{\log}(X \times \square^* \times \square^*, p)_0$ be the 3-iterated cochain complex whose $(r,-n,-m)$-th graded piece is the group $\mathcal{D}^r_{\log}(X \times \square^n \times \square^m, p)_0$ and whose differentials are $(d_{\mathcal{D}}, \delta, \delta)$. Let

$$\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_0 := s(\mathcal{D}^*_{\log}(X \times \square^* \times \square^*, p)_0) \tag{3.7.6}$$

be the associated simple complex. Observe that there is a cochain morphism

$$\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_0 \xrightarrow{\kappa} \mathcal{D}^*_{\mathbb{A}}(X,p)_0$$

sending $\alpha \in \mathcal{D}^r_{\log}(X \times \square^n \times \square^m, p)$ to $\alpha \in \mathcal{D}^r_{\log}(X \times \square^{n+m}, p)$ under the identification

$$\begin{array}{ccc} \square^{n+m} & \xrightarrow{\cong} & \square^n \times \square^m \\ (x_1,\dots,x_{n+m}) & \mapsto & ((x_1,\dots,x_n),(x_{n+1},\dots,x_{n+m})). \end{array}$$

The complex $\widehat{\mathcal{D}}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_0$ is defined analogously.

Let

$$s(\widehat{\mathcal{D}}^*_{\mathbb{A}}(X,p)_0 \otimes \widehat{\mathcal{D}}^*_{\mathbb{A}}(Y,q)_0) \xrightarrow{\bullet_{\mathbb{A}}} \widehat{\mathcal{D}}^*_{\mathbb{A}\times\mathbb{A}}(X \times Y, p)_0$$

be the morphism sending $\alpha \in \widehat{\mathcal{D}}^r_{\log}(X \times \square^n, p)_0$ and $\beta \in \widehat{\mathcal{D}}^s_{\log}(Y \times \square^m, q)_0$ to $\alpha \bullet_{\mathbb{A}} \beta \in \widehat{\mathcal{D}}^{r+s}_{\log}(X \times Y \times \square^n \times \square^m, p+q)_0$. Then, we have a factorization

$$\bullet_{\mathbb{A}} : s(\widehat{\mathcal{D}}^*_{\mathbb{A}}(X,p)_0 \otimes \widehat{\mathcal{D}}^*_{\mathbb{A}}(Y,q)_0) \xrightarrow{\bullet_{\mathbb{A}}} \widehat{\mathcal{D}}^*_{\mathbb{A}\times\mathbb{A}}(X \times Y, p)_0 \xrightarrow{\kappa} \widehat{\mathcal{D}}^*_{\mathbb{A}}(X \times Y, p+q)_0. \tag{3.7.7}$$

3.7.2 Product structure on the complex $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)$

In this section we define a product on the complex $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)$. It will be compatible with the product on $\mathcal{D}^*_{\mathbb{A}}(X,p)$, under the morphism ρ, and with the product on $\mathcal{H}^p(X,*)_0$ under g_1.

Let X and Y be two real varieties. For every p, let $\mathcal{Z}^p_{X,n}$ be the subset of codimension p subvarieties of $X\times\square^n$ intersecting properly the faces of $\square^n$. Let

$$\mathcal{Z}^{p,q}_{X,Y,n,m}=\mathcal{Z}^p_{X,n}\times\mathcal{Z}^q_{Y,m}\subseteq\mathcal{Z}^{p+q}_{X\times Y,n+m}$$

be the subset of the set of codimension $p+q$ subvarieties of $X\times Y\times\square^{n+m}$, intersecting properly the faces of $\square^{n+m}$, which are obtained as the cartesian product $Z\times W$ with $Z\in\mathcal{Z}^p_{X,n}$ and $W\in\mathcal{Z}^q_{Y,m}$.

For shorthand, we make the following identifications:

$$\begin{array}{rcl}\mathcal{Z}^q_{Y,m}=X\times\square^n\times\mathcal{Z}^q_{Y,m} & \subseteq & \mathcal{Z}^q_{X\times Y,n+m},\\ \mathcal{Z}^p_{X,n}=\mathcal{Z}^p_{X,n}\times Y\times\square^m & \subseteq & \mathcal{Z}^p_{X\times Y,n+m}.\end{array}$$

To ease the notation, we write temporarily

$$\square^{n,m}_{X,Y}:=X\times Y\times\square^n\times\square^m.$$

For every n,m,p,q, let $j^{p,q}_{X,Y}(n,m)$ be the morphism

$$\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y}\setminus\mathcal{Z}^p_{X,n},p+q)\oplus\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y}\setminus\mathcal{Z}^q_{Y,m},p+q)\xrightarrow{j^{p,q}_{X,Y}(n,m)}\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y}\setminus\mathcal{Z}^p_{X,n}\cup\mathcal{Z}^q_{Y,m},p+q)$$

defined as the morphism j in lemma 1.4.6.

Lemma 3.7.8. *There is a short exact sequence*

$$0\to\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y}\setminus\mathcal{Z}^{p,q}_{X,Y,n,m},p+q)\to\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y}\setminus\mathcal{Z}^p_{X,n},p+q)\oplus\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y}\setminus\mathcal{Z}^q_{Y,m},p+q)$$
$$\xrightarrow{j^{p,q}_{X,Y}(n,m)}\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y}\setminus\mathcal{Z}^p_{X,n}\cup\mathcal{Z}^q_{Y,m},p+q)\to 0.$$

Proof. It follows from lemma 1.4.6. □

By the quasi-isomorphism between the simple complex and the kernel of an epimorphism (see lemma 1.2.13), there is a quasi-isomorphism

$$\begin{array}{rcl}\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y}\setminus\mathcal{Z}^{p,q}_{X,Y,n,m},p+q) & \xrightarrow{\sim} & s(-j^{p,q}_{X,Y}(n,m))^*\\ \omega & \mapsto & (\omega,\omega,0),\end{array}$$

for every n,m. It induces a quasi-isomorphism

$$\mathcal{D}^*_{\log,\mathcal{Z}^{p,q}_{X,Y,n,m}}(\square^{n,m}_{X,Y},p+q)\xrightarrow{\sim}s(\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y},p+q)^*\xrightarrow{i^{p,q}_{X,Y}(n,m)}s(-j^{p,q}_{X,Y}(n,m)))^*,\quad(3.7.9)$$

where $i^{p,q}_{X,Y}(n,m)$ is defined by

$$\begin{array}{rcl}\mathcal{D}^*_{\log}(\square^{n,m}_{X,Y},p+q) & \xrightarrow{i^{p,q}_{X,Y}(n,m)} & s(-j^{p,q}_{X,Y}(n,m))^*\\ \omega & \mapsto & (\omega,\omega,0).\end{array}$$

Remark 3.7.10. Observe that the face and degeneracy maps δ_i^j induce a bicubical cochain complex structure on $s(i_{X,Y}^{p,q}(\cdot,\cdot))^*$. For every r, let $s(i_{X,Y}^{p,q}(*,*))_0^r$ denote the 2-iterated complex obtained by applying the normalized complex in both cubical directions. Consider the 3-iterated complex $s(i_{X,Y}^{p,q}(*,*))_0^*$ which, in degree $(r,-n,-m)$, is the group $s(i_{X,Y}^{p,q}(n,m))_0^r$ and the differential is (d_s,δ,δ). Denote by $s(i_{X,Y}^{p,q})_0^*$ the associated simple complex. Observe that the differential of $\alpha=(\alpha_0,\alpha_1,\alpha_2,\alpha_3)\in s(i_{X,Y}^{p,q})_0^r$ is given by

$$d_s'(\alpha_0,\alpha_1,\alpha_2,\alpha_3)=(d_{\mathcal{D}}\alpha_0,\alpha_0-d_{\mathcal{D}}\alpha_1,\alpha_0-d_{\mathcal{D}}\alpha_2,-\alpha_1+\alpha_2+d_{\mathcal{D}}\alpha_3).$$

We define the complexes

$$\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^p}(X,p)_0 \quad\text{and}\quad \mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y,p+q)_0$$

analogous to the complex $\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_0$ of remark 3.7.5. That is:

- $\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^p}(X,p)_0$ is the simple complex associated to the 3-iterated complex whose $(r,-n,-m)$ graded piece is $\mathcal{D}^r_{\log,\mathcal{Z}^p}(X\times\square^n\times\square^m,p)_0$.
- $\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y,p+q)_0$ is the simple associated to the 3-iterated complex whose $(r,-n,-m)$ graded piece is $\mathcal{D}^r_{\log,\mathcal{Z}^{p,q}_{X,Y,n,m}}(X\times Y\times\square^n\times\square^m,p+q)_0$.

We obtain morphisms

$$\begin{aligned}\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^p}(X,p)_0 &\xrightarrow{\rho} \mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_0\xrightarrow{\kappa}\mathcal{D}^*_{\mathbb{A}}(X,p)_0,\\ \mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y,p+q)_0 &\xrightarrow{\rho} \mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X\times Y,p)_0\xrightarrow{\kappa}\mathcal{D}^*_{\mathbb{A}}(X\times Y,p)_0,\\ s(i_{X,Y}^{p,q})_0^* &\xrightarrow{\rho} \mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X\times Y,p)_0\xrightarrow{\kappa}\mathcal{D}^*_{\mathbb{A}}(X\times Y,p)_0.\end{aligned}$$

There are also natural maps

$$\begin{aligned}\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y,p+q)_0 &\rightarrow \mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p+q}}(X\times Y,p+q)_0,\\ \mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p+q}}(X\times Y,p+q)_0 &\xrightarrow{\kappa} \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^{p+q}}(X\times Y,p+q)_0.\end{aligned}$$

Lemma 3.7.11. *The natural map*

$$\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y,p+q)_0\xrightarrow{\sim}s(i_{X,Y}^{p,q})_0^* \tag{3.7.12}$$

is a quasi-isomorphism. Moreover, it commutes with ρ.

Proof. It follows from the quasi-isomorphism (3.7.9), and lemma 1.2.10. □

Definition 3.7.13. Let $\bullet_{\mathbb{A}}$ be the map

$$\mathcal{D}^r_{\log,\mathcal{Z}^p}(X\times\square^n,p)_0\otimes\mathcal{D}^s_{\log,\mathcal{Z}^q}(Y\times\square^m,q)_0 \xrightarrow{\bullet_{\mathbb{A}}} s(i_{X,Y}^{p,q}(n,m))_0^{r+s}$$

defined by sending

$$(\omega,g)\otimes(\omega',g')\mapsto(-1)^{ns}(\omega\bullet\omega',(g\bullet\omega',(-1)^r\omega\bullet g'),(-1)^{r-1}g\bullet g').$$

Lemma 3.7.14. *The map $\bullet_{\mathbb{A}}$ defines a pairing of complexes*

$$s(\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \otimes \mathcal{D}^*_{\mathbb{A},\mathcal{Z}^q}(Y,q)_0) \xrightarrow{\bullet_{\mathbb{A}}} s(i^{p,q}_{X,Y})^*_0.$$

Proof. Let $(\omega, g) \in \mathcal{D}^r_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0$ and $(\omega', g') \in \mathcal{D}^s_{\log,\mathcal{Z}^q}(Y \times \square^m, q)_0$. Then, we have to see that

$$d'_s((\omega,g) \bullet_{\mathbb{A}} (\omega',g')) = d'_s(\omega,g) \bullet_{\mathbb{A}} (\omega',g') + (-1)^{r-n}(\omega,g) \bullet_{\mathbb{A}} d'_s(\omega',g').$$

This is equivalent to checking the two equalities:

$$\begin{aligned} d_s((\omega,g) \bullet_{\mathbb{A}} (\omega',g')) &= d_s(\omega,g) \bullet_{\mathbb{A}} (\omega',g') + (-1)^{r-n}(\omega,g) \bullet_{\mathbb{A}} d_s(\omega',g') \\ \delta((\omega,g) \bullet_{\mathbb{A}} (\omega',g')) &= (-1)^s \delta(\omega,g) \bullet_{\mathbb{A}} (\omega',g') + (-1)^n(\omega,g) \bullet_{\mathbb{A}} \delta(\omega',g'). \end{aligned}$$

The proof of the second equality follows analogously to the proof of lemma 3.7.1. The first equality is a direct computation. $\square$

Therefore, the external product on $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^*}(\cdot,*)_0$ is given, in the derived category of complexes, by

$$\begin{tikzcd} \mathcal{D}^r_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \otimes \mathcal{D}^s_{\mathbb{A},\mathcal{Z}^q}(Y,q)_0 \arrow[r, "\bullet_{\mathbb{A}}"] & s(i^{p,q}_{X,Y})^{r+s}_0 & \\ & \mathcal{D}^{r+s}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X \times Y, p+q)_0 \arrow[u, "\sim"] \arrow[r, "\kappa"] & \mathcal{D}^{r+s}_{\mathbb{A},\mathcal{Z}^{p+q}}(X \times Y, p+q)_0. \end{tikzcd}$$

The internal product is then obtained by composing with the pull-back along the diagonal map.

3.7.3 Product structure on the higher arithmetic Chow groups

At this point, we have defined a product structure for each of the complexes in the diagram $\widehat{\mathcal{Z}}^p(X,*)_0$. In order to define the product structure on the diagram, it remains to see that the product on $\mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ is compatible with the product on $\mathcal{H}^p(X,n)_0$, under the quasi-isomorphism g_1.

Let $\omega \in s(i^{p,q}_{X\times Y})^{2p+2q-l}_0$ and let

$$(\omega^0_l, \dots, \omega^l_l) \in \bigoplus_{j=0}^{l} s(i^{p,q}_{X,Y}(j, l-j))^{2p+2q}_0$$

be the components of ω corresponding to the degree $(2p+2q, -j, j-l)$. In particular, $d_s\omega^j_l = 0$. Since there is a quasi-isomorphism

$$\mathcal{D}^*_{\log,\mathcal{Z}^{p,q}_{X,Y}}(X \times Y \times \square^l, p+q)_0 \xrightarrow{\sim} s(i^{p,q}_{X,Y}(j, l-j))^*_0,$$

the cohomology class $[\omega_l^j]$ in the complex $s(i^{p,q}_{X,Y}(j,l-j))^*_0$ defines a cohomology class in $H^*(\mathcal{D}^*_{\log,\mathcal{Z}^{p,q}_{X,Y}}(X\times Y\times\square^l,p+q)_0)$. Hence, it defines a cohomology class $[\omega_l^j]\in\mathcal{H}^{p+q}(X\times Y,l)$, which defines a chain morphism

$$g_1 : s(i^{p,q}_{X,Y})_0^{2p+2q-*}\to\mathcal{H}^{p+q}(X\times Y,*)_0.$$

Proposition 3.7.15. *Let $Z\in\mathcal{Z}^p_{n,X}$ and $T\in\mathcal{Z}^q_{m,Y}$. Let $[(\omega_Z,g_Z)]\in\mathcal{H}^p(X,n)_0$ represent the class of a cycle $z\in Z^p(X,n)_0$ and $[(\omega_T,g_T)]\in\mathcal{H}^q(Y,m)_0$ represent the class of a cycle $t\in Z^q(Y,m)_0$. Then,*

$$[(\omega_Z,g_Z)\bullet_{\mathbb{A}}(\omega_T,g_T)]\in\mathcal{H}^{p+q}(X\times Y,n+m)_0$$

represents the class of the cycle $z\times t$ in $Z^{p+q}(X\times Y,n+m)_0$.

Proof. It follows from [24], Theorem 4.2.3 and [13], Theorem 7.7. □

For every p,q, the structure of the product on each of the complexes of diagram $\widehat{\mathcal{Z}}^p(X,*)_0$ is described, in bidegree (n,m), by the following diagram:

$$\begin{tikzcd}[column sep=small]
 & \mathcal{H}^p(X,n)_0\otimes\mathcal{H}^q(Y,m)_0 \arrow[dd, "\times"] & & \widehat{\mathcal{D}}^{2p-n}_{\mathbb{A}}(X,p)_0\otimes\widehat{\mathcal{D}}^{2q-m}_{\mathbb{A}}(Y,q)_0 \arrow[dd, "\bullet_{\mathbb{A}}"] \\
Z^p(X,n)_0\otimes Z^q(Y,m)_0 \arrow[ur, "f_1"] \arrow[dd, "\times"] & & \mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0\otimes\mathcal{D}^{2q-m}_{\mathbb{A},\mathcal{Z}^q}(Y,q)_0 \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\rho"] \arrow[dd, "\bullet_{\mathbb{A}}"] & \\
 & \mathcal{H}^{p+q}(X\times Y,n+m)_0 \arrow[dd, equal] & & \widehat{\mathcal{D}}^{2(p+q)-n-m}_{\mathbb{A}\times\mathbb{A}}(X\times Y,p+q)_0 \arrow[dd, equal] \\
Z^{p+q}(X\times Y,n+m)_0 \arrow[ur, "f_1"] \arrow[dd, equal] & & s(i^{p,q}_{X,Y})_0^{2p+2q-n-m} \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\rho"] & \\
 & \mathcal{H}^{p+q}(X\times Y,n+m)_0 \arrow[dd, equal] & & \widehat{\mathcal{D}}^{2(p+q)-n-m}_{\mathbb{A}\times\mathbb{A}}(X\times Y,p+q)_0 \arrow[dd, "\kappa"] \\
Z^{p+q}(X\times Y,n+m)_0 \arrow[ur, "f_1"] \arrow[dd, equal] & & \mathcal{D}^{2(p+q)-n-m}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y,p+q)_0 \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\rho"] \arrow[uu, "\sim"] \arrow[dd, "\kappa"] & \\
 & \mathcal{H}^{p+q}(X\times Y,n+m)_0 & & \widehat{\mathcal{D}}^{2(p+q)-n-m}_{\mathbb{A}}(X\times Y,p+q)_0 \\
Z^{p+q}(X\times Y,n+m)_0 \arrow[ur, "f_1"] & & \mathcal{D}^{2(p+q)-n-m}_{\mathbb{A},\mathcal{Z}^{p+q}}(X\times Y,p+q)_0 \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\rho"] &
\end{tikzcd}$$

This diagram induces a morphism in the derived category of chain complexes

$$s(\widehat{\mathcal{Z}}^p(X,*)_0\otimes\widehat{\mathcal{Z}}^q(Y,*)_0)\xrightarrow{\cup}\widehat{Z}^{p+q}(X\times Y,*)_0.$$

By section 3.1.1, for any $\beta \in \mathbb{Z}$ there is a morphism $\star_\beta$

$$\widehat{Z}^p(X,*)_0 \otimes \widehat{Z}^q(Y,*)_0 \xrightarrow{\star_\beta} s(\widehat{\mathcal{Z}}^p(X,*)_0 \otimes \widehat{\mathcal{Z}}^q(Y,*)_0).$$

The composition of $\star_\beta$ with $\cup$ induces a product

$$\widehat{CH}^p(X,n) \otimes \widehat{CH}^q(Y,m) \xrightarrow{\cup} \widehat{CH}^{p+q}(X \times Y, n+m),$$

independent of β.

Finally the pull-back by the diagonal map $X \xrightarrow{\Delta} X \times X$ gives an internal product on $\widehat{CH}^p(X,*)$:

$$\widehat{CH}^p(X,n) \otimes \widehat{CH}^q(X,m) \xrightarrow{\cup} \widehat{CH}^{p+q}(X \times X, n+m) \xrightarrow{\Delta^*} \widehat{CH}^{p+q}(X, n+m).$$

In the derived category of complexes, the product is given by the composition

$$\begin{tikzcd}
\widehat{Z}^p(X,n)_0 \otimes \widehat{Z}^q(X,m)_0 \arrow[d, "\star_\beta"] & & \\
s(\widehat{\mathcal{Z}}^p(X,n)_0 \otimes \widehat{\mathcal{Z}}^q(X,m)_0) \arrow[r, "\cup"] & \widehat{Z}^{p+q}(X \times X, n+m)_0 & \\
& \widehat{Z}^{p+q}_\Delta(X \times X, n+m)_0 \arrow[u, "\sim"'] \arrow[r, "\Delta^*"] & \widehat{Z}^{p+q}(X, n+m)_0.
\end{tikzcd}$$

Remark 3.7.16. It follows from the definition that, for $n = 0$, the product $\cup$ agrees with the product on the arithmetic Chow group $\widehat{CH}^p(X)$ defined by Burgos in [13].

3.8 Commutativity of the product

Let X, Y be arithmetic varieties over a field F. In this section, we prove that the pairing defined in the previous section on the higher arithmetic Chow groups is commutative, in the sense detailed below.

We first introduce some notation:

▷ If B_*, C_* are chain complexes, let

$$\sigma : s(B_* \otimes C_*) \to s(C_* \otimes B_*)$$

be the map sending $b \otimes c \in B_n \otimes C_m$ to $(-1)^{nm} c \otimes b \in C_m \otimes B_n$.

▷ Let $\sigma_{X,Y}$ be the morphism

$$\sigma_{X,Y} : X \times Y \to Y \times X$$

interchanging X with Y.

We will prove that there is a commutative diagram

$$\begin{tikzcd}
\widehat{CH}^p(X,n)\otimes\widehat{CH}^q(Y,m) \arrow[r, "\cup"] \arrow[d, "\sigma"'] & \widehat{CH}^{p+q}(X\times Y,n+m) \arrow[d, "\sigma^*"] \\
\widehat{CH}^q(Y,m)\otimes\widehat{CH}^p(X,n) \arrow[r, "\cup"] & \widehat{CH}^{p+q}(Y\times X,n+m)
\end{tikzcd}$$

In particular, the internal product on the higher arithmetic Chow groups will be graded commutative with respect to the degree n and commutative with respect to the degree p. That is, if $W\in\widehat{CH}^p(X,n)$ and $Z\in\widehat{CH}^q(X,m)$, then

$$W\cup Z=(-1)^{nm}Z\cup W.$$

Recall that, by definition, the product factorizes as

$$\widehat{CH}^p(X,n)\otimes\widehat{CH}^q(Y,m)\xrightarrow{\star_\beta}H_{n+m}(s(\widehat{\mathcal{Z}}^p(X,*)_0\otimes\widehat{\mathcal{Z}}^q(Y,*)_0))\xrightarrow{\cup}\widehat{CH}^{p+q}(X\times Y,n+m).$$

Since the first morphism gives the desired commutative diagram (see lemma 3.1.10), all that remains is to check the commutativity for

$$s(\widehat{\mathcal{Z}}^p(X,*)_0\otimes\widehat{\mathcal{Z}}^q(Y,*)_0)\overset{\cup}{\dashrightarrow}\widehat{\mathcal{Z}}^{p+q}(X\times Y,*)_0. \tag{3.8.1}$$

We want to see that, in the derived category of chain complexes, there is a commutative diagram

$$\begin{tikzcd}
s(\widehat{\mathcal{Z}}^p(X,*)_0\otimes\widehat{\mathcal{Z}}^q(Y,*)_0) \arrow[r, "\sigma"] \arrow[d, "\cup"'] & s(\widehat{\mathcal{Z}}^q(Y,*)_0\otimes\widehat{\mathcal{Z}}^p(X,*)_0) \arrow[d, "\cup"] \\
\widehat{\mathcal{Z}}^{p+q}(X\times Y,*)_0 \arrow[r, "\sigma^*_{X,Y}"'] & \widehat{\mathcal{Z}}^{p+q}(Y\times X,*)_0.
\end{tikzcd}$$

The obstruction to strict commutativity comes from the change of coordinates

$$\begin{array}{ccc}
\square^{n+m}=\square^m\times\square^n & \xrightarrow{\sigma_{n,m}} & \square^n\times\square^m=\square^{n+m} \\
(y_1,\dots,y_m,x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_n,y_1,\dots,y_m).
\end{array} \tag{3.8.2}$$

Recall that the product is described by the big diagram in 3.7.3. In order to prove the commutativity, we change the second and third row diagrams of the big diagram, by more suitable diagrams. These changes do not modify the definition of the product, but ease the study of the commutativity.

3.8.1 Alternative description of the product on $\widehat{CH}^*(X,*)$

New diagram for the second row. We define the complex $Z^p_{\mathbb{A}\times\mathbb{A}}(X,n)_0$ analogously to the way we defined the complex $\widehat{\mathcal{D}}_{\mathbb{A}\times\mathbb{A}}(X,p)_0$ (see 3.7.5). Let

$$Z^p(X,n,m)_0 := Z^p(X,n+m)_0,$$

and let

$$\delta^1 = \sum_{i=1}^{n}(\delta_i^0-\delta_i^1), \qquad \delta^2 = \sum_{i=n+1}^{n+m}(\delta_i^0-\delta_i^1).$$

Then, with differentials (δ^1,δ^2), $Z^p(X,*,*)_0$ is a 2-iterated chain complex. For the sake of simplicity, we denote both δ^1,δ^2 by δ.

Denote by $Z^p_{\mathbb{A}\times\mathbb{A}}(X,*)_0$ the associated simple complex. The chain complex $\mathcal{H}^p_{\mathbb{A}\times\mathbb{A}}(X,*)_0$ is defined analogously.

Let $\widehat{\mathcal{Z}}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0$ be the diagram

$$\begin{tikzcd}[column sep=small]
 & \mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0 & & \widehat{\mathcal{D}}^{2(p+q)-*}_{\mathbb{A}\times\mathbb{A}}(X\times Y,p+q)_0 \\
Z^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0 \arrow[ur, "f_1"] & & s(i^{p,q}_{X,Y})_0^{2(p+q)-*} \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\rho"'] &
\end{tikzcd}$$

This is the diagram fitting in the second row of the big diagram in 3.7.3. Denote by $\widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0$ the simple complex associated to this diagram.

New diagram for the third row. The third row of the new big diagram corresponds to a diagram whose complexes are obtained from the refined normalized complex of section 3.1.2. The point is that in these complexes most of the face maps are zero. This is the key point to construct explicit homotopies for the commutativity of the product. So, consider the following complexes:

- Let $Z^q(X,*,*)_{00}$ be the 2-iterated chain complex with

  $$Z^q(X,n,m)_{00} := \bigcap_{i\neq 0,n+1} \ker\delta_i^0 \subset Z^q(X,n+m)_0,$$

  and with differentials $(\delta',\delta'')=(-\delta_1^0,-\delta_{n+1}^0)$. Denote by $Z^q_{\mathbb{A}\times\mathbb{A}}(X,*)_{00}$ the associated simple complex.

- Let $\mathcal{D}^*_{\log}(X\times\square^*\times\square^*,p)_{00}$ be the 3-iterated complex whose $(r,-n,-m)$-graded piece is

  $$\mathcal{D}^r_{\log}(X\times\square^n\times\square^m,p)_{00} = \bigcap_{i\neq 0,n+1}\ker\delta_i^0 \subset \mathcal{D}^r_{\log}(X\times\square^{n+m},p)_0,$$

  and with differentials $(d_{\mathcal{D}},-\delta_1^0,-\delta_{n+1}^0)$. Let $\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_{00}$ be the associated simple complex. Let $\widehat{\mathcal{D}}^*_{\log}(X\times\square^*\times\square^*,p)_{00}$ be defined analogously.

► Let $\mathcal{D}^*_{\mathcal{Z}^{p,q}_{X,Y,*,*}}(X\times Y\times\square^*\times\square^*, p+q)_{00}$ be the 3-iterated complex whose $(r,-n,-m)$-graded piece is the subgroup of $\mathcal{D}^r_{\mathcal{Z}^{p,q}_{X,Y,n,m}}(X\times Y\times\square^{n+m}, p+q)_0$:

$$\mathcal{D}^r_{\mathcal{Z}^{p,q}_{X,Y,n,m}}(X\times Y\times\square^n\times\square^m, p+q)_{00} = \bigcap_{i\neq 0,n+1}\ker\delta_i^0$$

and has differentials $(d_{\mathcal{D}}, -\delta_1^0, -\delta_{n+1}^0)$. Let $\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y, p)_{00}$ be the associated simple complex.

Remark 3.8.3. Observe that there are induced morphisms

$$\begin{array}{rcl} Z^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00} & \xrightarrow{f_1} & \mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00}, \\ \mathcal{D}^{2(p+q)-*}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y, p+q)_{00} & \xrightarrow{g_1} & \mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00}, \\ \mathcal{D}^{2(p+q)-*}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y, p+q)_{00} & \xrightarrow{\rho} & \widehat{\mathcal{D}}^{2(p+q)-*}_{\mathbb{A}\times\mathbb{A}}(X\times Y, p+q)_{00} \end{array}$$

Then, let $\widehat{\mathcal{Z}}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00}$ be the diagram

$$\begin{tikzcd}[column sep=tiny] & \mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00} & & \widehat{\mathcal{D}}^{2(p+q)-*}_{\mathbb{A}\times\mathbb{A}}(X\times Y, p+q)_{00} \\ Z^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00} \arrow[ur, "f_1"] & & \mathcal{D}^{2(p+q)-*}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y, p+q)_{00} \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\rho"] & \end{tikzcd}$$

This is the diagram fitting in the third row of the new diagram analogous to 3.7.3. Let $\widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00}$ be the simple complex associated to this diagram.

Lemma 3.8.4. *Let X be an arithmetic variety over a field.*

(i) The natural chain morphisms

$$\begin{array}{rcl} Z^q_{\mathbb{A}\times\mathbb{A}}(X,*)_{00} & \xrightarrow{i} & Z^q_{\mathbb{A}\times\mathbb{A}}(X,*)_0, \\ Z^q_{\mathbb{A}\times\mathbb{A}}(X,*)_0 & \xrightarrow{\kappa} & Z^q(X,*)_0, \end{array}$$

are quasi-isomorphisms.

(ii) The natural cochain morphisms

$$\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_{00} \xrightarrow{i} \mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_0, \tag{3.8.5}$$

$$\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y, p+q)_{00} \xrightarrow{i} \mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y, p+q)_0, \tag{3.8.6}$$

$$\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X,p)_0 \xrightarrow{\kappa} \mathcal{D}^*_{\mathbb{A}}(X,p)_0, \tag{3.8.7}$$

are quasi-isomorphisms.

Proof. The proofs of the fact that the morphisms i are quasi-isomorphisms are analogous for the three cases. For every n, m, let $B(n,m)$ denote either $Z^p(X,n,m)$, $\mathcal{D}^r_{\log}(X \times \square^n \times \square^m, p)$ or $\mathcal{D}^r_{\log,\mathcal{Z}^{p,q}_{X,Y,n,m}}(X \times Y \times \square^n \times \square^m, p+q)$, for some r. The groups $B(n,m)_0$ and $B(n,m)_{00}$ are defined analogously.

Observe that for every n, m, $B(\cdot, m)$ and $B(n, \cdot)$ are cubical abelian groups. We want to see that there is a quasi-isomorphism

$$s(N_0^2 N_0^1 B(*,*)) \xrightarrow{i} s(N^2 N^1 B(*,*)), \tag{3.8.8}$$

where superindex 1 refers to the cubical structure given by the first index n and superindex 2 to the cubical structure given by the second index m.

We first prove that there is a quasi-isomorphism

$$s(N^2 N_0^1 B(*,*)) \xrightarrow{\sim} s(N^2 N^1 B(*,*)). \tag{3.8.9}$$

Let m be a fixed index. Then, $B(\cdot, m)$ is a cubical abelian group. Hence, by lemma 3.1.14 with maps h_j as in (3.2.3), there is a quasi-isomorphism

$$N_0^1 B(*,m) \xrightarrow{i} N^1 B(*,m).$$

By proposition 3.1.17, there is a quasi-isomorphism

$$N^2 N_0^1 B(*,m) \xrightarrow{i} N^2 N^1 B(*,m),$$

for every m. Hence, from lemma 1.2.10 we deduce (3.8.9).

To prove that the morphism (3.8.8) is a quasi-isomorphism, all that remains to see is that there is a quasi-isomorphism

$$s(N_0^2 N_0^1 B(*,*)) \xrightarrow{i} s(N^2 N_0^1 B(*,*)). \tag{3.8.10}$$

For every n fixed, $N_0^1 B(n,\cdot)$ is a cubical abelian group. Then, it follows from lemma 3.1.14, that, for every n, there is a quasi-isomorphism

$$N_0^2 N_0^1 B(n,*) \xrightarrow{i} N^2 N_0^1 B(n,*).$$

Then, by lemma 1.2.10, (3.8.10) is proved.

The proof of the fact that the morphisms in (3.8.5) and (3.8.7) are quasi-isomorphisms is also completely analogous. Therefore, we just prove the statement for the morphism (3.8.5). Consider the composition morphism

$$j : Z^q(X,m)_0 \to Z^q(X,0,m)_0 \to Z^q_{\mathbb{A}\times\mathbb{A}}(X,m)_0.$$

The composition of morphisms $Z^q(X,m)_0 \xrightarrow{j} Z^q_{\mathbb{A}\times\mathbb{A}}(X,m)_0 \xrightarrow{\kappa} Z^q(X,m)_0$ is the identity. Hence, it is enough to see that j is a quasi-isomorphism. Consider the 1st quadrant spectral sequence with $E^1_{n,m} = H_m(Z^q(X, n, *)_0)$. We will see that if $n \geq 1$, $E^1_{n,m} = 0$. By the homotopy invariance of higher Chow groups, the map

$$f : Z^q(X \times \square^n, *)_0 \xrightarrow{\delta_1^1 \cdots \delta_1^1} Z^q(X, *)_0$$

is a quasi-isomorphism. By proposition 3.1.17, it induces a quasi-isomorphism

$$f : Z^q(X \times \square^n, *)_0 = NZ^q(X \times \square^n, *)_0 \to NZ^q(X, *)_0$$

where the cubical structure on $Z^q(X, *)_0$ is the trivial one (see example 1.2.38). Since for a trivial cubical abelian group $NZ^q(X, *)_0 = 0$, we see that for $n > 0$, $H_m(Z^q(X, n, *)_0) = 0$ and hence

$$E^1_{n,m} = \begin{cases} 0 & \text{if } n > 0, \\ CH^q(X, m) & \text{if } n = 0. \end{cases}$$

□

It follows from the lemma that the product on the higher arithmetic Chow groups is represented by the diagram of complexes

$$\begin{tikzcd}
& \mathcal{H}^p(X,n)_0 \otimes \mathcal{H}^q(Y,m)_0 \arrow[dd, "\times"] & & \widehat{\mathcal{D}}^{2p-n}_{\mathbb{A}}(X,p)_0 \otimes \widehat{\mathcal{D}}^{2q-m}_{\mathbb{A}}(Y,q)_0 \arrow[dd, "\bullet_{\mathbb{A}}"] \\
Z^p(X,n)_0 \otimes Z^q(Y,m)_0 \arrow[ur, "f_1"] \arrow[dd, "\times"] & & \mathcal{D}^{2p-n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \otimes \mathcal{D}^{2q-m}_{\mathbb{A},\mathcal{Z}^q}(Y,q)_0 \arrow[ul, "g_1"', "\sim"] \arrow[ur, "\rho"] \arrow[dd, "\bullet_{p,q}"] & \\
& \mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X \times Y, n+m)_0 & & \widehat{\mathcal{D}}^{2(p+q)-n-m}_{\mathbb{A}\times\mathbb{A}}(X \times Y, p+q)_0 \\
Z^{p+q}_{\mathbb{A}\times\mathbb{A}}(X \times Y, n+m)_0 \arrow[ur, "f_1"] & & s(i^{p,q}_{X,Y})^{2p+2q-n-m}_0 \arrow[ul, "g_1"', "\sim"] \arrow[ur, "\rho"] & \\
& \mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X \times Y, n+m)_{00} \arrow[uu, "\sim"', "i"] \arrow[dd, equal] & & \widehat{\mathcal{D}}^{2(p+q)-n-m}_{\mathbb{A}\times\mathbb{A}}(X \times Y, p+q)_{00} \arrow[uu, "\sim"', "i"] \arrow[dd, equal] \\
Z^{p+q}_{\mathbb{A}\times\mathbb{A}}(X \times Y, n+m)_{00} \arrow[uu, "\sim"', "i"] \arrow[ur, "f_1"] \arrow[dd, equal] & & \mathcal{D}^{2(p+q)-n-m}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X \times Y, p+q)_{00} \arrow[uu, "\sim"', "i"] \arrow[ul, "g_1"', "\sim"] \arrow[ur, "\rho"] \arrow[dd, "\kappa"] & \\
& \mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X \times Y, n+m)_{00} \arrow[dd, "\kappa"] & & \widehat{\mathcal{D}}^{2(p+q)-n-m}_{\mathbb{A}\times\mathbb{A}}(X \times Y, p+q)_{00} \arrow[dd, "\kappa"] \\
Z^{p+q}_{\mathbb{A}\times\mathbb{A}}(X \times Y, n+m)_{00} \arrow[ur, "f_1"] \arrow[dd, "\kappa"] & & \mathcal{D}^{2(p+q)-n-m}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p+q}}(X \times Y, p+q)_{00} \arrow[ul, "g_1"', "\sim"] \arrow[ur, "\rho"] \arrow[dd, "\kappa"] & \\
& \mathcal{H}^{p+q}(X \times Y, n+m)_0 & & \widehat{\mathcal{D}}^{2(p+q)-n-m}_{\mathbb{A}}(X \times Y, p+q)_0 \\
Z^{p+q}(X \times Y, n+m)_0 \arrow[ur, "f_1"] & & \mathcal{D}^{2(p+q)-n-m}_{\mathbb{A},\mathcal{Z}^{p+q}}(X \times Y, p+q)_0 \arrow[ul, "g_1"', "\sim"] \arrow[ur, "\rho"] &
\end{tikzcd}$$

Let $\widehat{Z}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00}$ denote the simple of the diagram of the fourth row. Hence, in the derived category of complexes, this product is described by the composition

$$\begin{tikzcd} s(\widehat{\mathcal{Z}}^p(X,*)_0\otimes\widehat{\mathcal{Z}}^q(Y,*)_0) \arrow[r, "\cup"] & \widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0 & \\ & \widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00} \arrow[u, dashed, "i"] \arrow[r, "\kappa"] & \widehat{Z}^{p+q}(X\times Y,*)_0. \end{tikzcd}$$

3.8.2 Proof of the commutativity

In this section, we use the description of the product given in the previous section, in order to prove the commutativity of the product in the higher arithmetic Chow groups.

The morphism $\sigma^*_{X,Y,\square}$. Recall that the map $\sigma_{n,m}$ is defined by

$$\begin{array}{ccc} \square^{n+m}=\square^m\times\square^n & \xrightarrow{\sigma_{n,m}} & \square^n\times\square^m=\square^{n+m} \\ (y_1,\dots,y_m,x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_n,y_1,\dots,y_m). \end{array}$$

Let

$$\sigma_{X,Y,n,m}:X\times Y\times\square^m\times\square^n\to Y\times X\times\square^n\times\square^m$$

be the composition of $\sigma_{n,m}$ with $\sigma_{X,Y}$.

We define a morphism of diagrams

$$\widehat{\mathcal{Z}}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0 \xrightarrow{\sigma^*_{X,Y,\square}} \widehat{\mathcal{Z}}^{q,p}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_0$$

as follows:

- Let $\sigma^*_{X,Y,\square}:Z^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0\to Z^{p+q}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_0$ be the map sending

  $$Z\in Z^p(X\times Y,n,m)_0 \quad\mapsto\quad (-1)^{nm}\sigma^*_{X,Y,n,m}(Z)\in Z^{p+q}(Y\times X,m,n)_0.$$

  The morphism $\sigma^*_{X,Y,\square}:\mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0\to\mathcal{H}^{p+q}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_0$ is defined analogously.

- Let $\sigma^*_{X,Y,\square}:\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(X\times Y,p+q)_0\to\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}}(Y\times X,p+q)_0$ be the map that at the $(*,-n,-m)$ component is

  $$(-1)^{nm}\sigma^*_{X,Y,n,m}:\mathcal{D}^*_{\log}(X\times Y\times\square^n\times\square^m,p+q)_0\to\mathcal{D}^*_{\log}(Y\times X\times\square^m\times\square^n,p+q)_0.$$

  Observe that it is a cochain morphism.

- We define analogously the morphism

  $$\sigma^*_{X,Y,\square}:s(i^{p,q}_{X,Y})^*_0\to s(i^{q,p}_{Y,X})^*_0.$$

Observe that all these morphisms commute with the morphisms f_1, g_1 and ρ. Hence, they induce a morphism of diagrams and therefore a morphism on the associated simple complexes:

$$\widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0 \xrightarrow{\sigma^*_{X,Y,\Box}} \widehat{Z}^{q,p}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_0.$$

Observe that the morphism $\sigma^*_{X,Y,\Box}$ restricts to $\widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00}$ and to $\widehat{Z}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00}$.

Lemma 3.8.11. *The following diagram is commutative:*

$$\begin{tikzcd}
\widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0 \arrow[d, "\sigma^*_{X,Y,\Box}"'] & \widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00} \arrow[l, "i"'] \arrow[r, "\kappa"] \arrow[d, "\sigma^*_{X,Y,\Box}"'] & \widehat{Z}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00} \arrow[d, "\sigma^*_{X,Y,\Box}"'] \\
\widehat{Z}^{q,p}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_0 & \widehat{Z}^{q,p}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_{00} \arrow[l, "i"'] \arrow[r, "\kappa"] & \widehat{Z}^{p+q}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_{00}.
\end{tikzcd}$$

Proof. The statement follows from the definitions. □

Lemma 3.8.12. *The following diagram is commutative*

$$\begin{tikzcd}
s(\widehat{\mathcal{Z}}^p(X,*)_0\otimes\widehat{\mathcal{Z}}^q(Y,*)_0) \arrow[r, "\cup"] \arrow[d, "\sigma"'] & \widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_0 \arrow[d, "\sigma^*_{X,Y,\Box}"] \\
s(\widehat{\mathcal{Z}}^q(Y,*)_0\otimes\widehat{\mathcal{Z}}^p(X,*)_0) \arrow[r, "\cup"] & \widehat{Z}^{p,q}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_0.
\end{tikzcd}$$

Proof. It follows from the definition that the morphism $\sigma^*_{X,Y,\Box}$ commutes with the product $\times$ in $Z^*(X,*)_0$ and in $H^*(X,*_0)$. The fact that it commutes with $\bullet_{\mathbb{A}}$ and $\bullet_{p,q}$ is an easy computation. For instance, let $\alpha\otimes\beta\in\mathcal{D}^r_{\log}(X\times\Box^n,p)_0\otimes\mathcal{D}^s_{\log}(Y\times\Box^m,q)_0$. Then, $\sigma(\alpha\otimes\beta)=(-1)^{(r-n)(s-m)}\beta\otimes\alpha$ and:

$$\begin{aligned}
(-1)^{(r-n)(s-m)}\beta\bullet_{\mathbb{A}}\alpha &= (-1)^{(r-n)(s-m)+rm}p_{13}^*\beta\bullet p_{24}^*\alpha \\
&= (-1)^{(r-n)(s-m)+rm+rs}p_{24}^*\alpha\bullet p_{13}^*\beta \\
&= (-1)^{n(s-m)}\sigma^*_{X,Y,n,m}(p_{13}^*\alpha\bullet p_{24}^*\beta) \\
&= (-1)^{nm}\sigma^*_{X,Y,n,m}(\alpha\bullet_{\mathbb{A}}\beta)
\end{aligned}$$

as desired. □

A homotopy for the commutativity of the product. By lemmas 3.8.11 and 3.8.12, we are left to see that the diagram

$$\begin{tikzcd}
\widehat{Z}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00} \arrow[r, "\kappa"] \arrow[d, "\sigma^*_{X,Y,\Box}"'] & \widehat{Z}^{p+q}(X\times Y,*)_0 \arrow[d, "\sigma^*_{X,Y}"] \\
\widehat{Z}^{p+q}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_{00} \arrow[r, "\kappa"] & \widehat{Z}^{p+q}(Y\times X,*)_0
\end{tikzcd} \qquad (3.8.13)$$

is commutative up to homotopy. We follow the ideas used by Levine, in [41], §4, in order to prove the commutativity of the product on the higher algebraic Chow groups. We will end up with an explicit homotopy for the commutativity of diagram 3.8.13.

Remark 3.8.14. For any scheme X, consider the morphism

$$\widehat{Z}^p_{\mathbb{A}\times\mathbb{A}}(X,*)_{00} \xrightarrow{\sigma^*_\square} \widehat{Z}^p_{\mathbb{A}\times\mathbb{A}}(X,*)_{00}$$

induced by $(-1)^{n,m}\sigma^*_{n,m}$ at each component. Then, $\sigma^*_{X,Y,\square} = \sigma^*_{X,Y}\sigma^*_\square$ and hence, the commutativity of the diagram (3.8.13) follows from the commutativity of the diagram

$$\begin{tikzcd} \widehat{Z}^p_{\mathbb{A}\times\mathbb{A}}(X,*)_{00} \arrow[dr, "\kappa"] \arrow[dd, "\sigma^*_\square"'] & \\ & \widehat{Z}^p(X,*)_0. \\ \widehat{Z}^p_{\mathbb{A}\times\mathbb{A}}(X,*)_{00} \arrow[ur, "\kappa"'] & \end{tikzcd}$$

Let W_n be the closed subvariety of $\square^{n+1}\times\mathbb{P}^1$ defined by the equation

$$t_0(1-x_1)(1-x_{n+1}) = t_0 - t_1, \tag{3.8.15}$$

where $(t_0 : t_1)$ are the coordinates in $\mathbb{P}^1$ and $(x_1,\dots,x_{n+1})$ are the coordinates in $\square^{n+1}$. Since there is no solution for $t_0 = 0$, identifying $\square^1 \subseteq \mathbb{P}^1$ with the locus of $\mathbb{P}^1$ with $t_0 \neq 0$, there is an isomorphism $W_n \cong \square^n \times \square^1$.

Fix

$$\begin{array}{ccc} \square^{n+1} & \xrightarrow{\varphi_n} & W_n \\ (x_1,\dots,x_{n+1}) & \mapsto & (x_1,\dots,x_{n+1}, x_1+x_{n+1}-x_1x_{n+1}). \end{array}$$

to be the isomorphism and let

$$\pi_n : W_n \to \square^n$$

be the projection defined by

$$(x_1,\dots,x_{n+1},t) \mapsto (x_2,\dots,x_{n-1},t).$$

Let τ be the permutation

$$\begin{array}{ccc} \square^n & \xrightarrow{\tau} & \square^n \\ (x_1,\dots,x_n) & \mapsto & (x_2,\dots,x_n,x_1). \end{array}$$

It is easy to check that the following identities are satisfied:

$$\begin{aligned} \pi_n\varphi_n\delta_i^0 &= \begin{cases} id & \text{if } i=1, \\ \delta^0_{i-1}\pi_{n-1}\varphi_{n-1} & \text{if } i=2,\dots,n, \\ \tau & \text{if } i=n+1. \end{cases} \\ \pi_n\varphi_n\delta_i^1 &= \begin{cases} \delta^1_n & \text{if } i=1, \\ \delta^1_{i-1}\pi_{n-1}\varphi_{n-1} & \text{if } i=2,\dots,n, \\ \tau\delta^1_1 & \text{if } i=n+1. \end{cases} \end{aligned} \tag{3.8.16}$$

Let W_n^X be the pull-back of W_n to $X \times \square^n$. Then, the maps

$$\pi_n : W_n^X \to X \times \square^n, \quad \text{and} \quad \varphi_n : X \times \square^{n+1} \to W_n^X$$

are defined accordingly.

Proposition 3.8.17. *Let X be a quasi-projective regular scheme over a field k.*

(i) The scheme W_n is a flat regular scheme over $\square^n$.

(ii) There is a well-defined map

$$\begin{array}{rcl} Z^q(X,n) & \xrightarrow{h_n} & Z^q(X,n+1) \\ Y & \mapsto & \varphi_n^* \pi_n^*(Y). \end{array}$$

Proof. See [41], Lemma 4.1. □

Remark 3.8.18. Let $\sigma_{n,m}$ be the map defined in (3.8.2). Observe that it is decomposed as

$$\sigma_{n,m} = \tau \circ \overset{m}{\ldots} \circ \tau.$$

Therefore,

$$\sigma_{n,m}^* = \tau^* \circ \overset{m}{\ldots} \circ \tau^*.$$

For every $n \geq 1$, we define the morphisms

$$\begin{array}{rcl} \mathcal{H}^p(X,n) & \xrightarrow{h_n} & \mathcal{H}^p(X,n+1), \\ \mathcal{D}^*_{\log}(X \times \square^n, p) & \xrightarrow{h_n} & \mathcal{D}^*_{\log}(X \times \square^{n+1}, p), \\ \mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p) & \xrightarrow{h_n} & \mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^{n+1}, p), \end{array}$$

by $h_n = \varphi_n^* \pi_n^*$. By proposition 3.8.17, (ii), these morphisms are well defined.

Lemma 3.8.19. *Let α be an element of $Z^q(X,n)_0$, $\mathcal{H}^p(X,n)_0$, $\mathcal{D}^*_{\log}(X \times \square^n, p)_0$ or $\mathcal{D}^*_{\log,\mathcal{Z}^p}(X \times \square^n, p)_0$. Then, the following equality is satisfied*

$$\delta h_n(\alpha) + \sum_{i=1}^{n-1} (-1)^i h_{n-1} \delta_i^0(\alpha) = -\alpha + (-1)^{n-1} \tau^*(\alpha).$$

Proof. By hypothesis, $\delta_i^1(\alpha) = 0$ for all $i = 1, \ldots, n$. Then, by the pull-back of the equalities (3.8.16), we see that $\delta_i^1 \varphi_n^* \pi_n^*(\alpha) = 0$. Therefore, using (3.8.16),

$$\begin{aligned} \delta h_n(\alpha) &= \sum_{i=1}^{n+1} \sum_{j=0,1} (-1)^{i+j} \delta_i^j \varphi_n^* \pi_n^*(\alpha) = \sum_{i=1}^{n+1} (-1)^i \delta_i^0 \varphi_n^* \pi_n^*(\alpha) \\ &= -\alpha + \sum_{i=2}^{n} (-1)^i \varphi_n^* \pi_{n-1}^* \delta_{i-1}^0(\alpha) + (-1)^{n-1} \tau^*(\alpha) \\ &= -\alpha - \sum_{i=1}^{n-1} (-1)^i h_{n-1} \delta_i^0(\alpha) + (-1)^{n-1} \tau^*(\alpha), \end{aligned}$$

as desired. □

Proposition 3.8.20. *Let X be an arithmetic variety over a field. Then, up to homotopy, the following diagram is commutative*

$$\begin{tikzcd} \widehat{Z}^p_{\mathbb{A}\times\mathbb{A}}(X,n)_{00} \arrow[d, "\sigma^*_{\square}"'] \arrow[dr, "\kappa"] & \\ & \widehat{Z}^p(X,n)_0. \\ \widehat{Z}^p_{\mathbb{A}\times\mathbb{A}}(X,n)_{00} \arrow[ur, "\kappa"'] & \end{tikzcd}$$

Proof. We start by defining maps

$$\begin{array}{rcl} Z^p(X,n,m)_{00} & \xrightarrow{H_{n,m}} & Z^p(X,n+m+1)_0, \\ \mathcal{H}^p(X,n,m)_{00} & \xrightarrow{H_{n,m}} & \mathcal{H}^p(X,n+m+1)_0, \\ \mathcal{D}^*_{\log}(X\times\square^n\times\square^m,p)_{00} & \xrightarrow{H_{n,m}} & \mathcal{D}^*_{\log}(X\times\square^{n+m+1},p)_0, \\ \mathcal{D}^*_{\log,\mathcal{Z}^p}(X\times\square^n\times\square^m,p)_{00} & \xrightarrow{H_{n,m}} & \mathcal{D}^*_{\log,\mathcal{Z}^p}(X\times\square^{n+m+1},p)_0. \end{array}$$

By construction, these maps will commute with f_1, g_1 and ρ. This will allow us to define the homotopy for the commutativity of the diagram in the statement.

For all n,m, the maps $H_{n,m}$ will all be defined in the same way. Therefore, let $B(X,n,m)_{00}$ denote either $Z^p(X,n,m)_{00}$, $\mathcal{H}^p(X,n,m)_{00}$, $\mathcal{D}^*_{\log}(X\times\square^n\times\square^m,p)_{00}$, or $\mathcal{D}^*_{\log,\mathcal{Z}^p}(X\times\square^n\times\square^m,p)_{00}$. For the last two cases, $B(X,n,m)_{00}$ is a cochain complex, while for the first two cases, it is a group. Analogously, denote by $B(X,n+m+1)_0$ the groups/complexes that are the target of $H_{n,m}$. The map $H_{n,m}$ will be a cochain complex for the last two cases.

Let $\alpha\in B(X,n,m)_{00}$. Then, let $H_{n,m}(\alpha)\in B(X,n+m+1)_0$ be defined by

$$H_{n,m}(\alpha)=\begin{cases} \sum_{i=0}^{n-1}(-1)^{(m+i)(n+m-1)}h_{n+m+1}((\tau^*)^{m+i}(\alpha)), & n\neq 0,\\ 0 & n=0.\end{cases} \tag{3.8.21}$$

From the definition it follows that:

- If $B(X,n,m)_{00}$ is $\mathcal{D}^*_{\log}(X\times\square^n\times\square^m,p)_{00}$, or $\mathcal{D}^*_{\log,\mathcal{Z}^p}(X\times\square^n\times\square^m,p)_{00}$, then

  $$d_{\mathcal{D}}H_{n,m}(\alpha)=H_{n,m}d_{\mathcal{D}}(\alpha),$$

  i.e. $H_{n,m}$ is a cochain morphism.

- $f_1H_{n,m}=H_{n,m}f_1$, $g_1H_{n,m}=H_{n,m}g_1$ and $\rho H_{n,m}=H_{n,m}\rho$.

Recall that in all these complexes,

$$\begin{array}{rcl} \delta'(\alpha) & = & -\delta^0_1(\alpha)\in B(X,n-1,m)_{00}, \\ \delta''(\alpha) & = & -\delta^0_{n+1}(\alpha)\in B(X,n,m-1)_{00}. \end{array}$$

Lemma 3.8.22. *For every $\alpha\in B(X,n,m)_{00}$ we have*

$$\delta H_{n,m}(\alpha)-H_{n-1,m}\delta^0_1(\alpha)-(-1)^nH_{n,m-1}\delta^0_{n+1}(\alpha)=\alpha-(-1)^{nm}\sigma^*_{n,m}(\alpha).$$

Proof. If $n = 0$, since $\alpha = \sigma_{0,m}(\alpha)$ and $H_{0,m}(\alpha) = 0$ the equality is satisfied. For simplicity, for every $i = 0, \dots, n-1$, we denote

$$H^i_{n,m}(\alpha) = (-1)^{(m+i)(n+m-1)} h_{n+m+1}((\tau^*)^{m+i}(\alpha)) \in B(X, n+m+1)_0.$$

An easy computation shows that

$$\delta^0_j \tau^*(\alpha) = \begin{cases} \tau^* \delta^0_{j-1}(\alpha) & \text{if } j \neq 1, \\ \delta^0_n(\alpha) & \text{if } j = 1, \end{cases}$$

and hence,

$$\delta^0_j (\tau^*)^i(\alpha) = \begin{cases} (\tau^*)^i \delta^0_{j-i}(\alpha) & \text{if } j > i, \\ (\tau^*)^{i-1} \delta^0_n(\alpha) & \text{if } j = i, \\ (\tau^*)^{i-1} \delta^0_{n-i+j}(\alpha) & \text{if } j < i. \end{cases}$$

Therefore,

$$\begin{aligned} \delta H^i_{n,m}(\alpha) &= \sum_{j=1}^{n+m+1} (-1)^{j+(m+i)(n+m-1)} \delta^0_j h_{n+m+1}((\tau^*)^{m+i}(\alpha)) \\ &= (-1)^{1+(m+i)(n+m-1)} (\tau^*)^{m+i}(\alpha) \\ &\quad + (-1)^{(m+i+1)(n+m-1)} (\tau^*)^{m+i+1}(\alpha) \\ &\quad + \sum_{j=2}^{n+m} (-1)^{j+(m+i)(n+m-1)} h_{n+m}(\delta^0_{j-1} (\tau^*)^{m+i}(\alpha)). \end{aligned}$$

Recall that the only non-zero faces of α are δ^0_1 and δ^0_{n+1}. Then, in the last summand the terms correspond to $j - 1 = 2, \dots, n+m$ and $i = 0, \dots, n-1$. Therefore, from the equalities (3.8.16), we see that the only non-zero faces are the faces corresponding to the indices $j = m+i+2$ and $j = i+2$. In these cases, they take the values $(\tau^*)^{m+i}\delta^0_1$ and $(\tau^*)^{m+i-1}\delta^0_{n+1}$ respectively. Therefore, if $i \neq n-1$, we obtain

$$\begin{aligned} \delta H^i_{n,m}(\alpha) &= -(-1)^{(m+i)(n+m-1)} (\tau^*)^{m+i}(Z) \\ &\quad + (-1)^{(m+i+1)(n+m-1)} (\tau^*)^{m+i+1}(\alpha) \\ &\quad + (-1)^{(m+i)(n+m-2)} h_{n+m}((\tau^*)^{m+i} \delta^0_1(\alpha)) \\ &\quad + (-1)^{i+(m+i)(n+m-1)} h_{n+m}((\tau^*)^{m-1+i} \delta^0_{n+1}(\alpha)). \end{aligned}$$

Observe that $(-1)^{i+(m+i)(n+m-1)} = (-1)^{(m+i-1)(n+m)+n}$. Therefore, the last summand in the previous equality is exactly

$$H^i_{n-1,m}(\delta^0_1(\alpha)) + (-1)^n H^i_{n,m-1}(\delta^0_{n+1}(\alpha)).$$

If $i = n-1$, then $\delta^0_{j-1}(\tau^*)^{m+i}(\alpha) = 0$, for $j = 2, \dots, n-m$. Therefore,

$$\begin{aligned} \delta H^{n-1}_{n,m}(\alpha) &= (-1)^{1+(m+n-1)(n+m-1)} (\tau^*)^{m+n-1}(\alpha) \\ &\quad + (-1)^{(m+n)(n+m-1)} (\tau^*)^{m+n}(\alpha) \\ &\quad + (-1)^{n-1+(m+n-1)(n+m-1)} h_{n+m}((\tau^*)^{m-1+i} \delta^0_{n+1}(\alpha)) \\ &= -(-1)^{(m+n-1)(n+m-1)} (\tau^*)^{m+n-1}(\alpha) + \alpha \\ &\quad + (-1)^{n+(m+n-2)(n+m)} h_{n+m}((\tau^*)^{m-1+i} \delta^0_{n+1}(\alpha)). \end{aligned}$$

Finally, we have seen that

$$\begin{aligned}\delta H_{n,m}(\alpha) &= -(-1)^{m(n+m-1)}(\tau^*)^m(\alpha)+\sum_{i=0}^{n-2} H^i_{n-1,m}(\delta_1^0(\alpha))\\ &\quad +\sum_{i=0}^{n-1}(-1)^n H^i_{n,m-1}(\delta^0_{n+1}(\alpha))+\alpha,\end{aligned}$$

and since $(-1)^{m(n+m-1)}=(-1)^{nm}$, we obtain the equality

$$\delta H_{n,m}(\alpha)-H_{n-1,m}(\delta_1^0(\alpha))-(-1)^n H_{n,m-1}(\delta^0_{n+1}(\alpha))=\alpha-(-1)^{nm}\sigma^*_{n,m}(\alpha).$$

□

Let

$$Z^p_{\mathbb{A}\times\mathbb{A}}(X,*)_{00}\xrightarrow{H} Z^p(X,*+1)_0,\qquad H^p_{\mathbb{A}\times\mathbb{A}}(X,*)_{00}\xrightarrow{H} H^p(X,*+1)_0,$$

be the maps which are $H_{n,m}$ on the (n,m)-component. Let

$$\widehat{\mathcal{D}}^{2p-*}_{\mathbb{A}\times\mathbb{A}}(X,p)_{00}\xrightarrow{H}\widehat{\mathcal{D}}^{2p-*-1}_{\mathbb{A}}(X,p)_0,\qquad \mathcal{D}^{2p-*}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^p}(X,p)_{00}\xrightarrow{H}\mathcal{D}^{2p-*-1}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0,$$

be the maps which are $(-1)^r H_{n,m}$ on the $(r,-n,-m)$-component. Observe that now

$$d_{\mathcal{D}}H=-Hd_{\mathcal{D}}.$$

Let

$$H:\widehat{Z}^p_{\mathbb{A}\times\mathbb{A}}(X,n)_{00}\to\widehat{Z}^p(X,n+1)_0$$

be defined by

$$H(Z,\alpha_0,\alpha_1,\alpha_2)=(H(Z),H(\alpha_0),-H(\alpha_1),-H(\alpha_2)).$$

Let $x=(Z,\alpha_0,\alpha_1,\alpha_2)\in\widehat{Z}^p_{\mathbb{A}\times\mathbb{A}}(X,n)_{00}$. Then,

$$\begin{aligned}DH(x)&=(\delta H(Z),d_sH(\alpha_0),f_1H(Z)-g_1H(\alpha_0)+\delta H(\alpha_1),\rho H(\alpha_0)+d_sH(\alpha_2))\\ HD(x)&=(H\delta(Z),Hd_s(\alpha_0),-Hf_1(Z)+Hg_1(\alpha_0)+H\delta(\alpha_1),-H\rho(\alpha_0)+Hd_s(\alpha_2)).\end{aligned}$$

Observe that for $\alpha_0\in\mathcal{D}^r_{\log,\mathcal{Z}^p}(X\times\square^n\times\square^m,p)_{00}$, we have

$$\begin{aligned}Hd_s(\alpha_0)&=Hd_{\mathcal{D}}(\alpha_0)+(-1)^rH\delta(\alpha_0)=-d_{\mathcal{D}}H(\alpha_0)+(-1)^rH\delta(\alpha_0),\\ d_sH(\alpha_0)&=d_{\mathcal{D}}H(\alpha_0)+(-1)^r\delta H(\alpha_0).\end{aligned}$$

The same remark applies to $\alpha_2\in\mathcal{D}^r_{\log}(X\times\square^n\times\square^m,p)_{00}$. Therefore, by lemma 3.8.22, we obtain

$$DH(x)+HD(x)=x-\sigma^*_{\square}(x).$$

□

Corollary 3.8.23. *The following diagram is commutative up to homotopy*

$$\begin{tikzcd} \widehat{Z}^{p+q}_{\mathbb{A}\times\mathbb{A}}(X\times Y,*)_{00} \arrow[r, "\kappa"] \arrow[d, "\sigma^*_{X,Y,\square}"'] & \widehat{Z}^{p+q}(X\times Y,*)_0 \arrow[d, "\sigma^*_{X,Y}"] \\ \widehat{Z}^{p+q}_{\mathbb{A}\times\mathbb{A}}(Y\times X,*)_{00} \arrow[r, "\kappa"] & \widehat{Z}^{p+q}(Y\times X,*)_0 \end{tikzcd}$$

Proof. It follows from proposition 3.8.20. □

Corollary 3.8.24. *Let X, Y be arithmetic varieties.*

(i) Under the canonical isomorphism $X\times Y\cong Y\times X$, the pairing

$$\widehat{CH}^p(X,n)\otimes\widehat{CH}^q(Y,m)\xrightarrow{\cup}\widehat{CH}^{p+q}(X\times Y,n+m),$$

is graded commutative with respect to the degree n and commutative with respect to the degree p.

(ii) The internal pairing

$$\widehat{CH}^p(X,n)\otimes\widehat{CH}^q(X,m)\xrightarrow{\cup}\widehat{CH}^{p+q}(X,n+m),$$

is graded commutative with respect to the degree n and commutative with respect to the degree p.

3.9 Associativity

In this section we prove that the product for the higher arithmetic Chow groups is associative. First of all, observe that the product on $Z^*(X,*)_0$ is strictly associative. Hence, all that remains is to study the associativity of the product in the complexes with differential forms. The key point will be proposition 1.4.1.

Denote by h the homotopy for the associativity of the product in the Deligne complex of differential forms of proposition 1.4.1. Let X, Y, Z be complex algebraic manifolds. Then, the external product $\bullet_{\mathbb{A}}$ is associative, in the sense that there is a commutative diagram up to homotopy:

$$\begin{tikzcd}[column sep=small] & \mathcal{D}^r_{\mathbb{A}}(X,p)_0\otimes\mathcal{D}^s_{\mathbb{A}}(Y,q)_0\otimes\mathcal{D}^t_{\mathbb{A}}(Z,l)_0 \arrow[dl, "\bullet_{\mathbb{A}}\otimes id"'] \arrow[dr, "id\otimes\bullet_{\mathbb{A}}"] & \\ \mathcal{D}^{r+s}_{\mathbb{A}}(X\times Y,p+q)_0\otimes\mathcal{D}^t_{\mathbb{A}}(Z,l)_0 \arrow[dr, "\bullet_{\mathbb{A}}"'] & & \mathcal{D}^r_{\mathbb{A}}(X,p)_0\otimes\mathcal{D}^{s+t}_{\mathbb{A}}(Y\times Z,q+l)_0 \arrow[dl, "\bullet_{\mathbb{A}}"] \\ & \mathcal{D}^{r+s+t}_{\mathbb{A}}(X\times Y\times Z,p+q+l)_0 & \end{tikzcd} \tag{3.9.1}$$

This follows from the fact that the homotopy h is functorial (see [13]).

Proposition 3.9.2. *Let X, Y, Z be complex algebraic manifolds. Then, there is a commutative diagram, up to homotopy:*

$$\begin{tikzcd}
 & \mathcal{D}^r_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \otimes \mathcal{D}^s_{\mathbb{A},\mathcal{Z}^q}(Y,q)_0 \otimes \mathcal{D}^t_{\mathbb{A},\mathcal{Z}^l}(Z,l)_0 \arrow[dl, "\bullet_{\mathbb{A}}\otimes id"'] \arrow[dr, "id\otimes\bullet_{\mathbb{A}}"] & \\
\mathcal{D}^{r+s}_{\mathbb{A},\mathcal{Z}^{p+q}}(X\times Y,p+q)_0 \otimes \mathcal{D}^t_{\mathbb{A},\mathcal{Z}^l}(Z,l)_0 \arrow[dr, "\bullet_{\mathbb{A}}"'] & & \mathcal{D}^r_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \otimes \mathcal{D}^{s+t}_{\mathbb{A},\mathcal{Z}^{q+l}}(Y\times Z,q+l)_0 \arrow[dl, "\bullet_{\mathbb{A}}"] \\
 & \mathcal{D}^{r+s+t}_{\mathbb{A},\mathcal{Z}^{p+q+l}}(X\times Y\times Z,p+q+l)_0 &
\end{tikzcd}$$

Proof. In order to prove the proposition, we need to introduce some new complexes, which are analogous to $s(i^{p,q}_{X,Y})^*$, but with the three varieties X, Y, Z. Due to the similarity, we will not explain all the details. We leave the details to the reader.

We write $\square^{n,m,d}_{X,Y,Z} = X \times Y \times Z \times \square^{n+m+d}$. Let

$$A^* = \mathcal{D}^*_{\log}(\square^{n,m,d}_{X,Y,Z} \setminus \mathcal{Z}^p_{X,n}, k) \oplus \mathcal{D}^*_{\log}(\square^{n,m,d}_{X,Y,Z} \setminus \mathcal{Z}^q_{Y,m}, k) \oplus \mathcal{D}^*_{\log}(\square^{n,m,d}_{X,Y,Z} \setminus \mathcal{Z}^l_{Z,d}, k),$$

and

$$B^* = \mathcal{D}^*_{\log}(\square^{n,m,d}_{X,Y,Z} \setminus \mathcal{Z}^{p,q}_{X,Y,n,m}, k) \oplus \mathcal{D}^*_{\log}(\square^{n,m,d}_{X,Y,Z} \setminus \mathcal{Z}^{p,l}_{X,Z,n,d}, k) \oplus \mathcal{D}^*_{\log}(\square^{n,m,d}_{X,Y,Z} \setminus \mathcal{Z}^{q,l}_{Y,Z,m,d}, k),$$

and consider the sequence of morphisms of complexes

$$A^* \xrightarrow{i} B^* \xrightarrow{j} \mathcal{D}^*_{\log}(\square^{n,m,d} \setminus \mathcal{Z}^{p,q,l}_{X,Y,Z}, k).$$

By analogy with the definition of $s(-j^{p,q}_{X,Y}(n,m))^*$, denote by $s(-j^{p,q,l}_{X,Y,Z}(n,m,d))^*$ the simple complex associated to this sequence of morphisms (see remark 1.2.14). Consider the morphism

$$\begin{array}{ccc}
\mathcal{D}^*_{\log}(\square^{n,m,d}_{X,Y,Z}, k) & \xrightarrow{i^{p,q,l}_{X,Y,Z}(n,m,d)} & s(-j^{p,q,l}_{X,Y,Z}(n,m,d))^* \\
\omega & \mapsto & (\omega,\omega,\omega,0,0,0,0).
\end{array}$$

Observe that for every n, m, d, the simple of this morphism is a cochain complex. Moreover, considering the normalized complex associated to the cubical structure at every component of $s(i^{p,q,l}_{X,Y,Z}(\cdot,\cdot,\cdot))^*$, we obtain the cochain complex $s(i^{p,q,l}_{X,Y,Z})^*_0$ (analogous to the construction of $s(i^{p,q}_{X,Y})^*_0$ in remark 3.7.10).

Let $\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q,l}_{X,Y,Z}}(X\times Y\times Z, p+q+l)_0$ be the complex analogous to $\mathcal{D}^*_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}_{X,Y}}(X\times Y, p+q)_0$, but with the cartesian product of 3 varieties. It is the simple complex associated to the analogous 4-iterated complex (see page 139).

Observe that there is a quasi-isomorphism

$$\mathcal{D}^*_{\mathbb{A}\times\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q,l}_{X,Y,Z}}(X\times Y\times Z, p+q+l)_0 \xrightarrow{\sim} s(i^{p,q,l}_{X,Y,Z})^*_0.$$

We define a pairing

$$s(i^{p,q}_{X,Y}(n,m))^r_0 \otimes \mathcal{D}^{s,d}_{\mathbb{A},\mathcal{Z}^l}(Z,l)_0 \xrightarrow{\bullet} s(i^{p,q,l}_{X,Y,Z}(n,m,d))^{r+s}_0$$

by

$$\begin{aligned}(a,(b,c),d)\bullet(a',b') \quad &= \quad (-1)^{(n+m)s}(a\bullet a',(b\bullet a',c\bullet a',(-1)^r a\bullet b'),\\ &\qquad (d\bullet a',(-1)^{r-1}b\bullet b',(-1)^{r-1}c\bullet b'),(-1)^{r-2}d\bullet b').\end{aligned}$$

Define analogously a pairing

$$\mathcal{D}^{r,n}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0\otimes s(i^{q,l}_{Y,Z}(m,d))^s_0 \xrightarrow{\bullet} s(i^{p,q,l}_{X,Y,Z}(n,m,d))^{r+s}_0$$

by

$$\begin{aligned}(a,b)\bullet(a',(b',c'),d') \quad &= \quad (-1)^{ns}(a\bullet a',(b\bullet a',(-1)^r a\bullet b',(-1)^r a\bullet c'),\\ &\qquad ((-1)^{r-1}b\bullet b',(-1)^{r-1}b\bullet c',a\bullet d'),b\bullet d').\end{aligned}$$

It is easy to check that these two morphisms are chain morphisms.

Lemma 3.9.3. *The diagram*

$$\begin{tikzcd} & \mathcal{D}^r_{\mathbb{A},\mathcal{Z}^p}(X,p)_0\otimes\mathcal{D}^s_{\mathbb{A},\mathcal{Z}^q}(Y,q)_0\otimes\mathcal{D}^t_{\mathbb{A},\mathcal{Z}^l}(Z,l)_0 \arrow[dl, "\bullet_{p,q}\otimes id"'] \arrow[dr, "id\otimes\bullet_{\mathbb{A}}"] & \\ s(i^{p,q}_{X,Y})^{r+s}_0\otimes\mathcal{D}^t_{\mathbb{A},\mathcal{Z}^l}(Z,l)_0 \arrow[dr, "\bullet"'] & & \mathcal{D}^r_{\mathbb{A},\mathcal{Z}^p}(X,p)_0\otimes s(i^{q,l}_{Y,Z})^{s+t}_0 \arrow[dl, "\bullet"] \\ & s(i^{p,q,l}_{X,Y,Z})^{r+s+t}_0 & \end{tikzcd} \tag{3.9.4}$$

is commutative up to homotopy.

Proof. Let

$$\begin{aligned}(\omega_1,g_1) &\in \mathcal{D}^r_{\log,\mathcal{Z}^p}(X\times\square^n,p)_0,\\ (\omega_2,g_2) &\in \mathcal{D}^s_{\log,\mathcal{Z}^q}(Y\times\square^m,q)_0,\\ (\omega_3,g_3) &\in \mathcal{D}^t_{\log,\mathcal{Z}^l}(Z\times\square^d,l)_0.\end{aligned}$$

Then, the composition of the morphisms on the left side of the diagram is

$$\begin{aligned}&(-1)^{(n+m)t+ns}((\omega_1\bullet\omega_2)\bullet\omega_3,((g_1\bullet\omega_2)\bullet\omega_3,(-1)^r(\omega_1\bullet g_2)\bullet\omega_3,\\ &(-1)^{r+s}(\omega_1\bullet\omega_2)\bullet g_3),((-1)^{r-1}(g_1\bullet g_2)\bullet\omega_3,(-1)^{r+s-1}(g_1\bullet\omega_2)\bullet g_3,\\ &(-1)^{s-1}(\omega_1\bullet g_2)\bullet g_3),(-1)^{s-1}(g_1\bullet g_2)\bullet g_3).\end{aligned}$$

The composition of the morphisms on the right side of the diagram is

$$\begin{aligned}&(-1)^{(n+m)t+ns}(\omega_1\bullet(\omega_2\bullet\omega_3),(g_1\bullet(\omega_2\bullet\omega_3),(-1)^r\omega_1\bullet(g_2\bullet\omega_3),\\ &(-1)^{r+s}\omega_1\bullet(\omega_2\bullet g_3)),((-1)^{r-1}g_1\bullet(g_2\bullet\omega_3),(-1)^{r+s-1}g_1\bullet(\omega_2\bullet g_3),\\ &(-1)^{s-1}\omega_1\bullet(g_2\bullet g_3)),(-1)^{s-1}g_1\bullet(g_2\bullet g_3)).\end{aligned}$$

Then, the homotopy for the commutativity of the diagram is given by

$$\begin{aligned} H_{n,m,d} \quad = \quad & (-1)^{(n+m)t+ns}((h(\omega_1 \otimes \omega_2 \otimes \omega_3), h(g_1 \otimes \omega_2 \otimes \omega_3), \\ & (-1)^r h(\omega_1 \otimes g_2 \otimes \omega_3), (-1)^{r+s} h(\omega_1 \otimes \omega_2 \otimes g_3)), \\ & ((-1)^{r-1} h(g_1 \otimes g_2 \otimes \omega_3), (-1)^{r+s-1} h(g_1 \otimes \omega_2 \otimes g_3), \\ & (-1)^{s-1} h(\omega_1 \otimes g_2 \otimes g_3)), (-1)^{s-1} h(g_1 \otimes g_2 \otimes g_3)). \end{aligned}$$

Observe that it gives indeed a homotopy, since H and δ commute. $\square$

Finally, the claim of proposition 3.9.2 follows from the commutative diagram (all squares, apart from the one marked with # are strictly commutative),

$$\begin{tikzcd}
 & \mathcal{D}^r_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \otimes \mathcal{D}^s_{\mathbb{A},\mathcal{Z}^q}(Y,q)_0 \otimes \mathcal{D}^t_{\mathbb{A},\mathcal{Z}^l}(Z,l)_0 \arrow[dl, "\bullet_{p,q}\otimes id"'] \arrow[dr, "id\otimes\bullet_{q,l}"] & \\
s(i^{p,q}_{X,Y})^{r+s}_0 \otimes \mathcal{D}^t_{\mathbb{A},\mathcal{Z}^l}(Z,l)_0 \arrow[dr, "\bullet"'] & \# & \mathcal{D}^r_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \otimes s(i^{q,l}_{Y,Z})^{s+t}_0 \arrow[dl, "\bullet"] \\
 & s(i^{p,q,l}_{X,Y,Z})^{r+s+t}_0 \arrow[dd, equal] & \\
\mathcal{D}^{r+s}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q}}(X\times Y,p+q)_0 \otimes \mathcal{D}^t_{\mathbb{A},\mathcal{Z}^l}(Z,l)_0 \arrow[uu, "\sim"] \arrow[dd] & & \mathcal{D}^r_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \otimes \mathcal{D}^{s+l}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{q,t}}(Y\times Z,q+l)_0 \arrow[uu, "\sim"'] \arrow[dd] \\
 & s(i^{p,q,l}_{X,Y,Z})^{r+s+t}_0 \arrow[dl] \arrow[dr] & \\
s(i^{p+q,l}_{X\times Y,Z})^{r+s+t}_0 & & s(i^{p,q+l}_{X,Y\times Z})^{r+s+t}_0 \\
 & \mathcal{D}^{r+s+t}_{\mathbb{A}\times\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q,l}}(X\times Y\times Z,p+q+l)_0 \arrow[ul] \arrow[ur] \arrow[uu, "\sim"] \arrow[dl, "\kappa"'] \arrow[dr, "\kappa"] \arrow[dd, "\kappa"] & \\
\mathcal{D}^{r+s+t}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p+q,l}}(X\times Y\times Z,p+q+l)_0 \arrow[uu, "\sim"] \arrow[dr, "\kappa"] & & \mathcal{D}^{r+s+t}_{\mathbb{A}\times\mathbb{A},\mathcal{Z}^{p,q+l}}(X\times Y\times Z,p+q+l)_0 \arrow[uu, "\sim"'] \arrow[dl, "\kappa"'] \\
 & \mathcal{D}^{r+s+t}_{\mathbb{A},\mathcal{Z}^{p+q+l}}(X\times Y\times Z,p+q+l)_0. &
\end{tikzcd}$$

$\square$

Remark 3.9.5. Observe that the homotopy constructed in the proof of proposition 3.9.2 has no component in maximal degree, that is, in $\mathcal{D}^{2p+2q+2l}_{\mathbb{A},\mathcal{Z}^{p+q+l}}(X \times Y \times Z, p+q+l)_0$.

Corollary 3.9.6. *Let X, Y, Z be arithmetic varieties.*

(i) Under the canonical isomorphism $(X \times Y) \times Z \cong X \times (Y \times Z)$, the external pairing

$$\widehat{CH}^p(*,n) \otimes \widehat{CH}^q(*,m) \otimes \xrightarrow{\cup} \widehat{CH}^{p+q}(* \times *, n+m),$$

is associative.

(ii) The internal pairing

$$\widehat{CH}^p(X,n)\otimes \widehat{CH}^q(X,m) \xrightarrow{\cup} \widehat{CH}^{p+q}(X,n+m),$$

is associative.

Proof. It follows from (3.9.1) and proposition 3.9.2, together with remark 3.9.5 and the compatibility of the homotopies in (3.9.1) and proposition 3.9.2. For $n=m=l=0$, the associativity follows from equality (1.4.2). □

From sections 3.7, 3.8 and 3.9, we obtain the following theorem.

Theorem 3.9.7. *Let X be an arithmetic variety over an arithmetic field F. Then,*

$$\widehat{CH}^*(X,*) := \bigoplus_{p\geq 0, n\geq 0} \widehat{CH}^p(X,n)$$

is a commutative and associative ring with unity. Moreover, the morphism

$$\widehat{CH}^*(X,*) \ \xrightarrow{\zeta}\ CH^*(X,*),$$

of proposition 3.6.7, is a ring morphism.

3.10 Modified higher arithmetic Chow groups

In this final section, we sketch how a theory of higher arithmetic Chow groups could be constructed, following the pattern of the higher arithmetic K-theory defined by Takeda in [57].

In the previous sections, we developed the basis for a theory of higher arithmetic intersection theory, with the construction of the higher arithmetic Chow groups, and a product structure on them.

We focused on a theory that satisfied the following properties:

- Functoriality and product structure.
- For $n=0$, $\widehat{CH}^p(X,0)$ agrees with the arithmetic Chow group defined by Burgos in [13].
- Long exact sequence: There is a long exact sequence

$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$
$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0,$$

  where $\rho : CH^p(X,1) \to \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}}$ is the composition of the Beilinson regulator $CH^p(X,1) \to H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p))$ with the natural map $H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p)) \to \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}}$.

However, as considered by Takeda for the higher arithmetic K-theory, one might be interested in a theory where the last property (long exact sequence), is the following property:

- For every n, there is an exact sequence

$$CH^p(X,n+1) \xrightarrow{\rho} \mathcal{D}^{2p-n}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \to 0. \quad (3.10.1)$$

With the tools we have introduced in this chapter, such a theory can be easily constructed. Therefore, we show how this modified higher arithmetic Chow groups can be defined, prove that the exact sequences above are obtained, prove the agreement with the already defined group for $n=0$, but leave the functoriality and product structure to be checked. The ideas presented in the previous sections lead to these results. Note that since the product on the Deligne complex is not strictly associative, the product obtained on these modified higher arithmetic Chow groups will not be associative.

Observe that the second term in the exact sequence (3.10.1) consists of differential forms over X. However, the target complex of our construction of the regulator is the complex $\mathcal{D}^*_{\mathbb{A}}(X,p)_0$. In order to obtain the desired result, we use the construction of the regulator of section 3.5, restricting therefore to proper arithmetic varieties. Recall that for proper arithmetic varieties, we could construct a representative of the regulator with target complex $\mathcal{D}^*(X,p)$.

We start by giving the definition of the modified homology groups of a diagram of the form (3.1.6). Then, for every arithmetic variety X over a field, we consider the diagram

$$\begin{tikzcd} & \widetilde{\mathcal{H}}^p_{\mathbb{P}}(X,*) & & \mathcal{D}^{2p-*}(X,p) \\ \widetilde{Z}^p(X,*) \arrow[ur, "f_1"] & & \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \arrow[ul, "g_1"', "\sim"] \arrow[ur, "\bar{\rho}"] & \end{tikzcd}$$

given by the regulator in (3.5.14). The *modified higher arithmetic Chow groups* of X, $\widehat{CH}^p_{mod}(X,n)$, are the modified homology groups of the diagram.

We finish the section by proving that for $n=0$, we recover the arithmetic Chow group defined by Gillet and Soulé in [24].

3.10.1 Modified homology group of a diagram

Let

$$\mathcal{D}_* = \left(\begin{tikzcd} & B_* & & D_* \\ A_* \arrow[ur, "f_1"] & & C_* \arrow[ul, "g_1"', "\sim"] \arrow[ur, "\rho"] & \end{tikzcd} \right), \quad (3.10.2)$$

be a diagram as in (3.1.6), with g_1 a quasi-isomorphism. Recall from lemma 3.1.7 that there is a well-defined morphism

$$\begin{array}{rcl} H_*(A) & \xrightarrow{\rho} & H_*(D) \\ {[a]} & \mapsto & \rho g_1^{-1} f_1[a]. \end{array}$$

For every homological complex (E_*, d_E), let ZE_* denote the group of cycles in E_* and $\widetilde{E}_* = E/\operatorname{im} d_E$. Observe that there is a canonical map

$$H_*(E) \to \widetilde{E}_*.$$

Let $\widehat{Z}_n(\mathcal{D})$ be the *modified group of cycles of the diagram* $\mathcal{D}_*$ consisting of 4-ples $(a, c, b, d) \in ZA_n \oplus ZC_n \oplus B_{n+1} \oplus \widetilde{D}_{n+1}$ such that

$$f_1(a) - g_1(c) = d_B(b).$$

We define the equivalence relation by setting $(a, c, b, d) \sim 0$ if there exists a triple $(\alpha, \beta, \gamma) \in A_{n+1} \oplus C_{n+1} \oplus B_{n+2}$ such that

$$d_A(\alpha) = a, \qquad d_C(\beta) = c, \qquad f_1(\alpha) - g_1(\beta) - d_B(\gamma) = b, \quad \text{and} \quad \rho(\beta) = d.$$

Definition 3.10.3. Let $\mathcal{D}_*$ be a diagram as in (3.1.6). The *modified homology groups of the diagram* $\mathcal{D}_*$ are defined to be the quotient of $\widehat{Z}_n(\mathcal{D})$ by the equivalence relation $\sim$:

$$\widehat{H}_n(\mathcal{D}) = \widehat{Z}_n(\mathcal{D})/\sim, \qquad n \geq 0.$$

Let $\sigma_{>n} D_*$ be the bête truncation of the complex D_*, that is,

$$\sigma_{>n} D_r = \begin{cases} D_r & r > n, \\ 0 & r \leq n. \end{cases}$$

Let $\rho_{>n}$ be the composition of $\rho : C_* \to D_*$ with the canonical morphism $D_* \to \sigma_{>n} D_*$ and let

$$\widehat{\mathcal{D}}^n_* = \left(\begin{tikzcd} & B_* & & \sigma_{>n} D_* \\ A_* \arrow[ur, "f_1"] & & C_* \arrow[ul, "g_1"', "\sim"] \arrow[ur, "\rho_{>n}"] & \end{tikzcd} \right).$$

Then, it follows from the definition that

$$H_r(s(\widehat{\mathcal{D}}^n)_*) = \begin{cases} H_r(s(\mathcal{D})_*) & r > n, \\ \widehat{H}_n(\mathcal{D}) & r = n. \end{cases}$$

Consider the following morphisms:

$$\begin{array}{rcll} \widetilde{D}_{n+1} & \xrightarrow{a} & \widehat{H}_n(\mathcal{D}), & a(d) = [(0,0,0,-d)], \\ \widehat{H}_n(\mathcal{D}) & \xrightarrow{\zeta} & H_n(A_*), & \zeta[(a,c,b,d)] = [a], \\ \widehat{H}_n(\mathcal{D}) & \xrightarrow{\rho} & ZD_n & \rho[(a,c,b,d)] = \rho(c) - d_D(d). \end{array}$$

Observe that ρ is well defined: if $[(a, c, b, d)] = 0$, then there exists $\beta \in C_{n+1}$ such that $d_C(\beta) = c$ and $\rho(\beta) = d$. Hence,

$$\rho(c) = d_D \rho(\beta) = d_D(d).$$

Proposition 3.10.4. *Let $\mathcal{D}_*$ be a diagram as in* (3.1.6)*. Then, for every $n \geq 0$, there are exact sequences*

(a) $H_{n+1}(A) \xrightarrow{\rho} \widetilde{D}_{n+1} \xrightarrow{a} \widehat{H}_n(\mathcal{D}) \xrightarrow{\zeta} H_n(A) \to 0,$

(b) $0 \to H_n(s(\mathcal{D})_*) \to \widehat{H}_n(\mathcal{D}) \xrightarrow{\rho} ZD_n \to H_{n-1}(s(\mathcal{D})_*).$

Proof. The first exact sequence is obtained by lemma 3.1.7. The second exact sequence is obtained by considering the long exact sequence associated to the short exact sequence:

$$0 \to D_*/\sigma_{>n}D_*[-1] \to s(\mathcal{D})_* \to s(\widehat{\mathcal{D}}^n)_* \to 0.$$

□

Corollary 3.10.5. *There is a canonical isomorphism*

$$H_n(s(\mathcal{D})) = \ker(\rho : \widehat{H}_n(\mathcal{D}) \to D_n).$$

□

3.10.2 Modified higher arithmetic Chow groups

Let X be a proper arithmetic variety over a field and consider the diagram

$$\mathcal{Z}^p_{\mathbb{P}}(X,*) = \left(\begin{tikzcd} & \widetilde{\mathcal{H}}^p_{\mathbb{P}}(X,*) & & \mathcal{D}^{2p-*}(X,p) \\ \widetilde{Z}^p(X,*) \arrow[ur, "f_1"] & & \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\bar{\rho}"] & \end{tikzcd}\right).$$

Let $\widehat{\mathcal{D}}^{2p-*}(X,p)$ be the chain complex with:

$$\widehat{\mathcal{D}}^{2p-n}(X,p) = \begin{cases} \mathcal{D}^{2p-n}(X,p) & \text{if } n \neq 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Then, consider the induced diagram

$$\widehat{\mathcal{Z}}^p_{\mathbb{P}}(X,*) = \left(\begin{tikzcd} & \widetilde{\mathcal{H}}^p_{\mathbb{P}}(X,*) & & \widehat{\mathcal{D}}^{2p-*}(X,p) \\ \widetilde{Z}^p(X,*) \arrow[ur, "f_1"] & & \widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \arrow[ul, "g_1", "\sim"'] \arrow[ur, "\bar{\rho}"] & \end{tikzcd}\right).$$

Higher arithmetic Chow groups with projective lines. Consider the higher arithmetic Chow groups defined by the diagram $\widehat{\mathcal{Z}}^p_{\mathbb{P}}(X,*)$, that is,

$$\widehat{CH}^p_{\mathbb{P}}(X,n) = H_n(\widehat{\mathcal{Z}}^p_{\mathbb{P}}(X,*)).$$

Using the five lemma on the long exact sequences (3.1.8) associated to $\widehat{CH}^p_{\mathbb{P}}(X,n)$ and to $\widehat{CH}^p(X,n)$, one obtains an isomorphism

$$\widehat{CH}^p(X,n) \cong \widehat{CH}^p_{\mathbb{P}}(X,n).$$

Modified higher arithmetic Chow groups. We define the *modified higher arithmetic Chow groups* as the modified homology groups of the diagram $\mathcal{Z}^p_{\mathbb{P}}(X,*)$:

$$\widehat{CH}^p_{mod}(X,n) = \widehat{H}_n(\mathcal{Z}^p_{\mathbb{P}}(X,*)).$$

The next proposition follows from the results on the modified homology groups of a diagram discussed in the previous section.

Proposition 3.10.6. *Let X be a proper arithmetic variety over a field.*

(i) There is an identification

$$\widehat{CH}^p(X,n) = \ker(\bar{\rho} : \widehat{CH}^p_{mod}(X,n) \to \widehat{\mathcal{D}}^{2p-n}(X,p)),$$

where $\bar{\rho}[(Z,\alpha_1,\alpha_2,\alpha_3)] = \bar{\rho}(\alpha_2) - d_{\mathcal{D}}(\alpha_3)$.

(ii) $\widehat{CH}^p(X,0) = \widehat{CH}^p_{mod}(X,0)$.

(iii) For $n=0$, the map

$$\begin{array}{ccc} \widehat{CH}^p(X) & \to & \widehat{CH}^p_{mod}(X,0) \\ [(Z,(\omega,\tilde{g}))] & \mapsto & [(Z,(\omega,g),0,0)], \end{array}$$

is an isomorphism.

(iv) For every $n \geq 0$, there is an exact sequence

$$CH^p(X,n+1) \xrightarrow{\rho} \mathcal{D}^{2p-n}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \to 0.$$

Proof. The first statement is proposition 3.10.5. The third statement follows from the second and theorem 3.6.11. The second statement is a consequence of the definitions. The last statement follows from proposition 3.10.4. □

Remark 3.10.7. The functoriality and the product structure on these modified higher arithmetic Chow groups can be deduced by the same pattern as the functoriality and product structure on the higher arithmetic Chow groups. The key point is that the modified homology groups of a diagram are invariant if we change any of the complexes A_*, B_*, C_* be quasi-isomorphic complexes. One should expect the resulting product to be commutative, since the product on the Deligne complex is strictly commutative. However, since the product on the Deligne complex is not associative, one cannot expect the product on the modified higher arithmetic Chow groups to be associative.

Chapter 4

A chain morphism representing Adams operations on rational K-theory

In this chapter we construct an explicit chain morphism which induces the Adams operations on rational algebraic K-theory, for any regular noetherian scheme of finite Krull dimension.

Let X be a scheme and let $\mathcal{P}(X)$ be the category of locally free sheaves of finite rank on X. Consider the chain complex of cubes associated to the category $\mathcal{P}(X)$. In [47], McCarthy showed that the homology groups of this complex, with rational coefficients, are isomorphic to the rational algebraic K-groups of X (see section 1.3.3).

We start by showing that there is a normalized complex for the complex of cubes, in the style of the normalized complex associated to a cubical abelian group (see 1.2.34).

We then define Adams operations for *split cubes*, that is, for cubes which are split in all directions. By a purely combinatorial formula on the Adams operations of locally free sheaves, we give a formula for the Adams operations on split cubes. The key point is to use Grayson's idea of considering the secondary Euler characteristic class of the Koszul complex associated to a locally free sheaf of finite rank.

Finally, we assign to every cube of locally free sheaves on X, a collection of split cubes defined either on $X \times (\mathbb{P}^1)^*$ or on $X \times (\mathbb{A}^1)^*$. This is achieved by means of the *transgressions of cubes by affine or projective lines.*

The composition of the Adams operations for split cubes with the transgression morphism defines a chain morphism representing the Adams operations on the algebraic K-groups of a regular noetherian scheme. The results of chapter 2 are used to prove that our construction induces the Adams operations defined by Gillet and Soulé in [28].

The main application of our construction of Adams operations is the definition of a (pre)-λ-ring structure on the rational arithmetic K-groups of an arithmetic variety X. The details of this application are presented in the next chapter.

4.1 Iterated cochain complexes and the complex of cubes

In this section we introduce the chain complex of iterated cochain complexes. This complex can be viewed as a generalization of the complex of cubes and is generated by iterated complexes with arbitrary finite length.

We then prove that there is a normalized complex of cubes which is the analogue of the normalized chain complex associated to a cubical abelian group.

Recall that the notation on multi-indices was introduced in section 1.2.1.

4.1.1 The chain complex of iterated cochain complexes

Let X be a scheme and let $\mathcal{P} = \mathcal{P}(X)$ be the category of locally free sheaves of finite rank on X. Fix a universe $\mathcal{U}$ so that $\mathcal{P}(X)$ is $\mathcal{U}$-small for all X. We denote by $IC_n(X)$ the set of n-iterated cochain complexes over X, concentrated in non-negative degrees, of finite length and acyclic in all directions. Let $\mathbb{Z}IC_n(X)$ be the free abelian group generated by $IC_n(X)$. Then,

$$\mathbb{Z}IC_*(X) = \bigoplus_{n\geq 0} \mathbb{Z}IC_n(X)$$

is a graded abelian group, which can be made into a chain complex. For every $\boldsymbol{l} = (l_1, \ldots, l_n)$, we denote by

$$IC_n^{\boldsymbol{l}}(X) \subseteq IC_n(X)$$

the set of n-iterated cochain complexes of length l_i in the i-th direction.

Definition 4.1.1. Let $A^* \in IC_n^{\boldsymbol{l}}(X)$. For every $i = 1, \ldots, n$ and $j \in [0, l_i]$, the $(n-1)$-iterated cochain complex $\partial_i^j(A)^*$ is defined by

$$\partial_i^j(A)^{\boldsymbol{m}} := A^{s_i^j(\boldsymbol{m})} \in IC_{n-1}(X) \qquad \forall\ \boldsymbol{m}.$$

It is called the *j-th face of A^* in the i-th direction.* If $j > l_i$, we set

$$\partial_i^j(A) := 0.$$

It follows from the definition that for all $j \in [0, l_i]$ and $k \in [0, l_r]$,

$$\partial_i^j \partial_r^k = \partial_{r-1}^k \partial_i^j, \quad \text{if } i \leq r. \tag{4.1.2}$$

Then, there is a well-defined group morphism

$$\begin{array}{ccc} \mathbb{Z}IC_n(X) & \xrightarrow{d} & \mathbb{Z}IC_{n-1}(X) \\ A^* & \mapsto & \displaystyle\sum_{i=1}^{n} \sum_{j\geq 0} (-1)^{i+j} \partial_i^j(A)^*. \end{array}$$

Since $d^2 = 0$, the pair $(\mathbb{Z}IC_*(X), d)$ is a chain complex. It is called *the chain complex of iterated cochain complexes.*

Remark 4.1.3. Observe that we have obtained a *chain complex* whose n-th graded piece is generated by n-iterated *cochain complexes.* We will try to be very precise on this duality, so as not to confuse the reader.

Remark 4.1.4. Observe that the complex of cubes $\mathbb{Z}C_*(X)$, defined in section 1.3.3, is the complex of iterated cochain complexes obtained restricting to the iterated cochain complexes of length 2 in all directions, that is,

$$\mathbb{Z}C_n(X) = \mathbb{Z}IC_n^{2,\dots,2}(X).$$

4.1.2 The normalized complex of cubes

Let $\mathcal{P}$ be a small exact category in some universe $\mathcal{U}$. In this section, we show that there is a normalized complex for the complex of cubes, in the style of the normalized complex associated to a cubical abelian group (see 1.2.34). That is, we construct a complex $NC_*(\mathcal{P}) \subset \mathbb{Z}C_*(\mathcal{P})$, which maps isomorphically to $\widetilde{\mathbb{Z}}C_*(\mathcal{P})$.

In section 1.3.3, we defined morphisms

- *face maps:* for every $i = 1, \dots, n$ and $j = 0, 1, 2$,

$$\partial_i^j : \mathbb{Z}C_n(\mathcal{P}) \to \mathbb{Z}C_{n-1}(\mathcal{P}),$$

- *degeneracy maps:* for every $i = 1, \dots, n+1$ and $j = 0, 1$,

$$s_i^j : \mathbb{Z}C_n(\mathcal{P}) \to \mathbb{Z}C_{n+1}(\mathcal{P}),$$

satisfying, for any $k, l \in \{0, 1, 2\}$ and for all $u, v \in \{0, 1\}$, the following identities:

$$\begin{array}{rcl} \partial_i^l \partial_j^k & = & \begin{cases} \partial_j^k \partial_{i+1}^l & \text{if } j \leq i, \\ \partial_{j-1}^k \partial_i^l & \text{if } j > i. \end{cases} \\ \partial_i^0 s_i^0 & = & \partial_i^1 s_i^0 = id, \quad \partial_i^1 s_i^1 = \partial_i^2 s_i^1 = id, \quad \partial_i^2 s_i^0 = \partial_i^0 s_i^1 = 0, \\ \partial_i^l s_j^u & = & \begin{cases} s_j^u \partial_{i-1}^l & \text{if } j < i, \\ s_{j-1}^u \partial_i^l & \text{if } j > i. \end{cases} \\ s_i^u s_j^v & = & s_{j+1}^v s_i^u \text{ if } j \geq i. \end{array} \tag{4.1.5}$$

Proposition 4.1.6. *Let $NC_*(\mathcal{P}) \subset \mathbb{Z}C_*(\mathcal{P})$ be any of the following complexes:*

$$N_nC(\mathcal{P}) = \begin{cases} \bigcap_{i=1}^n \ker \partial_i^0 \cap \bigcap_{i=1}^n \ker \partial_i^2, \\ \bigcap_{i=1}^n \ker \partial_i^0 \cap \bigcap_{i=1}^n \ker(\partial_i^1 - \partial_i^0) = \bigcap_{i=1}^n \ker \partial_i^0 \cap \bigcap_{i=1}^n \ker \partial_i^1, \\ \bigcap_{i=1}^n \ker(\partial_i^1 - \partial_i^2) \cap \bigcap_{i=1}^n \ker \partial_i^2 = \bigcap_{i=1}^n \ker \partial_i^1 \cap \bigcap_{i=1}^n \ker \partial_i^2, \\ \bigcap_{i=1}^n \ker(\partial_i^1 - \partial_i^2) \cap \bigcap_{i=1}^n \ker(\partial_i^0 - \partial_i^1). \end{cases}$$

Then, the composition

$$NC_*(\mathcal{P}) \hookrightarrow \mathbb{Z}C_*(\mathcal{P}) \twoheadrightarrow \widetilde{\mathbb{Z}}C_*(\mathcal{P}) = \mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*(\mathcal{P})$$

is an isomorphism of chain complexes.

Proof. We will see that the complex of cubes can be obtained by associating two different cubical structures to the collection of abelian groups $\{\mathbb{Z}C_n(\mathcal{P})\}_n$. Then, applying twice the normalized construction to $\mathbb{Z}C.(\mathcal{P}) = \{\mathbb{Z}C_n(\mathcal{P})\}_n$, we obtain a subcomplex $NC_*(\mathcal{P}) \subset \mathbb{Z}C_*(\mathcal{P})$ isomorphic to $\mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*(\mathcal{P})$. The two different cubical structures of $\mathbb{Z}C.(\mathcal{P})$ are given as follows.

- For the *first structure* consider

$$\tilde{\partial}_i^0 = \partial_i^0, \quad \tilde{\partial}_i^1 = \partial_i^1 - \partial_i^2, \quad \text{and} \quad \tilde{s}_i = s_i^0.$$

- For the *second structure* consider

$$\tilde{\partial}_i^0 = \partial_i^2, \quad \tilde{\partial}_i^1 = \partial_i^1 - \partial_i^0, \quad \text{and} \quad \tilde{s}_i = s_i^1.$$

By the identities (4.1.5), both collections of faces and degeneracies satisfy the identities of a cubical structure on $\mathbb{Z}C.(\mathcal{P})$. Moreover, the differential of $\mathbb{Z}C_*(\mathcal{P})$ induced by both structures is exactly the differential of the complex of cubes, as stated in (1.3.12).

By the first structure, we obtain an isomorphism of chain complexes

$$N^1C_*(\mathcal{P}) \hookrightarrow \mathbb{Z}C_*(\mathcal{P}) \twoheadrightarrow \mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*^1(\mathcal{P})$$

where

$$N^1C_*(\mathcal{P}) = \bigcap_{i=1}^{n} \ker \partial_i^0 \quad \text{or} \quad \bigcap_{i=1}^{n} \ker(\partial_i^1 - \partial_i^2), \quad \text{and} \quad \mathbb{Z}D_n^1(\mathcal{P}) = \sum_{i=1}^{n} \operatorname{im} s_i^0.$$

The reader can check that the second structure induces a cubical structure on $N^1C.(\mathcal{P})$ and on $\mathbb{Z}C.(\mathcal{P})/\mathbb{Z}D^1_.(\mathcal{P})$ compatible with the map $N^1C.(\mathcal{P}) \to \mathbb{Z}C.(\mathcal{P})/\mathbb{Z}D^1_.(\mathcal{P})$. Therefore, there is an isomorphism of complexes

$$N^2N^1C_*(\mathcal{P}) \hookrightarrow N^2C_*(\mathcal{P}) \twoheadrightarrow N^2(\mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*^1(\mathcal{P})).$$

Applying lemma 1.2.36 to $\mathbb{Z}C.(\mathcal{P})/\mathbb{Z}D^1_.(\mathcal{P})$, we obtain an isomorphism of complexes

$$N^2(\mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*^1(\mathcal{P})) \hookrightarrow \mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*^1(\mathcal{P}) \twoheadrightarrow \frac{\mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*^1(\mathcal{P})}{\sum_{i=1}^{*} \operatorname{im} s_*^1}.$$

Since for every n, $\sum_{i=1}^n \operatorname{im} s_i^0 \cap \sum_{i=1}^n \operatorname{im} s_i^1 = \{0\}$, we obtain that

$$\frac{\mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*^1(\mathcal{P})}{\sum_{i=1}^{*} \operatorname{im} s_*^1} = \mathbb{Z}C_*(\mathcal{P})/\mathbb{Z}D_*(\mathcal{P}).$$

Hence, $NC_*(\mathcal{P}) = N^2N^1C_*(\mathcal{P})$ is isomorphic to $\widetilde{\mathbb{Z}}C_*(\mathcal{P})$. The four candidates in the statement of the proposition appear combining the two options for the normalized complex, for every structure. $\square$

Remark 4.1.7. There are actually two other possible cubical structures on $\mathbb{Z}C.(\mathcal{P})$. One can consider the structure with

$$\tilde{\partial}_i^0 = \partial_i^0 + \partial_i^2, \quad \tilde{\partial}_i^1 = \partial_i^1$$

and

$$\tilde{s}_i = s_i^0 \quad \text{or} \quad s_i^1.$$

Therefore, further normalized complexes are obtained. As long as we consider one cubical structure with $\tilde{s}_i = s_i^0$ and another cubical structure with $\tilde{s}_i = s_i^1$, we obtain different normalized complexes associated to $\mathbb{Z}C_*(\mathcal{P})$.

We fix, from now on, the *normalized chain complex* $NC_*(\mathcal{P})$ to be the one with n-th graded piece given by

$$N_nC(\mathcal{P}) := \bigcap_{i=1}^{n} \ker \partial_i^0 \cap \bigcap_{i=1}^{n} \ker \partial_i^2 \subset \mathbb{Z}C_n(\mathcal{P}),$$

and differential induced by the differential of $\mathbb{Z}C_*(\mathcal{P})$.

When $\mathcal{P} = \mathcal{P}(X)$, we simply write $N_*C(X)$.

The morphism Cub **and the normalized complex.** The morphism Cub defined in 1.3.14, can be described by means of the normalized chain complex associated to $S.(\mathcal{P})$ and to $C_*(\mathcal{P})$. Let

$$NS_n(\mathcal{P}) = \bigcap_{i=1}^{n} \ker \partial_i,$$

be the normalized complex associated to the simplicial abelian group $\mathbb{Z}S.(\mathcal{P})$, as described in 1.2.17.

Proposition 4.1.8. *There is a chain morphism*

$$\begin{array}{ccc} NS_*(\mathcal{P})[-1] & \xrightarrow{\mathrm{Cub}} & NC_*(\mathcal{P}) \\ E & \mapsto & \mathrm{Cub}\, E. \end{array}$$

Proof. It follows from the computation of the faces of $\mathrm{Cub}\, E$ given in proposition 1.3.13. □

Corollary 4.1.9. *The composition of the map induced by* Cub *in proposition 4.1.8, with the Hurewicz morphism, induces an isomorphism*

$$K_n(\mathcal{P})_{\mathbb{Q}} \xrightarrow{\mathrm{Cub}} H_n(NC_*(\mathcal{P}), \mathbb{Q}).$$

Proof. It follows from the commutative square

$$\begin{tikzcd} NS_*(\mathcal{P})[-1] \arrow[r, "\mathrm{Cub}"] \arrow[d, "\sim"'] & NC_*(\mathcal{P}) \arrow[d, "\sim"] \\ \mathbb{Z}S_*(\mathcal{P})[-1] \arrow[r, "\mathrm{Cub}"'] & \widetilde{\mathbb{Z}C}_*(\mathcal{P}), \end{tikzcd}$$

where the vertical arrows are quasi-isomorphisms. □

4.2 Adams operations for split cubes

Let X be any scheme. In this section, for every $k \geq 1$, we construct a chain morphism Ψ^k from the complex of split cubes to the complex of cubes on X.

We divide the construction into three steps. We first construct the chain complex of split cubes on X, $(\mathbb{Z}\,\mathrm{Sp}_*(X), d)$. We then define an intermediate chain complex $(\mathbb{Z}G^k(X)_*, d_s)$ and a chain morphism

$$\mathbb{Z}G^k(X)_* \xrightarrow{\mu\circ\varphi} \mathbb{Z}C_*(X).$$

Finally, for every n, we construct a morphism

$$\Psi^k : \mathbb{Z}\,\mathrm{Sp}_n(X) \to \mathbb{Z}G^k(X)_n.$$

Its composition with $\mu \circ \varphi$,

$$\mu \circ \varphi \circ \Psi^k : \mathbb{Z}\,\mathrm{Sp}_*(X) \to \mathbb{Z}C_*(X),$$

gives the definition of the Adams operations over split cubes.

As in the previous section, let X be a scheme and let $\mathcal{P} = \mathcal{P}(X)$ be the category of locally free sheaves of finite rank on X. Recall that the notation on multi-indices was introduced in section 1.2.1.

4.2.1 Split cubes

We introduce here the *complex of split cubes*, which plays a key role in the definition of the Adams operations for arbitrary cubes. Roughly speaking, these are the cubes which are split in all directions.

Direct sum cubes. For every $\boldsymbol{j} = (j_1, \ldots, j_n) \in \{0,1,2\}^n$, let $u_1 < \cdots < u_{s(\boldsymbol{j})}$ be the indices such that $j_{u_i} = 1$ and let

$$u(\boldsymbol{j}) = (u_1, \ldots, u_{s(\boldsymbol{j})}). \tag{4.2.1}$$

Observe that $s(\boldsymbol{j})$ is the length of $u(\boldsymbol{j})$.

Definition 4.2.2. Let $\{E^{\boldsymbol{i}}\}_{\boldsymbol{i}\in\{0,2\}^n}$ be a collection of locally free sheaves on X, indexed by $\{0,2\}^n$. Let $[E^{\boldsymbol{i}}]_{\boldsymbol{i}\in\{0,2\}^n}$ be the n-cube given by:

▷ The $\boldsymbol{j}$-component is

$$\bigoplus_{\boldsymbol{m}\in\{0,2\}^{s(\boldsymbol{j})}} E^{\sigma^{\boldsymbol{m}}_{u(\boldsymbol{j})}(\boldsymbol{j})}, \quad \boldsymbol{j} \in \{0,1,2\}^n.$$

▷ The morphisms are compositions of the following canonical morphisms:

$$\begin{array}{rclcrcl} A \oplus B & \twoheadrightarrow & A, & \qquad & A \oplus B & \xrightarrow{\cong} & B \oplus A, \\ A & \hookrightarrow & A \oplus B, & & A \oplus (B \oplus C) & \xrightarrow{\cong} & (A \oplus B) \oplus C. \end{array}$$

An n-cube of this form is called a *direct sum n-cube.*

Remark 4.2.3. In the previous definition, the direct sum is taken in the lexicographic order on $\{0,2\}^{s(\boldsymbol{j})}$.

Observe that, if $\boldsymbol{j} \in \{0,2\}^n$, then the $\boldsymbol{j}$-component of $[E^{\boldsymbol{i}}]_{\boldsymbol{i}\in\{0,2\}^n}$ is exactly $E^{\boldsymbol{j}}$. Hence, this n-cube has at the "corners" the given collection of objects, and we fill the "interior" with the appropriate direct sums.

Example 4.2.4. For $n = 1$, the 1-cube $[E^0, E^2]$ is the exact sequence

$$E^0 \to E^0 \oplus E^2 \to E^2.$$

Example 4.2.5. For $n = 2$, if $E^{00}, E^{02}, E^{20}, E^{22}$ are locally free sheaves on X, then the 2-cube $\begin{bmatrix} E^{00} & E^{02} \\ E^{20} & E^{22} \end{bmatrix}$ is the 2-cube

$$\begin{tikzcd}
E^{00} \arrow[r] \arrow[d] & E^{00} \oplus E^{02} \arrow[r] \arrow[d] & E^{02} \arrow[d] \\
E^{00} \oplus E^{20} \arrow[r] \arrow[d] & E^{00} \oplus E^{02} \oplus E^{20} \oplus E^{22} \arrow[r] \arrow[d] & E^{02} \oplus E^{22} \arrow[d] \\
E^{20} \arrow[r] & E^{20} \oplus E^{22} \arrow[r] & E^{22}.
\end{tikzcd}$$

Definition 4.2.6.

- Let E be an n-cube. The *direct sum n-cube associated to E*, $\mathrm{Sp}(E)$, is the n-cube
$$\mathrm{Sp}(E) := [E^{\boldsymbol{j}}]_{\boldsymbol{j}\in\{0,2\}^n}.$$

- A *split n-cube* is a couple (E, f), where E is an n-cube and $f : \mathrm{Sp}(E) \to E$ is an isomorphism of n-cubes such that $f^{\boldsymbol{j}} = id$ if $\boldsymbol{j} \in \{0,2\}^n$. The morphism f is called the *splitting* of (E, f).

- Let
$$\mathbb{Z}\,\mathrm{Sp}_n(X) := \mathbb{Z}\{\text{split } n - \text{cubes on } X\},$$
and let $\mathbb{Z}\,\mathrm{Sp}_*(X) = \bigoplus_n \mathbb{Z}\,\mathrm{Sp}_n(X)$.

Differential of split cubes. We endow $\mathbb{Z}\,\mathrm{Sp}_*(X)$ with a chain complex structure. That is, we define a differential morphism

$$\mathbb{Z}\,\mathrm{Sp}_n(X) \to \mathbb{Z}\,\mathrm{Sp}_{n-1}(X).$$

Let E be an arbitrary n-cube. Observe that if $j = 0, 2$, then, for all $l = 1, \dots, n$,

$$\partial_l^j\,\mathrm{Sp}(E) = \mathrm{Sp}(\partial_l^j E).$$

Therefore, if (E, f) is a split n-cube,

$$\partial_l^j(E, f) := (\partial_l^j E, \partial_l^j f)$$

is a split $(n-1)$-cube. By contrast, in general

$$\partial_l^1 \operatorname{Sp}(E) \neq \operatorname{Sp}(\partial_l^1 E). \tag{4.2.7}$$

However, if E is a split n-cube, $\partial_l^1 E$ is also isomorphic to $\operatorname{Sp}(\partial_l^1 E)$, i.e. it is also split.

In order to illustrate the forthcoming definition, we will start by defining the face $\partial_l^1(E, f) = (\partial_l^1 E, \hat{f})$ for $n = 2$. Let (E, f) be a split 2-cube:

$$\begin{tikzcd}
E^{00} \arrow[r] \arrow[d] & E^{00} \oplus E^{02} \arrow[r] \arrow[d] & E^{02} \arrow[d] \\
E^{00} \oplus E^{20} \arrow[r] \arrow[d] & E^{00} \oplus E^{02} \oplus E^{20} \oplus E^{22} \arrow[r] \arrow[d] & E^{02} \oplus E^{22} \arrow[d] \\
E^{20} \arrow[r] & E^{20} \oplus E^{22} \arrow[r] & E^{22}
\end{tikzcd}
\xrightarrow{f}
\begin{tikzcd}
E^{00} \arrow[r] \arrow[d] & E^{01} \arrow[r] \arrow[d] & E^{02} \arrow[d] \\
E^{10} \arrow[r] \arrow[d] & E^{11} \arrow[r] \arrow[d] & E^{12} \arrow[d] \\
E^{20} \arrow[r] & E^{21} \arrow[r] & E^{22}.
\end{tikzcd}$$

Then,

$$\partial_1^1(E) = E^{10} \to E^{11} \to E^{12}.$$

We define the morphism $\hat{f}^1 : E^{10} \oplus E^{12} \xrightarrow{\cong} E^{11}$, as the composition

$$E^{10} \oplus E^{12} \xrightarrow{(f^{10})^{-1} \oplus (f^{12})^{-1}} E^{00} \oplus E^{02} \oplus E^{20} \oplus E^{22} \xrightarrow{f^{11}} E^{11}.$$

Let (E, f) be a split n-cube. For every $\boldsymbol{j} \in \{0,1,2\}^n$, we define a morphism

$$\hat{f}^{\boldsymbol{j}} : \operatorname{Sp}(\partial_l^1 E)^{\boldsymbol{j}} \to (\partial_l^1 E)^{\boldsymbol{j}}$$

as the composition of the isomorphisms

$$\begin{tikzcd}
\bigoplus_{\boldsymbol{m} \in \{0,2\}^{s(\boldsymbol{j})}} (\partial_l^1 E)^{\sigma^{\boldsymbol{m}}_{u(\boldsymbol{j})}(\boldsymbol{j})} \arrow[r, "\hat{f}^{\boldsymbol{j}}"] \arrow[d, "\cong", "\oplus(\partial_l^1 f)^{-1}"'] & (\partial_l^1 E)^{\boldsymbol{j}} \\
\bigoplus_{\boldsymbol{m} \in \{0,2\}^{s(\boldsymbol{j})}} (\partial_l^0 E \oplus \partial_l^2 E)^{\sigma^{\boldsymbol{m}}_{u(\boldsymbol{j})}(\boldsymbol{j})} \arrow[r, "\cong"'] & \bigoplus_{\boldsymbol{m} \in \{0,2\}^{s(\boldsymbol{j})+1}} E^{\sigma^{\boldsymbol{m}}_{u(s_l^1(\boldsymbol{j}))}(s_l^1(\boldsymbol{j}))} \arrow[u, "f^{\boldsymbol{j}}"', "\cong"]
\end{tikzcd},$$

where the bottom arrow is the canonical isomorphism. Then, we define

$$\partial_l^1(E, f) := (\partial_l^1 E, \hat{f}).$$

With this definition of ∂_l^1, the commutation rule (4.1.2) is satisfied. Therefore, we have proved the following proposition.

Proposition 4.2.8. *The morphism*

$$d = \sum_{l=1}^{n} \sum_{i=0,1,2} (-1)^{i+l} \partial_l^i : \mathbb{Z}\,\mathrm{Sp}_n(X) \to \mathbb{Z}\,\mathrm{Sp}_{n-1}(X)$$

makes $\mathbb{Z}\,\mathrm{Sp}_*(X)$ into a chain complex. Moreover, the morphism $\mathbb{Z}\,\mathrm{Sp}_*(X) \to \mathbb{Z}C_*(X)$ obtained by forgetting the isomorphisms is a chain morphism.

□

Remark 4.2.9. Observe that due to (4.2.7), the morphism

$$\mathrm{Sp} : \mathbb{Z}C_*(X) \to \mathbb{Z}\,\mathrm{Sp}_*(X)$$

is not a chain morphism.

4.2.2 An intermediate complex for the Adams operations

Here we introduce the chain complex that serves as the target for the Adams operations defined on the chain complex of split cubes. We then construct a morphism from this new chain complex to the original chain complex of cubes $\mathbb{Z}C_*(X)$.

The intermediate complex. Let $k \geq 1$. For every $n \geq 0$ and $i = 1, \ldots, k-1$, we define

$$\begin{aligned} G_1^k(X)_n &:= IC_1^k(C_n(X)) \\ &:= \{\text{acyclic cochain complexes of length } k \text{ of } n-\text{cubes}\}, \\ G_2^{i,k}(X)_n &:= IC_2^{k-i,i}(C_n(X)) \\ &:= \{\text{2-iterated acyclic cochain complexes of lengths } (k-i,i) \\ &\qquad \text{of } n-\text{cubes}\}. \end{aligned}$$

The differential of $\mathbb{Z}C_*(X)$ induces a differential on the graded abelian groups

$$\mathbb{Z}G_2^{i,k}(X)_* := \bigoplus_n \mathbb{Z}G_2^{i,k}(X)_n \qquad \text{and} \qquad \mathbb{Z}G_1^k(X)_* := \bigoplus_n \mathbb{Z}G_1^k(X)_n.$$

That is, if $B \in G_2^{i,k}(X)_n$, then for every r, s, $B^{r,s}$ is an n-cube. Define $\partial_l^i(B)$ to be the 2-iterated cochain complex of lengths $(k-i,i)$ of $(n-1)$-cubes given by

$$\partial_l^i(B)^{r,s} := \partial_l^i(B^{r,s}) \in C_{n-1}(X), \qquad \text{for every } r, s.$$

Then the differential of B is defined as

$$d(B) = \sum_{i=1}^{n} \sum_{l=0}^{2} (-1)^{i+l} \partial_l^i(B).$$

If $A \in G_1^k(X)_n$, then for every r, A^r is an n-cube and the differential is defined analogously.

For every n, the simple complex associated to a 2-iterated cochain complex induces a morphism
$$\Phi^i : \mathbb{Z}G_2^{i,k}(X)_n \to \mathbb{Z}G_1^k(X)_n.$$
That is, for every $B \in G_2^{i,k}(X)_n$, $\Phi^i(B)$ is the exact sequence of n-cubes

$$\Phi^i(B) := 0 \to B^{00} \to \cdots \to \bigoplus_{j_1+j_2=j} B^{j_1,j_2} \to \cdots \to B^{k-i,i} \to 0$$

with morphisms given by

$$\begin{array}{rcl} B^{j_1,j_2} & \to & B^{j_1+1,j_2} \oplus B^{j_1,j_2+1} \\ b & \mapsto & d^1(b) + (-1)^{j_1} d^2(b). \end{array}$$

One can easily check that, for every $i = 1, \ldots, k-1$, Φ^i is a chain morphism.

We define a new chain complex by setting

$$\mathbb{Z}G^k(X)_n := \bigoplus_{i=1}^{k-1} \mathbb{Z}G_2^{i,k}(X)_{n-1} \oplus \mathbb{Z}G_1^k(X)_n.$$

If $B_i \in G_2^{i,k}(X)_{n-1}$, for $i = 1, \ldots, k-1$, and $A \in G_1^k(X)_n$, the differential is given by

$$d_s(B_1, \ldots, B_{k-1}, A) := (-dB_1, \ldots, -dB_{k-1}, \sum_{i=1}^{k-1} (-1)^i \Phi^i(B_i) + dA).$$

Since, for all i, the morphisms Φ^i are chain morphisms, $d^2 = 0$ and therefore $(\mathbb{Z}G^k(X)_* = \bigoplus_n \mathbb{Z}G^k(X)_n, d_s)$ is a chain complex.

A morphism to the complex of cubes. Our purpose here is to define a chain morphism from the chain complex $\mathbb{Z}G^k(X)_*$ to the complex of cubes $\mathbb{Z}C_*(X)$. It is constructed in two steps. First, we define a chain morphism from $\mathbb{Z}G^k(X)_*$ to the complex
$$\mathbb{Z}C_*^{arb}(X) := \mathbb{Z}IC_*^{\cdot,2,\ldots,2}(X).$$
This complex is the chain complex that in degree n consists of n-iterated cochain complexes of length 2 in directions $2, \ldots, n$ and arbitrary finite length in direction 1. Alternatively, it can be thought of as the complex of exact sequences of arbitrary finite length of $(n-1)$-cubes.

Then, we construct a morphism from $\mathbb{Z}C_*^{arb}(X)$ to $\mathbb{Z}C_*(X)$, using the splitting of an acyclic cochain complex into short exact sequences.

Let
$$A : 0 \to A^0 \to \cdots \to A^k \to 0 \in G_1^k(X)_n$$

be an acyclic cochain complex of n-cubes. We define $\varphi_1(A)$ to be the "secondary Euler characteristic class", i.e.

$$\varphi_1(A) = \sum_{p\geq 0}(-1)^{k-p+1}(k-p)A^p \in \mathbb{Z}C_n(X).$$

We choose the signs of this definition in order to agree with Grayson's definition of Adams operations for $n=0$ in [31]. Note that in loc. cit., an exact sequence is viewed as a chain complex, while here it is viewed as a cochain complex.

Recall that if $B_i \in G_2^{i,k}(X)_n$, then B_i is a 2-iterated acyclic cochain complex where $B_i^{j_1 j_2}$ is an n-cube, for every j_1, j_2. We attach to B_i a sum of exact sequences of n-cubes as follows.

$$\begin{aligned}\varphi_2(B_i) &= \sum_{j\geq 0}(-1)^{k-j+1}((k-i-j)B_i^{*,j} + (i-j)B_i^{j,*}) \\ &\quad + \sum_{s\geq 1}(-1)^{k-s}(k-s)\sum_{j\geq 0}(B_i^{s-j,j} \to \bigoplus_{j'\geq j} B_i^{s-j',j'} \to \bigoplus_{j'>j} B_i^{s-j',j'}).\end{aligned}$$

Roughly speaking, the first summand corresponds to the secondary Euler characteristic of the rows and the columns. The second summand appears as a correction factor for the fact that direct sums are not sums in $\mathbb{Z}C_n(X)$.

For every n, we define a morphism

$$\begin{array}{rcl}\mathbb{Z}G^k(X)_n & \xrightarrow{\varphi} & \mathbb{Z}C_n^{arb}(X) \\ (B_1,\dots,B_{k-1},A) & \mapsto & \varphi_1(A) + \sum_{i=1}^{k-1}(-1)^{i+1}\varphi_2(B_i).\end{array} \tag{4.2.10}$$

Lemma 4.2.11. *The morphism φ is a chain morphism.*

Proof. The lemma follows from the two equalities

$$\begin{aligned}d\varphi_1(A) &= \varphi_1(dA), \\ d\varphi_2(B_i) &= -\varphi_2(dB_i) - \varphi_1(\Phi^i(B_i)), \quad \forall\ i.\end{aligned}$$

The first equality holds as a direct consequence of the definition of φ_1. By the definition of the differential of $\mathbb{Z}G_2^{i,k}(X)_*$,

$$-\varphi_2(dB_i) = \sum_{l=2}^{n}\sum_{j=0}^{2}(-1)^{l+j}\partial_l^j\varphi_2(B_i).$$

Therefore, it remains to see that

$$\sum_{r\geq 0}(-1)^r\partial_1^r\varphi_2(B_i) = \varphi_1(\Phi^i(B_i)).$$

In other terms, writing

$$\begin{aligned}
(1) &:= \sum_{s\geq 1}(-1)^{k-s}(k-s)\sum_{j\geq 0}\sum_{r=0}^{2}(-1)^r\partial_1^r\big[B_i^{s-j,j}\to\bigoplus_{j'\geq j}B_i^{s-j',j'}\to\bigoplus_{j'>j}B_i^{s-j',j'}\big],\\
(2) &:= \sum_{r\geq 0}(-1)^r\partial_1^r\Big[\sum_{j\geq 0}(-1)^{k-j+1}((k-i-j)B_i^{*,j}+(i-j)B_i^{j,*})\Big],
\end{aligned}$$

we want to see that

$$(1)+(2)=\varphi_1(\Phi^i(B_i))=\sum_{s\geq 0}(-1)^{k-s+1}(k-s)\bigoplus_{j\geq 0}B_i^{s-j,j}. \tag{4.2.12}$$

By a telescopic argument, the first term is

$$\begin{aligned}
(1) &= \Big[\sum_{s\geq 1}(-1)^{k-s}(k-s)\sum_{j\geq 0}B_i^{s-j,j}\Big]-\Big[\sum_{s\geq 1}(-1)^{k-s}(k-s)\bigoplus_{j\geq 0}B_i^{s-j,j}\Big]\\
&= \sum_{s\geq 1}(-1)^{k-s}(k-s)\sum_{j\geq 0}B_i^{s-j,j}+\varphi_1(\Phi^i(B_i))-(-1)^{k-1}kB_i^{00}\\
&= \sum_{s\geq 0}(-1)^{k-s}(k-s)\sum_{j\geq 0}B_i^{s-j,j}+\varphi_1(\Phi^i(B_i)).
\end{aligned}$$

The second term is

$$\begin{aligned}
(2) &= \Big[\sum_{r\geq 0}(-1)^r\sum_{j\geq 0}(-1)^{k-j+1}(k-i-j)B_i^{r,j}\Big]\\
&\quad+\Big[\sum_{r\geq 0}(-1)^r\sum_{j\geq 0}(-1)^{k-j+1}(i-j)B_i^{j,r}\Big]\\
&= \sum_{s\geq 0}(-1)^{k-s+1}\sum_{j\geq 0}(k-i-j)B_i^{s-j,j}+\sum_{s\geq 0}(-1)^{k-s+1}\sum_{r\geq 0}(i-s+r)B_i^{s-r,r}\\
&= \sum_{s\geq 0}(-1)^{k-s+1}(k-s)\sum_{j\geq 0}B_i^{s-j,j}.
\end{aligned}$$

Adding the two expressions, (4.2.12) is proved. □

The final step is the construction of a morphism from $\mathbb{Z}C_*^{arb}(X)$ to $\mathbb{Z}C_*(X)$. Recall that, by definition, an element of $\mathbb{Z}C_m^{arb}(X)$ is a finite length exact sequence of $(m-1)$-cubes. The idea is to break this exact sequence into short exact sequences, obtaining a collection of short exact sequences of $(m-1)$-cubes, hence, a collection of m-cubes (see remark 1.3.11).

Let

$$0\to A^0\xrightarrow{f^0}\cdots\xrightarrow{f^{j-1}}A^j\xrightarrow{f^j}\cdots\xrightarrow{f^{r-1}}A^r\to 0$$

be an exact sequence of $(m-1)$-cubes i.e. an element of $C_m^{arb}(X)$. Let $\mu^j(A)$ be the short exact sequence of $(m-1)$-cubes defined by

$$\mu^j(A)\quad:\quad 0\to\ker f^j\to A^j\to\ker f^{j+1}\to 0,\quad j=0,\dots,r-1.$$

It is the m-cube that along the first direction is given by:

$$\partial_1^0(\mu^j(A)) = \ker f^j, \quad \partial_1^1(\mu^j(A)) = A^j, \quad \text{and} \quad \partial_1^2(\mu^j(A)) = \ker f^{j+1}.$$

We define μ by

$$\begin{array}{rcl} \mathbb{Z}C_m^{arb}(X) & \xrightarrow{\mu} & \mathbb{Z}C_m(X) \\ A & \mapsto & \sum_{j\geq 0}(-1)^{j-1}\mu^j(A). \end{array}$$

The next lemma follows from a direct computation.

Lemma 4.2.13. *The map μ is a chain morphism.*

□

4.2.3 Ideas of the definition of the Adams operations on split cubes

Let X be a scheme. In this section we give examples and the outline of the definition of Adams operations on split cubes. The starting point is Grayson's idea of using the Koszul complex.

Let $\mathbb{Z}SG^k(X)_*$ be the chain complex obtained like $\mathbb{Z}G^k(X)_*$ by considering split cubes. That is, considering the groups

$$\begin{array}{rcl} SG_1^k(X)_n & := & IC_1^k(\mathrm{Sp}_n(X)), \\ SG_2^{i,k}(X)_n & := & IC_2^{k-i,i}(\mathrm{Sp}_n(X)). \end{array}$$

Observe that there is a natural morphism

$$\mathbb{Z}SG^k(X)_* \to \mathbb{Z}G^k(X)_*$$

obtained by forgetting the splitting.

For every $k \geq 1$, we construct a morphism,

$$\mathbb{Z}\,\mathrm{Sp}_n(X) \xrightarrow{\Psi^k} \mathbb{Z}SG^k(X)_n,$$

which composed with $\mu \circ \varphi$, gives a morphism

$$\mathbb{Z}\,\mathrm{Sp}_n(X) \xrightarrow{\Psi^k} \mathbb{Z}C_n(X).$$

Definition 4.2.14. Let $E \in \mathrm{Sp}_0(X)$ and $k \geq 1$. We define

$$\Psi^k(E) \in SG^k(X)_0 = SG_1^k(X)_0$$

to be the $k-th$ *Koszul complex of* E, i.e. the exact sequence

$$0 \to \Psi^k(E)^0 \to \cdots \to \Psi^k(E)^k \to 0$$

with

$$\Psi^k(E)^p = E\cdot \overset{p}{\ldots} \cdot E \otimes E\wedge \overset{k-p}{\ldots} \wedge E = S^pE \otimes \bigwedge^{k-p} E.$$

Observe that, for $k = 1$, we have

$$\Psi^1(E) : 0 \to E \xrightarrow{=} E \to 0.$$

By definition, the Koszul complex is functorial. Moreover, it has a very good behavior with direct sums.

Lemma 4.2.15. *If E and F are two locally free sheaves on X, then there is a canonical isomorphism of exact sequences*

$$\Psi^k(E \oplus F) \cong \bigoplus_{m=0}^{k} \Psi^{k-m}(E) \otimes \Psi^m(F), \qquad \forall\ k. \tag{4.2.16}$$

□

This identification plays a key role in the construction of the Adams operations.

The definition of $\Psi^k(E)$ of a general split n-cube E is given by a combinatorial formula on the Adams operations $\Psi^k(E^{\boldsymbol{j}})$, $\boldsymbol{j} \in \{0, 1, 2\}^n$, of the locally free sheaves in the cube. In order to understand how the combinatorial formula of the definition 4.2.17 arises, we explain here the low degree cases. We give the detailed construction of the Adams operations for $n = 1$, with $k = 2, 3$, and for $n = 2$, $k = 2$. We extract from these examples the key facts that enable us to set the general formula.

Adams operations in the case $\mathbf{n = 1}$, $\mathbf{k = 2}$. Let $n = 1$ and $k = 2$, and let $E = [E^0, E^2]$. Recall that this notation means that E is the 1-cube $E^0 \to E^0 \oplus E^2 \to E^2$. Our aim is to define $\Psi^2(E)$ in such a way that its differential is exactly

$$-\Psi^2(E^0) + \Psi^2(E^0 \oplus E^2) - \Psi^2(E^2).$$

Consider the two exact sequences

$$\begin{aligned} C_0(E) &:= [\Psi^2(E^0), \Psi^1(E^0) \otimes \Psi^1(E^2)] \in G_1^2(X)_1, \\ C_1(E) &:= [\Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2), \Psi^2(E^2)] \in G_1^2(X)_1. \end{aligned}$$

Then,

$$\begin{aligned} d(C_0(E) + C_1(E)) = {} & -\Psi^2(E^0) - \Psi^1(E^0) \otimes \Psi^1(E^2) \\ & + \Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2) \oplus \Psi^2(E^2) - \Psi^2(E^2). \end{aligned}$$

Observe now that by the isomorphism (4.2.16),

$$\Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2) \oplus \Psi^2(E^2) \cong \Psi^2(E^0 \oplus E^2).$$

We define then $\widetilde{C}_1(E)$ to be the exact sequence $C_1(E)$ modified by means of this isomorphism, that is

$$\widetilde{C}_1(E) : \ \Psi^2(E^0) \oplus \Psi^1(E^0) \otimes \Psi^1(E^2) \to \Psi^2(E^0 \oplus E^2) \to \Psi^2(E^2).$$

Finally, observe that the extra term $\Psi^1(E^0)\otimes\Psi^1(E^2)$ is the simple complex associated to the 2-iterated complex of length $(1,1)$

$$\begin{tikzcd} E^0 \otimes E^0 \arrow[r] \arrow[d] & E^0 \otimes E^2 \arrow[d] \\ E^2 \otimes E^0 \arrow[r] & E^2 \otimes E^2 \end{tikzcd}$$

Hence, viewed as a 2-iterated complex, $\Psi^1(E^0)\otimes\Psi^1(E^2) \in G^1_{2,2}(X)_0$. We conclude that the differential of

$$\Psi^2(E) := (\Psi^1(E^0)\otimes\Psi^1(E^2), C_0(E) + \widetilde{C}_1(E)) \in \mathbb{Z}G_2^{1,2}(X)_0 \oplus \mathbb{Z}G_1^2(X)_1$$

is exactly $-\Psi^2(E^0) + \Psi^2(E^0\oplus E^2) - \Psi^2(E^2)$ as desired.

For an arbitrary split 1-cube (E,f), we define:

- $\widetilde{C}_0(E,f) := C_0(\mathrm{Sp}(E)) \in G_1^2(X)_1$.
- $\widetilde{C}_1(E,f)$ is the exact sequence obtained changing, via the given isomorphism $f : E^1 \cong E^0 \oplus E^2$, the terms $\Psi^2(E^0\oplus E^2)$ in $\widetilde{C}_1(\mathrm{Sp}(E))$ by $\Psi^2(E^1)$:

$$\begin{tikzcd} & \Psi^2(E^0\oplus E^2) \arrow[d, "\cong"] \arrow[dr] & \\ \Psi^2(E^0)\oplus\Psi^1(E^0)\otimes\Psi^1(E^2) \arrow[ur] \arrow[r] & \Psi^2(E^1) \arrow[r] & \Psi^2(E^2). \end{tikzcd}$$

We define then

$$\Psi^2(E,f) := (\Psi^1(E^0)\otimes\Psi^1(E^2), C_0(E,f) + \widetilde{C}_1(E,f)) \in \mathbb{Z}G_2^{1,2}(X)_0 \oplus \mathbb{Z}G_1^2(X)_1.$$

Adams operations in the case $\mathbf{n = 1}$, $\mathbf{k = 3}$. Let $E = [E^0, E^2]$ as above. Our aim now is to define $\Psi^3(E)$ in such a way that its differential is

$$-\Psi^3(E^0) + \Psi^3(E^0\oplus E^2) - \Psi^3(E^2).$$

We consider the exact sequences,

$$\begin{aligned} C_0(E) &:= [\Psi^3(E^0), \Psi^2(E^0)\otimes\Psi^1(E^2)], \\ C_1(E) &:= [\Psi^3(E^0)\oplus\Psi^2(E^0)\otimes\Psi^1(E^2), \Psi^1(E^0)\otimes\Psi^2(E^2)], \\ C_2(E) &:= [\Psi^3(E^0)\oplus\Psi^2(E^0)\otimes\Psi^1(E^2)\oplus\Psi^1(E^0)\otimes\Psi^2(E^2), \Psi^3(E^2)]. \end{aligned}$$

Then, we define $\widetilde{C}_2(E)$ to be the exact sequence obtained from $C_2(E)$ by changing

$$\Psi^3(E^0)\oplus\Psi^2(E^0)\otimes\Psi^1(E^2)\oplus\Psi^1(E^0)\otimes\Psi^2(E^2)\oplus\Psi^3(E^2)$$

with $\Psi^3(E^0\oplus E^2)$ by the isomorphism (4.2.16). That is, $\widetilde{C}_2(E)$ is the exact sequence

$$\Psi^3(E^0)\oplus\Psi^2(E^0)\otimes\Psi^1(E^2)\oplus\Psi^1(E^0)\otimes\Psi^2(E^2) \to \Psi^3(E^0\oplus E^2) \to \Psi^3(E^2).$$

Then, the differential of $C_0(E)+C_1(E)-\widetilde{C}_2(E)$ is

$$-\Psi^3(E^0)+\Psi^3(E^0\oplus E^2)-\Psi^3(E^2)-\Psi^2(E^0)\otimes\Psi^1(E^2)-\Psi^1(E^0)\otimes\Psi^2(E^2).$$

As in the previous examples, we have $\Psi^2(E^0)\otimes\Psi^1(E^2)\in\mathbb{Z}G_2^{1,3}(X)_0$ and $\Psi^1(E^0)\otimes\Psi^2(E^2)\in\mathbb{Z}G_2^{2,3}(X)_0$. Hence, the differential of

$$\Psi^3(E):=(\Psi^2(E^0)\otimes\Psi^1(E^2),\Psi^1(E^0)\otimes\Psi^2(E^2),C_0(E)+C_1(E)+\widetilde{C}_2(E))$$

is exactly $-\Psi^3(E^0)+\Psi^3(E^0\oplus E^2)-\Psi^3(E^2)$ as desired.

Finally, for an arbitrary split 1-cube (E,f),

$$\widetilde{C}_0(E,f):=C_0(\mathrm{Sp}(E)),\qquad \widetilde{C}_1(E,f):=C_1(\mathrm{Sp}(E)),$$

and $\widetilde{C}_2(E,f)$ is defined by changing the term $\Psi^3(E^0\oplus E^2)$ in $\widetilde{C}_2(\mathrm{Sp}(E))$ by $\Psi^3(E^1)$ by means of the isomorphism induced by f.

Adams operations in the case $\mathbf{n=2}$, $\mathbf{k=2}$. Let $E=\begin{bmatrix} E^{00} & E^{02} \\ E^{20} & E^{22}\end{bmatrix}$. Then, we define the following terms of $\mathbb{Z}G_1^2(X)_2$:

$$C_{00}(E):=\begin{bmatrix} \Psi^2(E^{00}) & \Psi^1(E^{00})\otimes\Psi^1(E^{02}) \\ \Psi^1(E^{00})\otimes\Psi^1(E^{20}) & \Psi^1(E^{00})\otimes\Psi^1(E^{22})\oplus\Psi^1(E^{02})\otimes\Psi^1(E^{20})\end{bmatrix},$$

$$C_{10}(E):=\begin{bmatrix} \begin{matrix}\Psi^2(E^{00})\\ \oplus\Psi^1(E^{00})\otimes\Psi^1(E^{20})\end{matrix} & \begin{matrix}\Psi^1(E^{00})\otimes\Psi^1(E^{02})\oplus\Psi^1(E^{00})\otimes\Psi^1(E^{22})\\ \oplus\Psi^1(E^{02})\otimes\Psi^1(E^{20})\end{matrix} \\ \Psi^2(E^{20}) & \Psi^1(E^{20})\otimes\Psi^1(E^{22})\end{bmatrix},$$

$$C_{01}(E):=\begin{bmatrix} \Psi^2(E^{00})\oplus\Psi^1(E^{00})\otimes\Psi^1(E^{02}) & \Psi^2(E^{02}) \\ \begin{matrix}\Psi^1(E^{00})\otimes\Psi^1(E^{20})\oplus\Psi^1(E^{00})\otimes\Psi^1(E^{22})\\ \oplus\Psi^1(E^{02})\otimes\Psi^1(E^{20})\end{matrix} & \Psi^1(E^{02})\otimes\Psi^1(E^{22})\end{bmatrix},$$

$$C_{11}(E):=\begin{bmatrix} \begin{matrix}\Psi^2(E^{00})\oplus\\ \Psi^1(E^{00})\otimes\Psi^1(E^{02})\oplus\Psi^1(E^{00})\otimes\Psi^1(E^{20})\\ \oplus\Psi^1(E^{00})\otimes\Psi^1(E^{22})\oplus\Psi^1(E^{02})\otimes\Psi^1(E^{20})\end{matrix} & \begin{matrix}\Psi^2(E^{02})\\ \oplus\Psi^1(E^{02})\otimes\Psi^1(E^{22})\end{matrix} \\ \Psi^2(E^{20})\oplus\Psi^1(E^{20})\otimes\Psi^1(E^{22}) & \Psi^2(E^{22})\end{bmatrix}.$$

The faces of each of these cubes are as follows (up to the isomorphism (4.2.16)):

▷ Terms that are summands of $\Psi^2(\partial_i^j E)$:

$$\begin{array}{rclrcl}
\partial_1^0C_{00}(E) &=& C_0(\partial_1^0E), & \partial_2^0C_{00}(E) &=& C_0(\partial_2^0E),\\
\partial_1^0C_{01}(E) &=& C_1(\partial_1^0E), & \partial_2^0C_{10}(E) &=& C_1(\partial_2^0E),\\
\partial_1^1C_{10}(E) &=& C_0(\partial_1^1E), & \partial_2^1C_{01}(E) &=& C_0(\partial_2^1E),\\
\partial_1^1C_{11}(E) &=& C_1(\partial_1^1E), & \partial_2^1C_{11}(E) &=& C_1(\partial_1^1E),\\
\partial_1^2C_{10}(E) &=& C_0(\partial_1^2E), & \partial_2^2C_{01}(E) &=& C_0(\partial_2^2E),\\
\partial_1^2C_{11}(E) &=& C_1(\partial_1^2E), & \partial_2^2C_{11}(E) &=& C_1(\partial_2^2E).
\end{array}$$

▷ Terms that are a direct sum of a tensor product of complexes:

$$\partial_1^2 C_{00}(E), \quad \partial_2^2 C_{00}(E), \quad \partial_2^2 C_{10}(E), \quad \partial_1^2 C_{01}(E).$$

▷ Terms that cancel each other:

$$\begin{array}{rclcrcl} \partial_1^1 C_{00}(E) & = & \partial_1^0 C_{10}(E), & \qquad & \partial_2^1 C_{00}(E) & = & \partial_2^0 C_{01}(E), \\ \partial_2^1 C_{10}(E) & = & \partial_2^0 C_{11}(E), & & \partial_1^1 C_{01}(E) & = & \partial_1^0 C_{11}(E). \end{array}$$

It follows that the differential of $C_{00}(E) + C_{10}(E) + C_{01}(E) + C_{11}(E)$ is $\Psi^2(dE)$ plus some terms which are a direct sum of a tensor product of complexes. These tensor product complexes can be viewed as 2-iterated cochain complexes of lengths $(1,1)$ of exact sequences. These terms are added in $\mathbb{Z}G_2^{1,2}(X)_1$.

Finally, for every split 2-cube (E,f), $\Psi^2(E,f)$ is defined by modifying the appropriate locally free sheaves in each $C_i(E)$ by means of the isomorphism f.

Outline of the definition of Ψ^k. From the given examples we see that the procedure is as follows:

▷ First, for every split n-cube (E,f), the direct sum n-cubes $C_i(E)$ are defined by a purely combinatorial formula on the Adams operations of the locally free sheaves $E^{\boldsymbol{j}}$, $\boldsymbol{j} \in \{0,2\}^n$.

▷ The previous construction is modified by the isomorphism (4.2.16).

▷ The entries of $C_i(E)$ which give the terms $C_i(\partial_l^1 E)$ in the differential, are modified by the morphisms induced by the isomorphism f.

From the examples, the key ideas that lead to the general combinatorial formula of $C_i(E)$ can be extracted:

▷ At each step, some entries in ∂_r^0 are constructed by taking the direct sum $\partial_r^0 \oplus \partial_r^2$ in a previous cube (where "previous" refers to the order $\leq$ for the subindices in $C_*(E)$).

▷ The new entries (not being direct sums of previous cubes) are direct sums of summands of the form $\Psi^{k_1}(E^{2\boldsymbol{n}_1}) \otimes \cdots \otimes \Psi^{k_r}(E^{2\boldsymbol{n}_r})$ satisfying:

(1) $\sum_s k_s = k$.

(2) In the position $2\boldsymbol{j}$ of the cube $C_{\boldsymbol{i}}(E)$, $\sum_s k_s \boldsymbol{n}_s = \boldsymbol{j} + \boldsymbol{i}$.

(3) Observe that in the example $n = 2$, all the entries in C_{00} are new, and the new entries for C_{10} are in the positions $(2,0), (2,2)$ and for C_{11} in $(2,2)$. Hence the new entries will correspond to the multi-indexes $\boldsymbol{j}$ such that $\boldsymbol{j} \geq \nu(\boldsymbol{i})$ (recall that $\nu(\boldsymbol{i})$ is the characteristic of the multi-index $\boldsymbol{i}$).

4.2.4 Definition of the cubes $C_i(E)$

Let $(E,f) \in \mathrm{Sp}_n(X)$ and fix $k \geq 1$. For every $\boldsymbol{i} \in [0,k-1]^n$, we define an exact sequence of direct sum cubes $C_{\boldsymbol{i}}(E) \in \mathbb{Z}SG_1^k(X)_n$. This definition is purely combinatorial and does not depend on the isomorphism f.

Let

$$L_k^r = \{\boldsymbol{k} = (k_1, \dots, k_r) \mid |\boldsymbol{k}| = k \text{ and } k_s \geq 1, \forall s\}$$

be the set of partitions of length r of k. Then, for every integer $n \geq 0$ and every multi-index $\boldsymbol{m}$ of length n, we define a new set of indices by:

$$\Lambda_k^n(\boldsymbol{m}) = \bigcup_{r \geq 1} \left\{ (\boldsymbol{k}, \boldsymbol{n}^1, \dots, \boldsymbol{n}^r) \in L_k^r \times (\{0,1\}^n)^r \;\middle|\; \sum k_s \boldsymbol{n}^s = \boldsymbol{m},\ \boldsymbol{n}^1 \prec \dots \prec \boldsymbol{n}^r \right\}.$$

Definition 4.2.17. Let $(E,f) \in \mathrm{Sp}_n(X)$. For every $\boldsymbol{i} \in [0,k-1]^n$, let $C_{\boldsymbol{i}}(E) \in SG_1^k(X)_n$ be the exact sequence of direct sum n-cubes, such that, for every $\boldsymbol{j} \in \{0,1\}^n$, the position $2\boldsymbol{j}$ is given as follows:

(i) If $\boldsymbol{j} \geq \nu(\boldsymbol{i})$, then

$$C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = \bigoplus_{\Lambda_k^n(\boldsymbol{j}+\boldsymbol{i})} \Psi^{k_1}(E^{2\boldsymbol{n}^1}) \otimes \dots \otimes \Psi^{k_r}(E^{2\boldsymbol{n}^r}). \tag{4.2.18}$$

(ii) If $\boldsymbol{j} \ngeq \nu(\boldsymbol{i})$, then

$$C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = \bigoplus_{\boldsymbol{j} \leq \boldsymbol{m} \leq \nu(\boldsymbol{i}) \cup \boldsymbol{j}} C_{\boldsymbol{i}-\nu(\boldsymbol{i})\cdot \boldsymbol{j}^c}(E)^{2\boldsymbol{m}}. \tag{4.2.19}$$

In order to simplify the notation, we will denote by r the length of $\boldsymbol{k} \in \Lambda_k^n(\boldsymbol{j}+\boldsymbol{i})$ in the future occurrences of the sum (4.2.18).

Observe that the definition of $C_i(E)$ for a split cube (E,f) does not depend on f.

Remark 4.2.20. First of all, observe that in equation (4.2.19), $\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot \boldsymbol{j}^c \in [0,k-1]^n$, i.e. for every s, $0 \leq (\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot \boldsymbol{j}^c)_s$:

- ▷ If $i_s = 0$, then $\nu(\boldsymbol{i})_s = 0$ and hence $(\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot \boldsymbol{j}^c)_s = 0$.
- ▷ If $i_s > 0$, then $\nu(\boldsymbol{i})_s = 1$ and since $(\boldsymbol{j}^c)_s = 0, 1$, we have $i_s \geq \nu(\boldsymbol{i})_s \cdot (\boldsymbol{j}^c)_s$.

Remark 4.2.21. Observe that equations (4.2.18) and (4.2.19) define $C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}}$ for all $\boldsymbol{j} \in \{0,1\}^n$. This follows from the following facts:

- ▷ Since $\nu(\boldsymbol{j}) \geq (0, \dots, 0)$, equation (4.2.18) defines $C_{\boldsymbol{i}}(E)$ for $|\,\boldsymbol{i}\,| = 0$.
- ▷ If $\boldsymbol{j} \ngeq \nu(\boldsymbol{i})$, then

  $$|\,\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot \boldsymbol{j}^c\,| < |\,\boldsymbol{i}\,|.$$

  Indeed, an equality would imply that $\nu(\boldsymbol{i}) \cdot \boldsymbol{j}^c = 0$ and hence that for all r such that $i_r \neq 0$, $\boldsymbol{j}_r^c = 0$, concluding that $\boldsymbol{j} \geq \nu(\boldsymbol{i})$.

Remark 4.2.22. Observe that equation (4.2.19) also holds trivially for $\boldsymbol{j} \geq \nu(\boldsymbol{i})$, because in this case $\nu(\boldsymbol{i}) \cdot \boldsymbol{j}^c = 0$ and $\nu(\boldsymbol{i}) \cup \boldsymbol{j} = \boldsymbol{j}$. We will use this observation in some proofs when only combinatorial questions are involved.

Remark 4.2.23. The direct sum of more than two terms means the consecutive direct sums of two objects under the lexicographic order in the subindices. In order to prove some equalities, it will be necessary to reorder the indices, and then return to the original order. For the sake of simplicity, we will not write the canonical isomorphisms used at every step and will just write equalities. The reader should bear this remark in mind throughout this section.

4.2.5 Faces of the cubes $C_{\boldsymbol{i}}(E)$

In this section, we compute the faces of the cubes $C_{\boldsymbol{i}}(E)$. We fix $k \geq 1$, $\boldsymbol{i} \in [0, k-1]^n$, a split n-cube $(E, f) \in \mathrm{Sp}_n(X)$ and $l \in \{1, \dots, n\}$.

Lemma 4.2.24.

$$\partial_l^0 C_{\boldsymbol{i}}(E) = \begin{cases} \partial_l^1 C_{\boldsymbol{i}-1_l}(E) & \text{if } i_l \neq 0, \\ C_{\partial_l(\boldsymbol{i})}(\partial_l^0 E) & \text{if } i_l = 0. \end{cases}$$

Proof. Assume that $i_l \neq 0$. It is enough to see that

$$\partial_l^0 C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = \partial_l^1 C_{\boldsymbol{i}-1_l}(E)^{2\boldsymbol{j}}, \qquad \forall\, \boldsymbol{j} \in \{0,1\}^{n-1}.$$

Observe that

$$\begin{aligned} \partial_l^0 C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} &= C_{\boldsymbol{i}}(E)^{2s_l^0(\boldsymbol{j})} = \bigoplus_{s_l^0(\boldsymbol{j}) \leq \boldsymbol{m} \leq \nu(\boldsymbol{i}) \cup s_l^0(\boldsymbol{j})} C_{\boldsymbol{i}-\nu(\boldsymbol{i})\cdot s_l^0(\boldsymbol{j})^c}(E)^{2\boldsymbol{m}}, \\ \partial_l^1 C_{\boldsymbol{i}-1_l}(E)^{2\boldsymbol{j}} &= C_{\boldsymbol{i}-1_l}(E)^{2s_l^0(\boldsymbol{j})} \oplus C_{\boldsymbol{i}-1_l}(E)^{2s_l^1(\boldsymbol{j})}, \end{aligned}$$

with

$$C_{\boldsymbol{i}-1_l}(E)^{2s_l^0(\boldsymbol{j})} = \bigoplus_{s_l^0(\boldsymbol{j}) \leq \boldsymbol{m} \leq \nu(\boldsymbol{i}-1_l) \cup s_l^0(\boldsymbol{j})} C_{\boldsymbol{i}-1_l-\nu(\boldsymbol{i}-1_l)\cdot s_l^0(\boldsymbol{j})^c}(E)^{2\boldsymbol{m}}, \tag{4.2.25}$$

$$C_{\boldsymbol{i}-1_l}(E)^{2s_l^1(\boldsymbol{j})} = \bigoplus_{s_l^1(\boldsymbol{j}) \leq \boldsymbol{m} \leq \nu(\boldsymbol{i}-1_l) \cup s_l^1(\boldsymbol{j})} C_{\boldsymbol{i}-1_l-\nu(\boldsymbol{i}-1_l)\cdot s_l^1(\boldsymbol{j})^c}(E)^{2\boldsymbol{m}}. \tag{4.2.26}$$

Let us compute each term separately. We start with (4.2.26). Since $s_l^1(\boldsymbol{j})_l = 1$, we see that $\boldsymbol{m}_l = 1$ for all indices $\boldsymbol{m}$ of the direct sum. Moreover, since $s_l^1(\boldsymbol{j})_l^c = 0$, $s_l^0(\boldsymbol{j})_l^c = 1$, and $\nu(\boldsymbol{i})_l = 1$, we obtain that

$$(\boldsymbol{i} - 1_l - \nu(\boldsymbol{i} - 1_l) \cdot s_l^1(\boldsymbol{j})^c)_l = i_l - 1_l = (\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot s_l^0(\boldsymbol{j})^c)_l.$$

Since it is clear that for all $t \neq l$, $(\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot s_l^0(\boldsymbol{j})^c)_t = (\boldsymbol{i} - 1_l - \nu(\boldsymbol{i} - 1_l) \cdot s_l^1(\boldsymbol{j})^c)_t$, we see that

$$\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot s_l^0(\boldsymbol{j})^c = \boldsymbol{i} - 1_l - \nu(\boldsymbol{i} - 1_l) \cdot s_l^1(\boldsymbol{j})^c.$$

Thus,

$$C_{\boldsymbol{i}-1_l}(E)^{2s_l^1(\boldsymbol{j})} = \bigoplus_{\substack{s_l^0(\boldsymbol{j})\leq \boldsymbol{m}\leq \nu(\boldsymbol{i})\cup s_l^0(\boldsymbol{j}),\\ \boldsymbol{m}_l=1}} C_{\boldsymbol{i}-\nu(\boldsymbol{i})\cdot s_l^0(\boldsymbol{j})^c}(E)^{2\boldsymbol{m}}.$$

All that remains is to see that

$$C_{\boldsymbol{i}-1_l}(E)^{2s_l^0(\boldsymbol{j})} = \bigoplus_{\substack{s_l^0(\boldsymbol{j})\leq \boldsymbol{m}\leq \nu(\boldsymbol{i})\cup s_l^0(\boldsymbol{j}),\\ \boldsymbol{m}_l=0}} C_{\boldsymbol{i}-\nu(\boldsymbol{i})\cdot s_l^0(\boldsymbol{j})^c}(E)^{2\boldsymbol{m}}.$$

We proceed by induction on i_l. If $i_l=1$ then $(\boldsymbol{i}-1_l)_l=0$ and hence $(\nu(\boldsymbol{i}-1_l)\cup s_l^0(\boldsymbol{j}))_l=0$ which means that $m_l=0$ for all multi-indices $\boldsymbol{m}$ in the direct sum (4.2.25). Moreover $(\boldsymbol{i}-1_l-\nu(\boldsymbol{i}-1_l)\cdot s_l^0(\boldsymbol{j})^c)_l=0$ and $(\boldsymbol{i}-\nu(\boldsymbol{i})\cdot s_l^0(\boldsymbol{j})^c)_l=0$. Therefore,

$$\boldsymbol{i}-1_l-\nu(\boldsymbol{i}-1_l)\cdot s_l^0(\boldsymbol{j})^c = \boldsymbol{i}-\nu(\boldsymbol{i})\cdot s_l^0(\boldsymbol{j})^c$$

and the equality is proven. Let $i_l>1$ and assume that the lemma is true for i_l-1. Then, since $i_l>1$, we have $\nu(\boldsymbol{i}-1_l)=\nu(\boldsymbol{i})$ and $\nu(\boldsymbol{i})_l=1$. Writing $\alpha=\boldsymbol{i}-1_l-\nu(\boldsymbol{i}-1_l)\cdot s_l^0(\boldsymbol{j})^c$, we obtain

$$\begin{aligned}
C_{\boldsymbol{i}-1_l}(E)^{2s_l^0(\boldsymbol{j})} &= \bigoplus_{\substack{s_l^0(\boldsymbol{j})\leq \boldsymbol{m}\leq \nu(\boldsymbol{i}-1_l)\cup s_l^0(\boldsymbol{j})\\ m_l=0}} C_\alpha(E)^{2\boldsymbol{m}} \oplus \bigoplus_{\substack{s_l^0(\boldsymbol{j})\leq \boldsymbol{m}\leq \nu(\boldsymbol{i}-1_l)\cup s_l^0(\boldsymbol{j})\\ m_l=1}} C_\alpha(E)^{2\boldsymbol{m}}\\
&= \bigoplus_{\boldsymbol{j}\leq \boldsymbol{n}\leq \nu(\partial_l(\boldsymbol{i}))\cup \boldsymbol{j}} (\partial_l^0\oplus\partial_l^1)C_\alpha(E)^{2\boldsymbol{n}}\\
&= \bigoplus_{\boldsymbol{j}\leq \boldsymbol{n}\leq \nu(\partial_l(\boldsymbol{i}))\cup \boldsymbol{j}} \partial_l^0 C_{\boldsymbol{i}-\nu(\boldsymbol{i})\cdot s_l^0(\boldsymbol{j})^c}(E)^{2\boldsymbol{n}}\\
&= \bigoplus_{\substack{s_l^0(\boldsymbol{j})\leq \boldsymbol{m}\leq \nu(\boldsymbol{i})\cup s_l^0(\boldsymbol{j}),\\ m_l=0}} C_{\boldsymbol{i}-\nu(\boldsymbol{i})\cdot s_l^0(\boldsymbol{j})^c}(E)^{2\boldsymbol{m}},
\end{aligned}$$

since $(\boldsymbol{i}-\nu(\boldsymbol{i})\cdot s_l^0(\boldsymbol{j})^c)_l=i_l-1$ and we can apply the induction hypothesis in the third equality.

Let us now prove the equality with $i_l=0$. Assume that $s_l^0(\boldsymbol{j})\geq\nu(\boldsymbol{i})$. Then,

$$\partial_l^0 C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = C_{\boldsymbol{i}}(E)^{2s_l^0(\boldsymbol{j})} = \bigoplus_{\Lambda_k^n(s_l^0(\boldsymbol{j})+\boldsymbol{i})} \Psi^{k_1}(E^{2\boldsymbol{n}^1})\otimes\cdots\otimes\Psi^{k_r}(E^{2\boldsymbol{n}^r}) = (*).$$

Since $(s_l^0(\boldsymbol{j})+\boldsymbol{i})_l=0$, $(\sum k_s\boldsymbol{n}^s)_l=0$. Hence, for all s, $\boldsymbol{n}_l^s=0$ and we obtain

$$(*) = \bigoplus_{\Lambda_k^{n-1}(\boldsymbol{j}+\partial_l(\boldsymbol{i}))} \Psi^{k_1}(\partial_l^0 E^{2\boldsymbol{n}^1})\otimes\cdots\otimes\Psi^{k_r}(\partial_l^0 E^{2\boldsymbol{n}^r}) = C_{\partial_l(\boldsymbol{i})}(\partial_l^0 E)^{2\boldsymbol{j}}.$$

Finally, if $s_l^0(\boldsymbol{j})\not\geq\nu(\boldsymbol{i})$, the direct sum (4.2.19) can be written in the form

$$\partial_l^0 C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = \bigoplus_{\boldsymbol{m},\boldsymbol{n}} C_{\boldsymbol{n}}(E)^{2\boldsymbol{m}}$$

with $\boldsymbol{m} \geq \nu(\boldsymbol{n})$ and $m_l = 0$. We deduce that $n_l = 0$ and hence,

$$\partial_l^0 C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = \bigoplus_{\boldsymbol{m},\boldsymbol{n}} C_{\boldsymbol{n}}(E)^{2\boldsymbol{m}} = \bigoplus_{\boldsymbol{m},\boldsymbol{n}} C_{\partial_l(\boldsymbol{n})}(\partial_l^0 E)^{2\boldsymbol{m}} = C_{\partial_l(\boldsymbol{i})}(\partial_l^0 E)^{2\boldsymbol{j}}.$$

This reduces the proof to the already considered case. □

Lemma 4.2.27. *If $\boldsymbol{i}_l = k-1$, then $\partial_l^2 C_{\boldsymbol{i}}(E) = C_{\partial_l(\boldsymbol{i})}(\partial_l^2 E)$.*

Proof. Arguing as in the proof of the previous lemma, we limit ourselves to proving the equality in the case where $s_l^1(\boldsymbol{j}) \geq \nu(\boldsymbol{i})$. In this situation, we obtain

$$(\partial_l^2 C_{\boldsymbol{i}}(E))^{2\boldsymbol{j}} = C_{\boldsymbol{i}}(E)^{2s_l^1(\boldsymbol{j})} = \bigoplus_{\Lambda_k^n(s_l^1(\boldsymbol{j})+\boldsymbol{i})} \Psi^{k_1}(E^{2\boldsymbol{n}^1}) \otimes \cdots \otimes \Psi^{k_r}(E^{2\boldsymbol{n}^r}).$$

Since $(s_l^1(\boldsymbol{j}) + \boldsymbol{i})_l = 1 + k - 1 = k$, we deduce that for all s, $\boldsymbol{n}_l^s = 1$ and therefore the lemma is proved. □

The next lemma determines the faces ∂_l^2 of $C_{\boldsymbol{i}}(E)$ whenever $i_l \neq k-1$.

Lemma 4.2.28. *Let $\boldsymbol{j} \in \{0,1\}^n$ with $j_l = 1$ and let $i_l \neq k-1$. Up to a canonical isomorphism, each of the direct summands of $C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}}$ is the tensor product of an exact sequence of length $(k - i_l - 1)$ by an exact sequence of length $(i_l + 1)$. Explicitly, in the equality*

$$C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = \bigoplus_{\Lambda_k^n(\boldsymbol{j}+\boldsymbol{i})} \Psi^{k_1}(E^{2\boldsymbol{n}^1}) \otimes \cdots \otimes \Psi^{k_r}(E^{2\boldsymbol{n}^r}),$$

the tensor product of the Koszul complexes corresponding to the multi-indices $\boldsymbol{n}$ with $n_l = 0$ gives the exact sequence of length $k - i_l - 1$, while the tensor product of the Koszul complexes corresponding to the multi-indices $\boldsymbol{n}$ with $n_l = 1$ gives the exact sequence of length $i_l + 1$.

Proof. If $\boldsymbol{j} \geq \nu(\boldsymbol{i})$, then,

$$C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = \bigoplus_{\Lambda_k^n(\boldsymbol{j}+\boldsymbol{i})} \Psi^{k_1}(E^{2\boldsymbol{n}^1}) \otimes \cdots \otimes \Psi^{k_r}(E^{2\boldsymbol{n}^r}).$$

Assume that one of the summands is not a tensor product. Then there exists a multi-index $\boldsymbol{n}$ with $k\boldsymbol{n} = \boldsymbol{j} + \boldsymbol{i}$. In particular, $(k \cdot \boldsymbol{n})_l = 1 + \boldsymbol{i}_l$. But $(k \cdot \boldsymbol{n})_l$ is either 0 or k, and by hypothesis, $1 \leq 1 + \boldsymbol{i}_l < 1 + k - 1 = k$, which is a contradiction. If $\boldsymbol{j} \not\geq \nu(\boldsymbol{i})$, then,

$$C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}} = \bigoplus_{\boldsymbol{j} \leq \boldsymbol{m} \leq \nu(\boldsymbol{i}) \cup \boldsymbol{j}} C_{\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot \boldsymbol{j}^c}(E)^{2\boldsymbol{m}}.$$

The condition $\boldsymbol{j} \leq \boldsymbol{m}$ implies that $\boldsymbol{m}_l = 1$. Moreover, $(\boldsymbol{i} - \nu(\boldsymbol{i}) \cdot \boldsymbol{j}^c)_l = i_l - \boldsymbol{j}_l^c \leq i_l < k-1$. This means that every direct summand is a tensor product of exact sequences. By induction on $|\boldsymbol{i}|$, this is true for any multi-index $\boldsymbol{i}$.

Let us prove that every direct summand can be seen as the tensor product of an exact sequence of length $k-i_l-1$, corresponding to the multi-indices $\boldsymbol{n}$ with $n_l=0$, and one of length i_l+1, corresponding to the multi-indices with $n_l=1$. By an induction argument, it is enough to prove the result in the case $\boldsymbol{j}\geq \nu(\boldsymbol{i})$. Let $(\boldsymbol{k},\boldsymbol{n}^1,\dots,\boldsymbol{n}^r)\in\Lambda_k^n(\boldsymbol{j}+\boldsymbol{i})$. Let $s_1,\dots,s_m$ be the indices such that $\boldsymbol{n}_l^{s_j}=1$ and let $s'_1,\dots,s'_{r-m}$ be the indices such that $\boldsymbol{n}_l^{s'_j}=0$. Since $\sum k_s\boldsymbol{n}_l^s=i_l+1$, we see that $\sum_{j=1}^m k_{s_j}=i_l+1$ and hence

$$T_1:=\Psi^{k_{s_1}}(E^{2\boldsymbol{n}^{s_1}})\otimes\dots\otimes\Psi^{k_{s_m}}(E^{2\boldsymbol{n}^{s_m}})$$

is an exact sequence of length i_l+1. Then,

$$T_0:=\Psi^{k_{s'_1}}(E^{2\boldsymbol{n}^{s'_1}})\otimes\dots\otimes\Psi^{k_{s'_{r-m}}}(E^{2\boldsymbol{n}^{s'_{r-m}}})$$

is an exact sequence of length $k-i_l-1$ and there is a canonical isomorphism

$$\Psi^{k_1}(E^{2\boldsymbol{n}^1})\otimes\dots\otimes\Psi^{k_r}(E^{2\boldsymbol{n}^r})\cong T_0\otimes T_1$$

as desired. □

Lemma 4.2.29. *If $i_l=k-1$, then*

$$\partial_l^1 C_{\boldsymbol{i}}(E)\cong C_{\partial_l(\boldsymbol{i})}(\partial_l^0E\oplus\partial_l^2E),$$

with the isomorphism induced by the canonical isomorphism of the Koszul complex of a direct sum in (4.2.16).

Proof. The first part of the lemma is lemma 4.2.24. Assume then that $i_l=k-1$. Applying lemmas 4.2.27 and 4.2.24 recursively, we obtain

$$\partial_l^1 C_{\boldsymbol{i}}(E)\cong\partial_l^0 C_{\boldsymbol{i}}(E)\oplus\partial_l^2 C_{\boldsymbol{i}}(E)=C_{\partial_l(\boldsymbol{i})}(\partial_l^0E)\oplus\bigoplus_{a\in[0,k-1]}\partial_l^2 C_{\boldsymbol{i}-a_l}(E). \tag{4.2.30}$$

Then, if $\boldsymbol{j}\in\{0,1\}^{n-1}$ satisfies $\boldsymbol{j}\geq\partial_l\nu(\boldsymbol{i})$, we obtain that

$$\begin{aligned}\partial_l^1 C_{\boldsymbol{i}}(E)^{2\boldsymbol{j}}={}&\textstyle\bigoplus_{\Lambda_k^{n-1}(\boldsymbol{j}+\partial_l(\boldsymbol{i}))}\Psi^{k_1}(\partial_l^0E^{2\boldsymbol{n}^1})\otimes\dots\otimes\Psi^{k_r}(\partial_l^0E^{2\boldsymbol{n}^r})\oplus\\ &\textstyle\bigoplus_{a\in[0,k-1]}\bigoplus_{\Lambda_k^n(s_l^1(\boldsymbol{j})+\boldsymbol{i}-a_l)}\Psi^{k_1}(E^{2\boldsymbol{n}^1})\otimes\dots\otimes\Psi^{k_r}(E^{2\boldsymbol{n}^r}).\end{aligned}$$

On the other hand, by the additivity of Ψ^k in (4.2.16), there are canonical isomorphisms

$$\begin{aligned}&\Psi^{k_1}((\partial_l^0E\oplus\partial_l^2E)^{2\boldsymbol{n}^1})\otimes\dots\otimes\Psi^{k_r}((\partial_l^0E\oplus\partial_l^2E)^{2\boldsymbol{n}^r})=\\ &\qquad\cong\bigoplus_{m_1=0}^{k_1}\cdots\bigoplus_{m_r=0}^{k_r}\begin{aligned}&\Psi^{k_1-m_1}(\partial_l^0E^{2\boldsymbol{n}^1})\otimes\dots\otimes\Psi^{k_r-m_r}(\partial_l^0E^{2\boldsymbol{n}^r})\\ &\otimes\Psi^{m_1}(\partial_l^2E^{2\boldsymbol{n}^1})\otimes\dots\otimes\Psi^{m_r}(\partial_l^2E^{2\boldsymbol{n}^r}).\end{aligned}\end{aligned}$$

Therefore, $C_{\partial_l(\boldsymbol{i})}(\partial_l^0 E \oplus \partial_l^2 E)^{2\boldsymbol{j}}$ is canonically isomorphic to

$$\bigoplus_{\Lambda_k^{n-1}(\boldsymbol{j}+\partial_l(\boldsymbol{i}))} \bigoplus_{m_1=0}^{k_1} \cdots \bigoplus_{m_r=0}^{k_r} \begin{array}{l} \Psi^{k_1-m_1}(E^{2s_l^0(\boldsymbol{n}^1)}) \otimes \cdots \otimes \Psi^{k_r-m_r}(E^{2s_l^0(\boldsymbol{n}^r)}) \\ \otimes \Psi^{m_1}(E^{2s_l^1(\boldsymbol{n}^1)}) \otimes \cdots \otimes \Psi^{m_r}(E^{2s_l^1(\boldsymbol{n}^r)}). \end{array}$$

The first summand in (4.2.30) corresponds to the indices $m_1, \ldots, m_r = 0$ in the latter sum. Therefore, we have to see that the second summand in (4.2.30) corresponds to the summand in the latter sum with at least one index $m_i \neq 0$. We will see that there is a bijection between the sets of multi-indices of each term.

For every collection $k_1, \ldots, k_r, \boldsymbol{n}^1, \ldots, \boldsymbol{n}^r, m_1, \ldots, m_r$ with not all $m_s = 0$, let $a = k - \sum m_s$. Since $\sum m_s \neq 0$, $a \in [0, k-1]$. Let $s_1, \ldots, s_t \in \{1, \ldots, r\}$ be the indices for which $k_{s_l} - m_{s_l} \neq 0$ and let $s'_1, \ldots, s'_m \in \{1, \ldots, r\}$ be the indices for which $m_{s'_l} \neq 0$. Then, to these data correspond the indices a, and

$$\begin{aligned} \{k'_1, \ldots, k'_{t+m}\} &= \{k_{s_1} - m_{s_1}, \ldots, k_{s_t} - m_{s_t}, m_{s'_1}, \ldots, m_{s'_m}\} \\ \hat{\boldsymbol{n}}^p &= \begin{cases} s_l^0(\boldsymbol{n}^{s_p}) & \text{if } p = 1, \ldots, t, \\ s_l^1(\boldsymbol{n}^{s_{p-t}}) & \text{if } p = t+1, \ldots, t+m. \end{cases} \end{aligned}$$

Conversely, let a, k_s and $\boldsymbol{n}^s$ be given. Then, we rearrange the collection $\boldsymbol{n}^s$ by the rule:

$$\boldsymbol{n}^1, \ldots, \boldsymbol{n}^y, \boldsymbol{n}^{y+1}, \ldots, \boldsymbol{n}^x, \boldsymbol{n}^{x+1}, \ldots, \boldsymbol{n}^{2x-y}, \boldsymbol{n}^{2x-y+1}, \ldots, \boldsymbol{n}^r$$

with

$$(\boldsymbol{n}^s)_l = \begin{cases} 0 & \text{if } s = 1, \ldots, x, \\ 1 & \text{if } s = x+1, \ldots, r, \end{cases}$$

The index y satisfies that for $s = 1, \ldots, y$ and for $s = 2x-y+1, \ldots, r$, there is no other index s' with $\partial_l(\boldsymbol{n}^s) = \partial_l(\boldsymbol{n}^{s'})$ and for $s = y+1, \ldots, x-y$, $\partial_l(\boldsymbol{n}^s) = \partial_l(\boldsymbol{n}^{s+x-y})$. Then, the corresponding multi-indices are

$$\begin{aligned} (\hat{\boldsymbol{n}}^1, \ldots, \hat{\boldsymbol{n}}^{r-x-y}) &= (\partial_l(\boldsymbol{n}^1), \ldots, \partial_l(\boldsymbol{n}^x), \partial_l(\boldsymbol{n}^{2x-y+1}), \ldots, \partial_l(\boldsymbol{n}^r)), \\ k'_s &= \begin{cases} k_s & \text{if } s = 1, \ldots, y, \\ k_s + k_{s+x-y} & \text{if } s = y+1, \ldots, x, \\ k_{s+x-y} & \text{if } s = x+1, \ldots, r-x-y. \end{cases} \\ m'_s &= \begin{cases} 0 & \text{if } s = 1, \ldots, y, \\ k_{s+x-y} & \text{if } s = y+1, \ldots, x, \\ k_{s+x-y} & \text{if } s = x+1, \ldots, r-x-y. \end{cases} \end{aligned}$$

The lemma follows from this correspondence. □

4.2.6 Definition of the cubes $\widetilde{C}_{\boldsymbol{i}}(E)$

At this point, we have defined the cubes $C_{\boldsymbol{i}}(E)$ for every split n-cube (E, f). Roughly speaking, all that remains is to change, by means of f, the terms corresponding to $\partial_l^0 E \oplus \partial_l^2 E$ by the terms in $\partial_l^1 E$.

By the collection of lemmas above, this will be the case whenever $i_l = k-1$ and $j_l = 1$. Therefore, let $\boldsymbol{j} \in \{0,1,2\}^n$ and $\boldsymbol{i} \in [0,k-1]^n$. Let

- ▷ $w(\boldsymbol{j}) = (w_1, \dots, w_{r_w(\boldsymbol{j})})$ where $w_1 < \dots < w_{r_w(\boldsymbol{j})}$ are the indices such that $j_{w_m} = 1$ and $i_{w_m} = k-1$.
- ▷ $v(\boldsymbol{j}) = (v_1, \dots, v_{r_v(\boldsymbol{j})})$ where $v_1 < \dots < v_{r_v(\boldsymbol{j})}$ are the indices such that $j_{v_m} = 1$ and $i_{v_m} \neq k-1$.

Then, by lemma 4.2.29,

$$C_{\boldsymbol{i}}(E)^{\boldsymbol{j}} \cong \bigoplus_{\boldsymbol{m}\in\{0,2\}^{r_v(\boldsymbol{j})}} C_{\partial_{w(\boldsymbol{j})}(\boldsymbol{i})}\Big(\bigoplus_{\boldsymbol{n}\in\{0,2\}^{r_w(\boldsymbol{j})}} E^{\sigma^{\boldsymbol{n}}_{w(\boldsymbol{j})}}\Big)^{\partial_{w(\boldsymbol{j})}(\sigma^{\boldsymbol{m}}_{v(\boldsymbol{j})}(\boldsymbol{j}))}.$$

Recall that there is an isomorphism

$$\bigoplus_{\boldsymbol{n}\in\{0,2\}^r} E^{\sigma^{\boldsymbol{n}}_{w(\boldsymbol{j})}} \overset{f}{\cong} \partial^{\boldsymbol{1}}_{w(\boldsymbol{j})}E.$$

This motivates the following definition.

Definition 4.2.31. Let (E,f) be a split n-cube and let $\boldsymbol{i} \in [0,k-1]^n$. The n-cube $\widetilde{C}_{\boldsymbol{i}}(E)$ is defined by:

$$\widetilde{C}_{\boldsymbol{i}}(E)^{\boldsymbol{j}} = \bigoplus_{\boldsymbol{m}\in\{0,2\}^{r_w(\boldsymbol{j})}} C_{\partial_{w(\boldsymbol{j})}(\boldsymbol{i})}\big(\partial^{\boldsymbol{1}}_{w(\boldsymbol{j})}E\big)^{\partial_{w(\boldsymbol{j})}(\sigma^{\boldsymbol{m}}_{v(\boldsymbol{j})}(\boldsymbol{j}))}. \tag{4.2.32}$$

The morphisms in $\widetilde{C}_{\boldsymbol{i}}(E)$ are given as follows.

(i) If l with $i_l = k-1$ does not exist, then the cube $\widetilde{C}_{\boldsymbol{i}}(E)$ is split.

(ii) If $i_l = k-1$, then the morphisms in the cube are induced by the fixed isomorphisms $\partial^{\boldsymbol{1}}_{w(\boldsymbol{j})}(E) \cong \bigoplus_{\boldsymbol{n}\in\{0,2\}^{r_w(\boldsymbol{j})}} \partial^{\boldsymbol{n}}_{w(\boldsymbol{j})}E$ and the canonical isomorphisms in lemma 4.2.29.

Since all isomorphisms are fixed, the following proposition is a consequence of lemmas 4.2.24, 4.2.27, 4.2.28 and 4.2.29,

Proposition 4.2.33. *Let (E,f) be a split n-cube, $\boldsymbol{i} \in [0,k-1]^n$ and $l \in \{1,\dots,n\}$.*

(i) If $i_l = 0$, then $\partial^0_l \widetilde{C}_{\boldsymbol{i}}(E) = \widetilde{C}_{\partial_l(\boldsymbol{i})}(\partial^0_l E)$.

(ii) If $i_l \neq 0$, then $\partial^0_l \widetilde{C}_{\boldsymbol{i}}(E) = \partial^1_l \widetilde{C}_{\boldsymbol{i}-1_l}(E)$.

(iii) If $i_l = k-1$, then $\partial^2_l \widetilde{C}_{\boldsymbol{i}}(E) = \widetilde{C}_{\partial_l(\boldsymbol{i})}(\partial^2_l E)$ and $\partial^1_l \widetilde{C}_{\boldsymbol{i}}(E) = \widetilde{C}_{\partial_l(\boldsymbol{i})}(\partial^1_l E)$.

(iv) Lemma 4.2.28 remains valid for the cubes $\widetilde{C}_{\boldsymbol{i}}(E)$.

□

Remark 4.2.34. Let (E,f) be a split n-cube. Observe that by the choice of isomorphisms, for every $\boldsymbol{j}$ with $j_l = 1$ and $\boldsymbol{i}$ with $i_l = k-1$, the arrows

$$\widetilde{C}_{\boldsymbol{i}}(E)^{\sigma_l^0(\boldsymbol{j})} \to \widetilde{C}_{\boldsymbol{i}}(E)^{\boldsymbol{j}} \quad \text{and} \quad \widetilde{C}_{\boldsymbol{i}}(E)^{\boldsymbol{j}} \to \widetilde{C}_{\boldsymbol{i}}(E)^{\sigma_l^2(\boldsymbol{j})}$$

are induced by the arrows

$$\partial_l^0 E \to \partial_l^1 E, \quad \partial_l^1 E \to \partial_l^2 E \quad \text{and} \quad \partial_l^2 E \to \partial_l^0 E \oplus \partial_l^2 E \xrightarrow{f} \partial_l^1 E.$$

4.2.7 Definition of Ψ^k

In this section, we define a morphism

$$\mathbb{Z}\,\mathrm{Sp}_n(X) \xrightarrow{\Psi^k} \mathbb{Z}SG^k(X)_n$$

using the cubes $\widetilde{C}_{\boldsymbol{i}}(E)$ constructed in the previous section.

Recall that when $i_l \neq k-1$, the exact sequence $\partial_l^1 \widetilde{C}_{\boldsymbol{i}}(E)$ is canonically isomorphic to the simple associated to a 2-iterated cochain complex of $(n-1)$-cubes, of lengths $(k-i_l-1, i_l+1)$. We define

$$\begin{array}{rcl} \mathbb{Z}\,\mathrm{Sp}_n(X) & \xrightarrow{\Psi^k} & \mathbb{Z}SG^k(X)_n = \displaystyle\bigoplus_{m=1}^{k-1} \mathbb{Z}SG_2^{m,k}(X)_{n-1} \oplus \mathbb{Z}SG_1^k(X)_n \\ (E,f) & \mapsto & (\Psi_2^{1,k}(E), \dots, \Psi_2^{k-1,k}(E), \Psi_1^k(E)) \end{array}$$

with

$$\begin{array}{rcl} \Psi_1^k(E) & = & \displaystyle\sum_{\boldsymbol{i}\in[0,k-1]^n} \widetilde{C}_{\boldsymbol{i}}(E), \\ \Psi_2^{m,k}(E) & = & \displaystyle\sum_{l=1}^n (-1)^{m+l+1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n,\\ i_l=m-1}} \partial_l^1 \widetilde{C}_{\boldsymbol{i}}(E), \quad \text{for } m=1,\dots,k-1, \end{array}$$

where in the last equality we consider, by proposition 4.2.33 (iv), $\partial_l^1 \widetilde{C}_{\boldsymbol{i}}(E)$ as a 2-iterated complex.

Remark 4.2.35. Observe that in the last definition, considering $\partial_l^1 \widetilde{C}_{\boldsymbol{i}}(E)$ as a tensor product of complexes involves changing the order in the summations. To be precise, recall that the terms corresponding to the indices $\boldsymbol{n}$ with $n_l = 0$ form the first complex in the tensor product, and the ones with $n_l = 1$ form the second complex. The tensor product of the Koszul complexes in each summand of $\widetilde{C}_{\boldsymbol{i}}(E)$ is ordered by the lexicographic order. Hence, the two orders would only agree for $l=1$. For instance,

▷ the face ∂_2^2 of the cube $C_{00}(E)$ (see example $k=2, n=2$), is

$$[\Psi^1(E^{00}) \otimes \Psi^1(E^{02}), \Psi^1(E^{00}) \otimes \Psi^1(E^{22}) \oplus \Psi^1(E^{02}) \otimes \Psi^1(E^{20})],$$

▷ this complex, viewed as a tensor product of complexes, is

$$\left[\begin{array}{ccccc} E^{00}\otimes E^{02} \to E^{00}\otimes E^{02} & & E^{00}\otimes E^{22}\oplus E^{20}\otimes E^{02} \longrightarrow E^{00}\otimes E^{22}\oplus E^{20}\otimes E^{02} \\ \downarrow \qquad\qquad \downarrow & , & \downarrow \qquad\qquad\qquad\qquad \downarrow \\ E^{00}\otimes E^{02} \to E^{00}\otimes E^{02} & & E^{00}\otimes E^{22}\oplus E^{20}\otimes E^{02} \longrightarrow E^{00}\otimes E^{22}\oplus E^{20}\otimes E^{02} \end{array}\right]$$

Notice the difference in the order of $\Psi^1(E^{20})\otimes\Psi^1(E^{02})$.

Hence, strictly speaking, Ψ^k cannot be a chain morphism. However, for every split n-cube, the composition of Ψ^k with φ leads to a collection of n-cubes (see the definition of φ in (4.2.10)). The locally free sheaves of these cubes are direct sums, tensor products, exterior products and symmetric products of the locally free sheaves $E^{\boldsymbol{j}}$.

We can map, with the corresponding canonical isomorphism, every n-cube of $\varphi\circ\Psi^k(E)$ to the n-cube whose summands are all ordered by the lexicographic order. Then, $\varphi\circ\Psi^k$ is a chain morphism.

This trick can only be performed after the composition with φ, and cannot be corrected in the definition of Ψ^k.

Proposition 4.2.36. *Let E be a split n-cube. Then, there is a canonical isomorphism*

$$d_s\Psi^k(E)\cong_{can}\Psi^k(dE).$$

Proof. We have to see that

$$\Psi_2^{m,k}(dE) = -d\Psi_2^{m,k}(E), \quad \text{for } m=1,\dots,k-1, \tag{4.2.37}$$

$$\Psi_1^k(dE) = \sum_{m=1}^{k-1}(-1)^m\Phi^m(\Psi_2^{m,k}(E)) + d\Psi_1^k(E). \tag{4.2.38}$$

We start by proving (4.2.38). By definition,

$$d\Psi_1^k(E) = \sum_{\boldsymbol{i}\in[0,k-1]^n} d\widetilde{C}_{\boldsymbol{i}}(E) = \sum_{\boldsymbol{i}\in[0,k-1]^n}\sum_{l=1}^{n}\sum_{s=0}^{2}(-1)^{l+s}\partial_l^s\widetilde{C}_{\boldsymbol{i}}(E).$$

Then, by lemma 4.2.33,

$$\begin{aligned} d\Psi_1^k(E) \cong_{can} \sum_{l=1}^{n}(-1)^l\Bigg[& \sum_{\boldsymbol{i}\in[0,k-1]^n, i_l=0}\widetilde{C}_{\partial_l(\boldsymbol{i})}(\partial_l^0E) - \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n,\\ i_l=k-1}}\widetilde{C}_{\partial_l(\boldsymbol{i})}(\partial_l^1E) \\ & + \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n\\ i_l=k-1}}\widetilde{C}_{\partial_l(\boldsymbol{i})}(\partial_l^1E) + \sum_{m=0}^{k-2}\sum_{\substack{\boldsymbol{i}\in[0,k-1]^n\\ i_l=m}}\Phi^{i_l+1}(\partial_l^1\widetilde{C}_{\boldsymbol{i}}(E))\Bigg]. \end{aligned}$$

Therefore,

$$\begin{aligned}
d\Psi_1^k(E) \quad &\cong_{can} \quad \sum_{l=1}^{n}\sum_{s=0}^{2}(-1)^{l+s} \sum_{\boldsymbol{i}\in[0,k-1]^{n-1}} \widetilde{C}_{\boldsymbol{i}}(\partial_l^s E) \\
&\quad + \sum_{l=1}^{n}(-1)^l \sum_{m=0}^{k-2} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n,\\ i_l=m}} \Phi^{m+1}(\partial_l^1 \widetilde{C}_{\boldsymbol{i}}(E)) \\
&= \quad \Psi_1^k(dE) - \sum_{m=1}^{k-1}(-1)^m \Phi^m(\Psi_2^{m,k}(E)),
\end{aligned}$$

and equality (4.2.38) is proven. Let us prove now (4.2.37). By definition,

$$\begin{aligned}
d\Psi_2^{m,k}(E) \cong_{can} &\sum_{l=1}^{n}(-1)^{m+l+1} \sum_{\boldsymbol{i}\in[0,k-1]^n,\ i_l=m-1} d\partial_l^1 \widetilde{C}_{\boldsymbol{i}}(E) \\
= &\sum_{l=1}^{n}(-1)^{m+l+1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n\\ i_l=m-1}} \sum_{r=1}^{n-1}\sum_{s=0}^{2}(-1)^{r+s}\partial_r^s\partial_l^1 \widetilde{C}_{\boldsymbol{i}}(E).
\end{aligned}$$

Recall that for all j, $\quad \partial_r^j\partial_l^1 = \begin{cases} \partial_l^1\partial_{r+1}^j, & \text{if } r \geq l, \\ \partial_{l-1}^1\partial_r^j, & \text{if } r < l. \end{cases}$

Hence, fixing $s = 1$,

$$\sum_{l=1}^{n}\sum_{r=1}^{n-1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n\\ i_l=m-1}} (-1)^{l+r}\partial_r^1\partial_l^1 \widetilde{C}_{\boldsymbol{i}}(E) = (1) + (2)$$

where

$$\begin{aligned}
(1) \quad &= \quad \sum_{l=1}^{n-1}\sum_{r=l}^{n-1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n\\ i_l=m-1}} (-1)^{l+r}\partial_l^1\partial_{r+1}^1 \widetilde{C}_{\boldsymbol{i}}(E), \\
(2) \quad &= \quad \sum_{l=2}^{n}\sum_{r=1}^{l-1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n\\ i_l=m-1}} (-1)^{l+r}\partial_{l-1}^1\partial_r^1 \widetilde{C}_{\boldsymbol{i}}(E).
\end{aligned}$$

By lemmas 4.2.24 and 4.2.27, we obtain

$$\begin{aligned}
(2) \quad \cong_{can} \quad & \sum_{l=2}^{n}\sum_{r=1}^{l-1}(-1)^{l+r} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n \\ i_l=m-1, i_r=k-1}} \partial^1_{l-1}\widetilde{C}_{\partial_r(\boldsymbol{i})}(\partial^1_r E) + \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n \\ i_l=m-1, i_r\neq k-1}} \partial^1_{l-1}\partial^1_r\widetilde{C}_{\boldsymbol{i}}(E) \\
= \quad & \sum_{r=1}^{n-1}\sum_{l=r}^{n-1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^{n-1} \\ i_l=m-1}} (-1)^{l+r+1}\partial^1_l\widetilde{C}_{\boldsymbol{i}}(\partial^1_r E) \\
& + \sum_{l=2}^{n}\sum_{r=1}^{l-1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n \\ i_l=m-1, i_r\neq k-1}} (-1)^{l+r}\partial^1_{l-1}\partial^1_r\widetilde{C}_{\boldsymbol{i}}(E) \\
= \quad & (a_2) + (b_2),
\end{aligned}$$

and, we also obtain that

$$\begin{aligned}
(1) \quad \cong_{can} \quad & \sum_{l=1}^{n-1}\sum_{r=l}^{n-1}(-1)^{l+r} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n, i_l=m-1, \\ i_{r+1}=k-1}} \partial^1_l\widetilde{C}_{\partial_{r+1}(\boldsymbol{i})}(\partial^1_{r+1} E) \\
& + \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n, i_l=m-1, \\ i_{r+1}\neq k-1}} \partial^1_l\partial^1_{r+1}\widetilde{C}_{\boldsymbol{i}}(E) \\
= \quad & \sum_{r=2}^{n}\sum_{l=1}^{r-1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^{n-1} \\ i_l=m-1}} (-1)^{l+r+1}\partial^1_l\widetilde{C}_{\boldsymbol{i}}(\partial^1_r E) \\
& + \sum_{l=1}^{n-1}\sum_{r=l}^{n-1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n, i_l=m-1, \\ i_{r+1}\neq k-1}} (-1)^{l+r}\partial^1_l\partial^1_{r+1}\widetilde{C}_{\boldsymbol{i}}(E) \\
= \quad & (a_1) + (b_1).
\end{aligned}$$

Therefore

$$(a_1) + (a_2) = \sum_{r=1}^{n}\sum_{l=1}^{n-1} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^{n-1} \\ i_l=m-1}} (-1)^{l+r+1}\partial^1_l\widetilde{C}_{\boldsymbol{i}}(\partial^1_r E).$$

The term (b_1) can be rewritten as

$$(b_1) = \sum_{l=1}^{n-1}\sum_{r=l+1}^{n} \sum_{\substack{\boldsymbol{i}\in[0,k-1]^n \\ i_l=m-1, i_r\neq k-1}} (-1)^{l+r+1}\partial^1_l\partial^1_r\widetilde{C}_{\boldsymbol{i}}(E),$$

and (b_2) as

$$\begin{aligned}(b_2) &= \sum_{l=2}^{n}\sum_{r=1}^{l-1}\sum_{\substack{\boldsymbol{i}\in[0,k-1]^n\\ i_l=m-1,i_r\neq k-1}}(-1)^{l+r}\partial_r^1\partial_l^1\widetilde{C}_{\boldsymbol{i}}(E)\\ &= \sum_{r=1}^{n-1}\sum_{l=r+1}^{n}\sum_{\substack{\boldsymbol{i}\in[0,k-1]^n\\ i_l=m-1,i_r\neq k-1}}(-1)^{l+r}\partial_r^1\partial_l^1\widetilde{C}_{\boldsymbol{i}}(E).\end{aligned}$$

Therefore, the sum $(b_1)+(b_2)$ is zero, and

$$\sum_{r=1}^{n-1}(-1)^r\partial_r^1\Psi_2^{k,m}(E)\cong_{can}-\sum_{r=1}^{n-1}(-1)^r\Psi_2^{k,m}(\partial_r^1E).$$

We proceed in the same way to prove the equalities for ∂_r^0 and ∂_r^2. For ∂_r^0, we split the sum in $i_r=0$ and $i_r\neq 0$. For ∂_r^2 in the index $k-1$ as in the case ∂_r^1.

□

For every n, let

$$\Psi^k:\mathbb{Z}\operatorname{Sp}_n(X)\to\mathbb{Z}C_n(X)$$

be the composition

$$\Psi^k:\mathbb{Z}\operatorname{Sp}_n(X)\xrightarrow{\Psi^k}\mathbb{Z}SG^k(X)_n\xrightarrow{\mu\circ\varphi}\mathbb{Z}C_*(X),$$

modified by the canonical isomorphisms, so that every direct sum of tensor, exterior and symmetric products is ordered by the lexicographic order of the corresponding multi-indices.

Corollary 4.2.39. *For every scheme X, there is a well-defined chain morphism*

$$\Psi^k:\mathbb{Z}\operatorname{Sp}_*(X)\to\mathbb{Z}C_n(X).$$

□

4.3 The transgression morphism

Fix $\mathcal{C}_B$ to be a category of schemes over a base scheme B. In this section, we introduce all the ingredients for the definition of Adams operations on the rational algebraic K-theory of a regular noetherian scheme. Let X be a scheme. We first define a chain complex $\widetilde{\mathbb{Z}C}_*^{\widetilde{\square}}(X)$ that is the target for the Adams operations. Then, we prove that it is quasi-isomorphic to the chain complex of cubes with rational coefficients. Hence, its rational homology groups are isomorphic to the rational K-groups. Finally, we define a morphism, *the transgression morphism*, from $\mathbb{Z}C_*(X)$ to a new chain complex $\mathbb{Z}\operatorname{Sp}_*^{\square}(X)$, whose image consists only of split cubes. Then, for each k, the morphism Ψ^k defined in the previous section induces a morphism

$$\Psi^k : \mathbb{Z}\,\mathrm{Sp}_*^{\square}(X) \to \widetilde{\mathbb{Z}C}_*^{\widetilde{\square}}(X).$$

Composing with the transgression morphism we obtain a chain complex (denoted, by abuse of notation, by Ψ^k):

$$\Psi^k : NC_*(X) \to \mathbb{Z}C_*(X) \to \widetilde{\mathbb{Z}C}_*^{\widetilde{\square}}(X).$$

4.3.1 The transgression chain complex

Let $\mathbb{P}^1 = \mathbb{P}^1_B$ be the projective line over the base scheme B and let

$$\square = \mathbb{P}^1 \setminus \{1\} \cong \mathbb{A}^1.$$

As stated in section 3.2.1, the cartesian products $(\mathbb{P}^1)^{\cdot}$ and $\square^{\cdot}$ have a cocubical scheme structure.

Let $X \times (\mathbb{P}^1)^n$ and $X \times \square^n$ denote $X \times_B (\mathbb{P}^1)^n$ and $X \times_B \square^n$ respectively. Since most of the constructions will be analogous for $\mathbb{P}^1$ and for $\square$, we write

$$\mathbb{B} = \mathbb{P}^1 \text{ or } \square.$$

For $i = 1, \ldots, n$ and $j = 0, 1$, consider the chain morphisms induced on the complex of cubes

$$\begin{aligned} \delta_i^j = (Id \times \delta_j^i)^* \quad &: \quad \mathbb{Z}C_*(X \times \mathbb{B}^n) \to \mathbb{Z}C_*(X \times \mathbb{B}^{n-1}), \\ \sigma_i = (Id \times \sigma^i)^* \quad &: \quad \mathbb{Z}C_*(X \times \mathbb{B}^{n-1}) \to \mathbb{Z}C_*(X \times \mathbb{B}^n). \end{aligned}$$

These maps endow $\mathbb{Z}C_*(X \times \mathbb{B}^{\cdot})$ with a cubical chain complex structure. Observe that if $(x_1 : y_1), \ldots, (x_n : y_n)$ are homogeneous coordinates of $(\mathbb{P}^1)^n$, then δ_i^0 corresponds to the restriction map to the hyperplane $x_i = 0$ and δ_i^1 corresponds to the restriction map to the hyperplane $y_i = 0$. On the affine lines, with coordinates $(t_1, \ldots, t_n)$ where $t_i = \frac{x_i}{x_i - y_i}$, the map δ_i^0 corresponds to the restriction map to the hyperplane $t_i = 0$ and the map δ_i^1 to the restriction map to the hyperplane $t_i = 1$.

Let $\mathbb{Z}C^{\mathbb{B}}_{*,*}(X)$ be the 2-iterated chain complex given by

$$\mathbb{Z}C^{\mathbb{B}}_{r,n}(X) := \mathbb{Z}C_r(X \times \mathbb{B}^n),$$

and differentials

$$\begin{aligned} d &= d_{C_*(X \times \mathbb{B}^n)}, \\ \delta &= \sum (-1)^{i+j} \delta_i^j. \end{aligned}$$

Denote by $(\mathbb{Z}C^{\mathbb{B}}_*(X), d_s)$ the associated simple complex.

Observe that, by functoriality, the face and degeneracy maps ∂_i^j and s_i^j, as defined in section 1.3.3, commute with δ_i^j and σ_i. Therefore, there are analogous 2-iterated chain complexes

$$\begin{aligned} \widetilde{\mathbb{Z}}C^{\mathbb{B}}_{r,n}(X) &:= \mathbb{Z}C_r(X \times \mathbb{B}^n)/\mathbb{Z}D_r(X \times \mathbb{B}^n), \\ NC^{\mathbb{B}}_{r,n}(X) &:= NC_r(X \times \mathbb{B}^n). \end{aligned}$$

Recall from 4.1.2 that $NC_*(X \times \mathbb{B}^n)$ is the normalized complex of cubes in $X \times \mathbb{B}^n$ and from 1.3.3 that $\mathbb{Z}D_*(X \times \mathbb{B}^n)$ is the complex of degenerate cubes in $X \times \mathbb{B}^n$. That is, we consider the normalized complex of cubes and the quotient by degenerate cubes to the first direction of the 2-iterated complex $\mathbb{Z}C^{\mathbb{B}}_{r,n}(X)$.

In addition, the second direction of these 2-iterated complexes corresponds to the chain complex associated to a cubical abelian group. Therefore, one has to factor out by the degenerate elements.

Let $\mathbb{Z}C_*(X \times \square^n)_{deg} \subset \mathbb{Z}C_*(X \times \square^n)$ be the subcomplex consisting of the degenerate elements, i.e. that lie in the image of σ_i for some $i = 1, \dots, n$. Analogously, we define the complexes

$$NC^{\square}_{r,n}(X)_{\mathrm{deg}} = NC^{\square}_{r,n}(X) \cap \mathbb{Z}C_r(X \times \square^n)_{deg},$$

and

$$\widetilde{\mathbb{Z}}C^{\square}_{r,n}(X)_{\mathrm{deg}} = \mathbb{Z}C_r(X \times \square^n)_{deg}/\mathbb{Z}D_r(X \times \square^n)_{deg}$$

of degenerate elements in $NC^{\square}_{r,n}(X)$ and $\widetilde{\mathbb{Z}}C^{\square}_{r,n}(X)$ respectively.

We define the 2-iterated chain complexes,

$$\begin{aligned} \widetilde{\mathbb{Z}}C^{\widetilde{\square}}_{r,n}(X) &:= \widetilde{\mathbb{Z}}C^{\square}_{r,n}(X)/\widetilde{\mathbb{Z}}C^{\square}_{r,n}(X)_{\mathrm{deg}}, \\ NC^{\widetilde{\square}}_{r,n}(X) &:= NC^{\square}_{r,n}(X)/NC^{\square}_{r,n}(X)_{deg}. \end{aligned}$$

Denote by $(\widetilde{\mathbb{Z}}C^{\widetilde{\square}}_*(X), d_s)$ and $(NC^{\widetilde{\square}}_*(X), d_s)$ the simple complexes associated to these 2-iterated chain complexes.

Proposition 4.3.1. *If X is a regular noetherian scheme, the natural morphism of complexes*

$$NC_*(X) = NC^{\widetilde{\square}}_{*,0}(X) \to NC^{\widetilde{\square}}_*(X)$$

induces an isomorphism in homology groups with coefficients in $\mathbb{Q}$.

Proof. Consider the first quadrant spectral sequence with E^0 term given by

$$E^0_{r,n} = NC^{\widetilde{\square}}_{r,n}(X) \otimes \mathbb{Q}.$$

When it converges, it converges to the homology groups $H_*(NC^{\widetilde{\square}}_*(X), \mathbb{Q})$. If we see that for all $n > 0$ the rational homology of the complex $NC^{\widetilde{\square}}_{*,n}(X)$ is zero, the spectral sequence converges and the proposition is proven.

This is proved by an induction argument. For every $j = 1, \ldots, n$, let

$$NC^{\square,j}_{r,n}(X)_{deg} = \sum_{i=1}^{j} \sigma_i(NC^{\square}_{*,n-1}(X)) \subseteq NC^{\square}_{r,n}(X)_{deg}$$

and let $NC^{\widetilde{\square},j}_{*,n}(X)$ be the respective quotient. We will show that, for all $n > 0$ and $j = 1, \ldots, n$,

$$H_*(NC^{\widetilde{\square},j}_{*,n}(X), \mathbb{Q}) = 0.$$

For $j = 1$ and $n > 0$,

$$NC^{\widetilde{\square},1}_{*,n}(X) = NC^{\square}_{*,n}(X)/\sigma_1(NC^{\square}_{*,n-1}(X)).$$

By the homotopy invariance of algebraic K-theory of regular noetherian schemes (see section 1.3.4), the rational homology of this complex is zero. Then, if $j > 1$ and $n > 1$, $NC^{\widetilde{\square},j}_{*,n}(X)$ is the cokernel of the monomorphism

$$\begin{array}{ccc} NC^{\widetilde{\square},j-1}_{*,n-1}(X) & \xrightarrow{\sigma_j} & NC^{\widetilde{\square},j-1}_{*,n}(X) \\ E & \mapsto & \sigma_j(E). \end{array}$$

Since by the induction hypothesis both sides have zero rational homology, so does the cokernel. $\square$

Observe that in the proof of last proposition, the key point was that for regular noetherian schemes, the K-groups of $X \times \square^n$ are isomorphic to the K-groups of X. In the case of projective lines, the situation is slightly trickier, because the K-groups of $X \times \mathbb{P}^1$ are not isomorphic to the K-groups of X. We have to use the Dold-Thom isomorphism relating both groups. This implies that we shall also factor out by the class of the canonical bundle on $(\mathbb{P}^1)^n$.

Let $p_1, \ldots, p_n$ be the projections onto the i-th coordinate of $(\mathbb{P}^1)^n$. Consider the invertible sheaf $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^1}(1)$, the dual of the tautological sheaf of $\mathbb{P}^1$, $\mathcal{O}_{\mathbb{P}^1}(-1)$. We define then the 2-iterated chain complexes

$$\begin{aligned} NC^{\mathbb{P}}_{r,n}(X)_{deg} &:= \sum_{i=1}^{n} \sigma_i(NC^{\mathbb{P}}_{r,n-1}(X)) + p_i^*\mathcal{O}(1) \otimes \sigma_i(NC^{\mathbb{P}}_{r,n-1}(X)), \\ NC^{\widetilde{\mathbb{P}}}_{r,n}(X) &:= NC^{\mathbb{P}}_{r,n}(X)/NC^{\mathbb{P}}_{r,n}(X)_{deg}. \end{aligned}$$

Denote by $(NC^{\widetilde{\mathbb{P}}}_{*}(X), d_s)$ the simple complex associated to this 2-iterated chain complex.

Proposition 4.3.2. *If X is a regular noetherian scheme, the natural morphism of complexes*

$$NC_*(X) = NC^{\widetilde{\mathbb{P}}}_{*,0}(X) \to NC^{\widetilde{\mathbb{P}}}_{*}(X)$$

induces an isomorphism on homology with coefficients in $\mathbb{Q}$.

Proof. The proof is analogous to the proof of the last proposition. By considering the spectral sequence associated with the homology of a 2-iterated complex, we just have to see that for all j,

$$H_*(NC^{\widetilde{\mathbb{P}},j}_{*,n}(X),\mathbb{Q}) = 0.$$

For $j = 1$ and $n > 0$, it follows from the Dold-Thom isomorphism on algebraic K-theory of regular noetherian schemes. For $j > 1$ and $n > 1$, $NC^{\widetilde{\mathbb{P}},j}_{*,n}(X)$ is the cokernel of the monomorphism

$$\begin{array}{rcl} NC^{\widetilde{\mathbb{P}},j-1}_{*,n-1}(X) \oplus N\mathbb{C}^{\widetilde{\mathbb{P}},j-1}_{*,n-1}(X) & \to & NC^{\widetilde{\mathbb{P}},j-1}_{*,n}(X) \\ (E_0,E_1) & \mapsto & \sigma_j(E_0) + p_j^*\mathcal{O}(1)\otimes\sigma_j(E_1). \end{array}$$

Since by the induction hypothesis both sides have zero rational homology, so does the cokernel. □

Remark 4.3.3. Let

$$\begin{array}{rcl} \mathbb{Z}C^{\mathbb{P}}_{r,n}(X)_{deg} & = & \displaystyle\sum_{i=1}^{n} \sigma_i(\mathbb{Z}C^{\mathbb{P}}_{r,n-1}(X)) + p_i^*\mathcal{O}(1)\otimes\sigma_i(\mathbb{Z}C^{\mathbb{P}}_{r,n-1}(X)), \\ \widetilde{\mathbb{Z}}C^{\mathbb{P}}_{r,n}(X)_{\mathrm{deg}} & = & \mathbb{Z}C^{\mathbb{P}}_{r,n}(X)_{deg}/\mathbb{Z}D_r(X\times(\mathbb{P}^1)^n)_{deg}, \end{array}$$

and let

$$\widetilde{\mathbb{Z}}C^{\widetilde{\mathbb{P}}}_{r,n}(X) := \widetilde{\mathbb{Z}}C^{\mathbb{P}}_{r,n}(X)/\widetilde{\mathbb{Z}}C^{\mathbb{P}}_{r,n}(X)_{\mathrm{deg}}.$$

Denote by $(\widetilde{\mathbb{Z}}C^{\widetilde{\mathbb{P}}}_*(X), d_s)$ the simple complex associated to this 2-iterated chain complex.

It follows from the definitions that there is an isomorphism

$$\widetilde{\mathbb{Z}}C^{\widetilde{\mathbb{P}}}_*(X) \cong NC^{\widetilde{\mathbb{P}}}_*(X).$$

4.3.2 The transgression of cubes by affine and projective lines

Let x and y be the global sections of $\mathcal{O}(1)$ given by the projective coordinates $(x:y)$ on $\mathbb{P}^1$. Let X be a scheme and let p_0 and p_1 be the projections from $X\times\mathbb{P}^1$ to X and $\mathbb{P}^1$ respectively. Then, for every locally free sheaf E on X, we denote

$$E(k) := p_0^*E\otimes p_1^*\mathcal{O}(k).$$

The following definition is taken from [15].

Definition 4.3.4. Let

$$E: 0\to E^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} E^2 \to 0$$

be a short exact sequence. The *first transgression by projective lines* of E, $\mathrm{tr}_1(E)$, is the kernel of the morphism

$$\begin{array}{rcl} E^1(1)\oplus E^2(1) & \to & E^2(2) \\ (a,b) & \mapsto & f^1(a)\otimes x - b\otimes y. \end{array}$$

Observe that this locally free sheaf on $X \times \mathbb{P}^1$ satisfies that

$$\begin{aligned}\delta_1^0 \operatorname{tr}_1(E) &= \operatorname{tr}_1(E)|_{x=0} = E^1, \\ \delta_1^1 \operatorname{tr}_1(E) &= \operatorname{tr}_1(E)|_{y=0} = \operatorname{im} f^0 \oplus E^2.\end{aligned}$$

By restriction to $\square$, we obtain the *transgression by affine lines.*

From now on, we restrict ourselves to the affine case. However, all the results can be written in terms of the complexes with projective lines.

Let E be an n-cube. We define the *first transgression* of E as the $(n-1)$-cube on $X \times \square^1$ given by

$$\operatorname{tr}_1(E)^{\boldsymbol{j}} := \operatorname{tr}_1(\partial^{\boldsymbol{j}}_{2,\dots,n} E), \qquad \text{for all } \boldsymbol{j} \in \{0,1,2\}^{n-1},$$

i.e. we take the transgression of the exact sequences in the first direction. Since tr_1 is a functorial exact construction, the *m-th transgression sheaf* can be defined recursively as

$$\operatorname{tr}_m(E) = \operatorname{tr}_1 \operatorname{tr}_{m-1}(E) = \operatorname{tr}_1 \overset{m}{\cdots} \operatorname{tr}_1(E).$$

It is a $(n-m)$-cube on $X \times \square^m$. In particular, $\operatorname{tr}_n(E)$ is a locally free sheaf on $X \times \square^n$.

Observe that the transgression is functorial, i.e., if $E \xrightarrow{\psi} F$ is a morphism of n-cubes, then there is an induced morphism

$$\operatorname{tr}_m(E) \xrightarrow{\operatorname{tr}_m(\psi)} \operatorname{tr}_m(F),$$

for every $m = 1, \dots, n$. In particular, for every n-cube E and $i = 1, \dots, n$, the morphism of $(n-1)$-cubes

$$\partial_i^0 E \xrightarrow{f_i^0} \partial_i^1 E,$$

induces a morphism

$$\operatorname{tr}_m(\partial_i^0 E) \xrightarrow{\operatorname{tr}_m(f_i^0)} \operatorname{tr}_m(\partial_i^1 E),$$

for $m = 1, \dots, n-1$.

Lemma 4.3.5. *For every n-cube E and $i = 1, \dots, n$, the following identities hold:*

$$\delta_i^0 \operatorname{tr}_n(E) = \operatorname{tr}_{n-1}(\partial_i^1 E), \tag{4.3.6}$$
$$\delta_i^1 \operatorname{tr}_n(E) \cong \operatorname{im} \operatorname{tr}_{n-1}(f_i^0) \oplus \operatorname{tr}_{n-1}(\partial_i^2 E), \tag{4.3.7}$$

with the isomorphism being canonical, i.e. a combination of commutativity and associativity isomorphisms for direct sums, distributivity isomorphism for the tensor product of a direct sum and commutativity isomorphisms of the pull-back of a direct sum with the direct sum of the pull-back.

Proof. The proof is straightforward. For $n = 1$ it follows from the definition. Therefore,

$$\begin{aligned}\delta_i^0 \operatorname{tr}_n(E) &= \delta_i^0 \operatorname{tr}_1 \overset{n}{\cdots} \operatorname{tr}_1(E) \\ &= \operatorname{tr}_{n-i} \partial_i^1 \operatorname{tr}_{i-1}(E) = \operatorname{tr}_{n-1}(\partial_i^1 E).\end{aligned}$$

For the second statement, observe first of all that there is a canonical isomorphism $\mathrm{tr}_n(A\oplus B)\cong \mathrm{tr}_n(A)\oplus \mathrm{tr}_n(B)$. It follows by recurrence from the case $n=1$. So, let E,F be two short exact sequences. Then, $\mathrm{tr}_1(E\oplus F)$ is the kernel of the map

$$(E_1\oplus F_1)(1)\oplus(E_2\oplus F_2)(1)\to(E_2\oplus F_2)(2),$$

while $\mathrm{tr}_1(E)\oplus\mathrm{tr}_1(F)$ is the direct sum of the kernels of the maps

$$E_1(1)\oplus E_2(1)\to E_2(2),\quad F_1(1)\oplus F_2(1)\to F_2(2).$$

Hence, there is clearly a canonical isomorphism. Therefore

$$\begin{aligned}\delta_i^1\,\mathrm{tr}_n(E) &= \delta_i^1\,\mathrm{tr}_1 \overset{n}{\cdots} \mathrm{tr}_1(E)\\ &= \mathrm{tr}_{n-i}(\operatorname{im}\mathrm{tr}_{i-1}(f_i^0)\oplus\mathrm{tr}_{i-1}(\partial_i^2 E))\\ &\cong \operatorname{im}\mathrm{tr}_{n-1}(f_i^0)\oplus\mathrm{tr}_{n-1}(\partial_i^2 E).\end{aligned}$$

□

Remark 4.3.8. Since the isomorphisms in (4.3.7) are canonical, if $E\xrightarrow{\psi}F$ is a morphism of n-cubes, for every i we obtain a commutative diagram

$$\begin{tikzcd}
\delta_i^1\,\mathrm{tr}_n(E) \arrow[rr, "\delta_i^1\,\mathrm{tr}_n(\psi)"] \arrow[d, "\cong"'] & & \delta_i^1\,\mathrm{tr}_n(F) \arrow[d, "\cong"] \\
\operatorname{im}\mathrm{tr}_{n-1}(f_i^0)\oplus\mathrm{tr}_{n-1}(\partial_i^2 E) \arrow[rr] & & \operatorname{im}\mathrm{tr}_{n-1}(f_i^0)\oplus\mathrm{tr}_{n-1}(\partial_i^2 F).
\end{tikzcd}$$

Observe that for every n-cube E, the $(n-m)$-cube $\mathrm{tr}_m(E)$ is obtained applying tr_1 on the directions $1,\dots,m$. In the next definition, we generalize this construction by specifying the directions in which we apply the first transgression.

Definition 4.3.9. Let $\boldsymbol{i}=(i_1,\dots,i_{n-m})$, with $1\le i_1<\dots<i_{n-m}\le n$. We define, $\mathrm{tr}_m^{\boldsymbol{i}}(E)\in C_{n-m,m}^{\square}(X)$, by

$$\mathrm{tr}_m^{\boldsymbol{i}}(E)^{\boldsymbol{j}}=\mathrm{tr}_m(\partial_{\boldsymbol{i}}^{\boldsymbol{j}}E),\qquad \text{for all } \boldsymbol{j}\in\{0,1,2\}^{n-m}.$$

Observe that this means that we consider the first transgression iteratively in the directions not in the multi-index $\boldsymbol{i}$, from the highest index to the lowest.

By convention, the transgressions without super-index correspond to the multi-index $\boldsymbol{i}=(m+1,\dots,n)$. The following lemma is a direct consequence of lemma 4.3.5.

Lemma 4.3.10. *Let $\boldsymbol{i}=(i_1,\dots,i_{n-m})$ with $1\le i_1<\dots<i_{n-m}\le n$. Then, for every fixed $r=1,\dots,m$, let $\boldsymbol{i}'=\boldsymbol{i}-\mathbf{1}_{r+1}^{n-m}$ and let $(v_1,\dots,v_m)$ be the ordered multi-index with the entries in $\{1,\dots,n\}\setminus\{i_1,\dots,i_{n-m}\}$. Then,*

$$\begin{aligned}\delta_r^0\,\mathrm{tr}_m^{\boldsymbol{i}}(E) &= \mathrm{tr}_{m-1}^{\boldsymbol{i}'}(\partial_{v_r}^1 E),\\ \delta_r^1\,\mathrm{tr}_m^{\boldsymbol{i}}(E) &\cong \operatorname{im}\mathrm{tr}_{m-1}^{\boldsymbol{i}'}(f_{v_r}^0)\oplus\mathrm{tr}_{m-1}^{\boldsymbol{i}'}(\partial_{v_r}^2 E)\qquad\textit{(canonically)}.\end{aligned}$$

□

4.3.3 Cubes with canonical kernels

We introduce here a new subcomplex of $\mathbb{Z}C_*(X)$, consisting of the *cubes with canonical kernels*. In this new class of cubes, the transgressions behave almost like a chain morphism. Namely, if E is an n-cube with canonical kernels, we will have

$$\begin{array}{rcl}\delta_i^0 \operatorname{tr}_n(E) & = & \operatorname{tr}_{n-1}(\partial_i^1 E), \\ \delta_i^1 \operatorname{tr}_n(E) & \cong & \operatorname{tr}_{n-1}(\partial_i^0 E) \oplus \operatorname{tr}_{n-1}(\partial_i^2 E).\end{array} \tag{4.3.11}$$

Definition 4.3.12. Let E be an n-cube. We say that E has *canonical kernels* if for every $i = 1, \ldots, n$ and $\boldsymbol{j} \in \{0,1,2\}^{n-1}$, there is an inclusion $(\partial_i^0 E)^{\boldsymbol{j}} \subset (\partial_i^1 E)^{\boldsymbol{j}}$ of sets and moreover the morphism

$$f_i^0 : \partial_i^0 E \to \partial_i^1 E$$

is the canonical inclusion of cubes.

Let $KC_n(X) \subseteq C_n(X)$ be the subset of all cubes with canonical kernels. The differential of $\mathbb{Z}C_*(X)$ induces a differential on $\mathbb{Z}KC_*(X)$ making the inclusion arrow a chain morphism.

Let E be a 1-cube i.e. a short exact sequence $E^0 \xrightarrow{f^0} E^1 \xrightarrow{f^1} E^2$. Then, we define

$$\begin{array}{rcl}\lambda_1^0(E) & : & 0 \to 0 \to E^0 \xrightarrow{f^0} \operatorname{im} f^0 \to 0, \\ \lambda_1^1(E) & : & 0 \to \ker f^1 \to E^1 \xrightarrow{f^1} E^2 \to 0.\end{array}$$

Both of them are 1-cubes with canonical kernels. Then, we define

$$\lambda_1(E) = \lambda_1^1(E) - \lambda_1^0(E) \in \mathbb{Z}KC_1(X).$$

For an arbitrary n-cube $E \in C_n(X)$ and for every $i = 1, \ldots, n$, let $\lambda_i^0(E)$ and $\lambda_i^1(E)$ be the n-cubes which along the i-th direction are:

$$\begin{array}{rclcrcl}\partial_i^0 \lambda_i^0(E) & = & 0 & \qquad & \partial_i^0 \lambda_i^1(E) & = & \operatorname{im} f_i^0 \\ \partial_i^1 \lambda_i^0(E) & = & \partial_i^0 E & & \partial_i^1 \lambda_i^1(E) & = & \partial_i^1 E \\ \partial_i^2 \lambda_i^0(E) & = & \operatorname{im} f_i^0 & & \partial_i^2 \lambda_i^1(E) & = & \partial_i^2 E,\end{array}$$

Then, we define

$$\begin{array}{rcl}\lambda_i(E) & = & -\lambda_i^0(E) + \lambda_i^1(E), \quad i = 1, \ldots, n, \\ \lambda(E) & = & \begin{cases} \lambda_n \cdots \lambda_1(E), & \text{if } n > 0, \\ E & \text{if } n = 0.\end{cases}\end{array}$$

Proposition 4.3.13. *The map*

$$\lambda : \mathbb{Z}C_n(X) \to \mathbb{Z}KC_n(X)$$

is a morphism of complexes.

Proof. First of all, observe that the image by λ of any n-cube E, is a sum of n-cubes with canonical kernels. It is a consequence of the fact that for any commutative square of epimorphisms,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "j"'] & D, \end{tikzcd}$$

the set equality $\ker(\ker g \to \ker h) = \ker(\ker f \to \ker j)$ holds. The equality $d\lambda(E) = \lambda d(E)$ follows from the equalities

$$\begin{array}{rcl} \partial_i^j \lambda(E) & = & \partial_i^j(\lambda_n \cdots \lambda_i^1 \cdots \lambda_1(E)) - \partial_i^j(\lambda_n \cdots \lambda_i^0 \cdots \lambda_1(E)), \\ \partial_i^0(\lambda_n \cdots \lambda_i^1 \cdots \lambda_1(E)) & = & \partial_i^2(\lambda_n \cdots \lambda_i^0 \cdots \lambda_1(E)), \\ \partial_i^j(\lambda_n \cdots \lambda_i^1 \cdots \lambda_1(E)) & = & \lambda_{n-1} \cdots \lambda_1(\partial_i^j E), \quad j = 1, 2, \\ \partial_i^1(\lambda_n \cdots \lambda_i^0 \cdots \lambda_1(E)) & = & \lambda_{n-1} \cdots \lambda_1(\partial_i^0 E). \end{array}$$

$\square$

4.3.4 The transgression morphism

Observe that the face maps δ_i^j of $\square^{\cdot}$ induce morphisms on the complex of split cubes

$$\delta_i^j : \mathbb{Z}\,\mathrm{Sp}_r(X \times \square^n) \to \mathbb{Z}\,\mathrm{Sp}_r(X \times \square^{n-1}).$$

Let $\mathbb{Z}\,\mathrm{Sp}_{*,*}^{\square}(X)$ be the 2-iterated chain complex given by

$$\mathbb{Z}\,\mathrm{Sp}_{r,n}^{\square}(X) = \mathbb{Z}\,\mathrm{Sp}_r(X \times \square^n),$$

and differentials

$$\begin{array}{rcl} d & = & d_{\mathrm{Sp}_*(X \times \square^n)}, \\ \delta & = & \sum (-1)^{i+j} \delta_i^j. \end{array}$$

Let $(\mathbb{Z}\,\mathrm{Sp}_*^{\square}(X), d_s)$ be the associated simple complex. Using the transgressions, we define here a morphism of complexes

$$\mathbb{Z}KC_*(X) \xrightarrow{T} \mathbb{Z}\,\mathrm{Sp}_*^{\square}(X)$$

which composed with λ gives the *transgression morphism*

$$\mathbb{Z}C_*(X) \xrightarrow{T} \mathbb{Z}\,\mathrm{Sp}_*^{\square}(X).$$

For every n-cube E with canonical kernels, the component of $T(E)$ in $\mathbb{Z}\,\mathrm{Sp}_{0,n}^{\square}(X)$ is exactly $(-1)^n \operatorname{tr}_n(E)$. However, the assignment

$$E \mapsto (-1)^n \operatorname{tr}_n(E)$$

is not a chain morphism. The failure comes from equality (4.3.11), since, first of all, the equality holds only up to some canonical isomorphisms, and second, a direct sum is not a sum in the complex of cubes. We will add some “correction cubes” in $\mathbb{Z}\,\mathrm{Sp}^{\square}_{n-m,m}(X)$, with $m \neq n$, in order to obtain a chain morphism T.

We start by constructing the morphism step by step in the low degree cases, deducing from the examples the key ideas.

The transgression morphism for $n = 1, 2$. Let E be a 1-cube with canonical kernels. Then,

$$\delta\,\mathrm{tr}_1(E) = -\delta_1^0\,\mathrm{tr}_1(E) + \delta_1^1\,\mathrm{tr}_1(E) = -E^1 + \delta_1^1\,\mathrm{tr}_1(E).$$

We know that there is a canonical isomorphism (which in this case is the identity):

$$\delta_1^1\,\mathrm{tr}_1(E) \cong E^0 \oplus E^2.$$

Hence, the differential of

$$T(E) := (-\,\mathrm{tr}_1(E), E^0 \to \delta_1^1\,\mathrm{tr}_1(E) \to E^2) \in \mathbb{Z}C^{\square}_{0,1}(X) \oplus \mathbb{Z}C^{\square}_{1,0}(X)$$

is exactly $E^1 - E^0 - E^2 \in \mathbb{Z}C_0(X)$.

Let E be a 2-cube with canonical kernels. Then,

$$\begin{aligned} \delta\,\mathrm{tr}_2(E) &= -\delta_1^0\,\mathrm{tr}_2(E) + \delta_1^1\,\mathrm{tr}_2(E) + \delta_2^0\,\mathrm{tr}_2(E) - \delta_2^1\,\mathrm{tr}_2(E) \\ &= -\,\mathrm{tr}_1(\partial_1^1 E) + \delta_1^1\,\mathrm{tr}_2(E) + \mathrm{tr}_1(\partial_2^1 E) - \delta_2^1\,\mathrm{tr}_2(E). \end{aligned}$$

Let

$$T_1^i(E) = \mathrm{tr}_1(\partial_i^0 E) \to \delta_i^1\,\mathrm{tr}_2(E) \to \mathrm{tr}_1(\partial_i^2 E),$$

where the arrows are defined by the canonical isomorphism $\delta_i^1\,\mathrm{tr}_2(E) \cong \mathrm{tr}_1(\partial_i^0 E) \oplus \mathrm{tr}_1(\partial_i^2 E)$. Let

$$T_2(E) = \begin{tikzcd} E^{00} \arrow[r] \arrow[d] & \delta_1^1\,\mathrm{tr}_1(E^{*0}) \arrow[r] \arrow[d] & E^{20} \arrow[d] \\ \delta_1^1\,\mathrm{tr}_1(E^{0*}) \arrow[r] \arrow[d] & \delta_1^1\delta_2^1\,\mathrm{tr}_2(E) \arrow[r] \arrow[d] & \delta_1^1\,\mathrm{tr}_1(E^{2*}) \arrow[d] \\ E^{02} \arrow[r] & \delta_1^1\,\mathrm{tr}_1(E^{*2}) \arrow[r] & E^{22}, \end{tikzcd}$$

with the arrows induced by the canonical isomorphisms of lemma 4.3.5. Then, for $n = 2$, we define

$$T(E) := (\mathrm{tr}_2(E), \sum_{i=1,2} (-1)^i T_1^i(E), T_2(E)) \in \mathbb{Z}C^{\square}_{0,2}(X) \oplus \mathbb{Z}C^{\square}_{1,1}(X) \oplus \mathbb{Z}C^{\square}_{2,0}(X).$$

By lemma 4.3.10, since E is a cube with canonical kernels, these cubes are all split. We fix the splittings to be the canonical isomorphisms of lemma 4.3.10. Then, in $\mathbb{Z}C^{\square}_{0,1}(X)\oplus \mathbb{Z}C^{\square}_{1,0}(X)$,

$$\begin{aligned} d_sT(E) &= (\delta \operatorname{tr}_2(E) + \sum_{i=1,2}(-1)^i dT_1^i(E), -\sum_{i=1,2}(-1)^i\delta T_1^i(E) + dT_2(E)) \\ &= \sum_{i=1,2}\sum_{j=0,1}(-1)^{i+j}T(\partial_i^j E). \end{aligned}$$

The transgression morphism. Recall that if $\boldsymbol{j}\in\{0,1,2\}^m$, we defined in section 4.2.1

$$s(\boldsymbol{j}) = \#\{r|\ j_r = 1\},$$

and the multi-index $u(\boldsymbol{j}) = (u_1,\dots,u_{s(\boldsymbol{j})})$ with u_i the indices such that $j_{u_i} = 1$ and ordered by $u_1 < \cdots < u_{s(\boldsymbol{j})}$. Consider the set of multi-indices

$$J_n^m := \{\boldsymbol{i} = (i_1,\dots,i_{n-m})|\ 1\le i_1 < \cdots < i_{n-m}\le n\}.$$

Then, for every $\boldsymbol{i}\in J_n^m$ and $\boldsymbol{j}\in\{0,1,2\}^{n-m}$, we define

$$\boldsymbol{i}(\boldsymbol{j}) = (i_{u_1}, i_{u_2}-1,\dots,i_{u_l}-l+1,\dots,i_{u_{s(\boldsymbol{j})}}-s(\boldsymbol{j})+1).$$

Definition 4.3.14. Let E be an n-cube with canonical kernels. For every $0\le m\le n$ and $\boldsymbol{i}\in J_n^m$, we define $T^{\boldsymbol{i}}_{n-m,m}(E)\in\mathbb{Z}C^{\square}_{n-m,m}(X)$ as the $(n-m)$-cube on $X\times\square^m$ given by:

- If $\boldsymbol{j}\in\{0,2\}^{n-m}$ then

$$T^{\boldsymbol{i}}_{n-m,m}(E)^{\boldsymbol{j}} := \operatorname{tr}^{\boldsymbol{i}}_m(E)^{\boldsymbol{j}} = \operatorname{tr}_m(\partial^{\boldsymbol{j}}_{\boldsymbol{i}}E).$$

- If $\boldsymbol{j}\in\{0,1,2\}^{n-m}$ with $j_k = 1$ for some k, then we define

$$T^{\boldsymbol{i}}_{n-m,m}(E)^{\boldsymbol{j}} := \big(\delta^{\boldsymbol{1}}_{\boldsymbol{i}(\boldsymbol{j})}\operatorname{tr}^{\partial_{u(\boldsymbol{j})}(\boldsymbol{i})}_{m+s(\boldsymbol{j})}(E)\big)^{\partial_{u(\boldsymbol{j})}(\boldsymbol{j})}.$$

That is, we start by considering the first transgression of E, iteratively, in the directions s not in the multi-index $\boldsymbol{i}$ and in those i_s with $j_s = 1$. This gives a $(n-m-s(\boldsymbol{j}))$-cube on $X\times\square^{m+s(\boldsymbol{j})}$. Then, we apply δ^1_s for each affine component coming from a direction with $j_s = 1$. We obtain a $(n-m)$-cube on $X\times\square^m$.

Observe that in the above definition, the second case generalizes the first case.

Lemma 4.3.15. *For every n-cube E with canonical kernels and every $\boldsymbol{i}\in J_n^m$, the cube $T^{\boldsymbol{i}}_{n-m,m}(E)$ is split.*

Proof. If $\boldsymbol{j} \in \{0,1,2\}^{n-m}$, it follows by lemma 4.3.5 that,

$$T^{\boldsymbol{i}}_{n-m,m}(E)^{\boldsymbol{j}} = \left(\delta^{\boldsymbol{1}}_{\boldsymbol{i}(\boldsymbol{j})} \operatorname{tr}^{\partial_{u(j)}(\boldsymbol{i})}_{m+s(\boldsymbol{j})}(E)\right)^{\partial_{u(j)}(\boldsymbol{j})} \cong \bigoplus_{\boldsymbol{m}\in\{0,2\}^{s(\boldsymbol{j})}} \operatorname{tr}^{\boldsymbol{i}}_{m}(E)^{\sigma^{\boldsymbol{m}}_{u(j)}(\boldsymbol{j})}.$$

$\square$

Observe that for any morphism of n-cubes $E \to F$, there is a commutative square

$$\begin{tikzcd} T^{\boldsymbol{i}}_{n-m,m}(E)^{\boldsymbol{j}} \arrow[r] \arrow[d, "\cong"'] & T^{\boldsymbol{i}}_{n-m,m}(F)^{\boldsymbol{j}} \arrow[d, "\cong"] \\ \bigoplus_{\boldsymbol{m}\in\{0,2\}^{s(\boldsymbol{j})}} \operatorname{tr}^{\boldsymbol{i}}_{m}(E)^{\sigma^{\boldsymbol{m}}_{u(j)}(\boldsymbol{j})} \arrow[r] & \bigoplus_{\boldsymbol{m}\in\{0,2\}^{s(\boldsymbol{j})}} \operatorname{tr}^{\boldsymbol{i}}_{m}(F)^{\sigma^{\boldsymbol{m}}_{u(j)}(\boldsymbol{j})}. \end{tikzcd}$$

Finally, we consider

$$T_{n-m,m}(E) := \sum_{\boldsymbol{i}\in J^{n-m}_{n}} (-1)^{|\boldsymbol{i}|+\sigma(n-m)+m} T^{\boldsymbol{i}}_{n-m,m}(E) \in \mathbb{Z}\operatorname{Sp}^{\square}_{n-m,m}(X), \tag{4.3.16}$$

where $\sigma(k) = 1 + \cdots + k = \frac{k(k+1)}{2}$.

Proposition 4.3.17. *The map*

$$\begin{array}{rcl} \mathbb{Z}KC_n(X) & \xrightarrow{T} & \bigoplus_{m=0}^{n} \mathbb{Z}\operatorname{Sp}^{\square}_{n-m,m}(X) \\ E & \mapsto & T_{n-m,m}(E) \end{array}$$

is a chain morphism.

Proof. We have to see that T commutes with the differentials. Remember that the differential d_s in the simple complex is defined, in the $(n-m,m)$-component, by $d + (-1)^{n-m}\delta$. Therefore, we have to see that for every $m = 0,\ldots,n-1$, the equality

$$dT_{n-m,m} + (-1)^{n-m-1}\delta T_{n-m-1,m+1} = T_{n-m-1,m}d \tag{4.3.18}$$

holds. The right hand side is

$$\begin{aligned} T_{n-m-1,m}d &= \sum_{r=1}^{n}\sum_{j=0}^{2}(-1)^{r+j} T_{n-m-1,m}\partial^j_r \\ &= \sum_{r=1}^{n}\sum_{j=0}^{2}(-1)^{r+j} \sum_{\boldsymbol{i}\in J^{n-m-1}_{n-1}} (-1)^{|\boldsymbol{i}|+\sigma(n-m-1)+m} T^{\boldsymbol{i}}_{n-m-1,m}\partial^j_r. \end{aligned}$$

We compute the terms d and δ on the left hand side separately.

$$dT_{n-m,m} = \sum_{r=1}^{n-m} \sum_{\boldsymbol{i}\in J_n^{n-m}} (-1)^{|\boldsymbol{i}|+\sigma(n-m)+r+m}(\partial_r^0 - \partial_r^1 + \partial_r^2) T^{\boldsymbol{i}}_{n-m,m}.$$

By definition, $\partial_r^l T^{\boldsymbol{i}}_{n-m,m} = T^{\partial_r(\boldsymbol{i})-\boldsymbol{1}^{n-m}_{r+1}}_{n-m-1,m}\partial^l_{i_r}$ if $l = 0, 2$.

Since $(-1)^{\sigma(n-m)+n-m} = (-1)^{\sigma(n-m-1)}$, we obtain

$$\begin{aligned} dT_{n-m,m} &= \sum_{r=1}^{n}(-1)^r[T_{n-m-1,m}\partial^0_{i_r} + T_{n-m-1,m}\partial^2_{i_r}] - \\ &\quad - \sum_{\boldsymbol{i}\in J_n^{n-m}} \sum_{r=1}^{n}(-1)^{|\boldsymbol{i}|+r+\sigma(n-m)+m}\partial_r^1 T^{\boldsymbol{i}}_{n-m,m}. \end{aligned}$$

For the summand corresponding to the differential δ, one has that

$$(-1)^m \delta T_{n-m-1,m+1} = (1) + (2),$$

with

$$\begin{aligned} (1) &= -\sum_{r=1}^{m} \sum_{\boldsymbol{i}\in J_n^{n-m-1}} (-1)^{|\boldsymbol{i}|+\sigma(n-m-1)+r}\delta_r^0 T^{\boldsymbol{i}}_{n-m-1,m+1}, \\ (2) &= \sum_{r=1}^{m} \sum_{\boldsymbol{i}\in J_n^{n-m-1}} (-1)^{|\boldsymbol{i}|+\sigma(n-m-1)+r}\delta_r^1 T^{\boldsymbol{i}}_{n-m-1,m+1}. \end{aligned}$$

For every $\boldsymbol{j} \in \{0,1,2\}^{n-m-1}$, we obtain by definition

$$(\delta^0_r T^{\boldsymbol{i}}_{n-m,m})^{\boldsymbol{j}} = \left(\delta_r^0 \delta^{\boldsymbol{1}}_{\boldsymbol{i}(\boldsymbol{j})}\, \mathrm{tr}^{\partial_{u(\boldsymbol{j})}(\boldsymbol{i})}_{m+s(\boldsymbol{j})}\right)^{\partial_{u(\boldsymbol{j})}(\boldsymbol{j})}.$$

Using the commutation rules of the faces δ^*_* and ∂^*_*, we obtain that

$$\delta^0_r T^{\boldsymbol{i}}_{n-m,m} = T^{\boldsymbol{i}-\boldsymbol{1}^{n-m-1}_{l+1}}_{n-m-1,m}\partial^1_{r+l},$$

where l is the maximal between the indices k, such that $r+k-1 \geq i_k$. Therefore,

$$\begin{aligned} (1) &= \sum_{r=1}^{m}\sum_{l=0}^{n-m-1} \sum_{\substack{\boldsymbol{i}\in J_n^{n-m-1} \\ l=\max\{k|r+k-1\geq i_k\}}} (-1)^{|\boldsymbol{i}|+\sigma(n-m-1)+r+1} T^{\boldsymbol{i}-\boldsymbol{1}^{n-m-1}_{l+1}}_{n-m-1,m}\partial^1_{r+l} \\ &= \sum_{s=1}^{n}\sum_{l=0}^{n-m-1} \sum_{\substack{\boldsymbol{i}\in J_n^{n-m-1} \\ l=\max\{k|s-l+k-1\geq i_k\}}} (-1)^{|\boldsymbol{i}|+\sigma(n-m-1)+1+s-l} T^{\boldsymbol{i}-\boldsymbol{1}^{n-m-1}_{l+1}}_{n-m-1,m}\partial^1_{s} \\ &= \sum_{s=1}^{n} \sum_{\boldsymbol{i}\in J_{n-1}^{n-m-1}} (-1)^{|\boldsymbol{i}|+\sigma(n-m-1)+n-m+s} T^{\boldsymbol{i}}_{n-m-1,m}\partial^1_s \\ &= (-1)^{n+1}\sum_{s=1}^{n}(-1)^{s+1}T_{n-m-1,m}\partial^1_s. \end{aligned}$$

All that remains is to see that

$$(2) = (-1)^{n-m-1} \sum_{\boldsymbol{i} \in J_n^{n-m}} \sum_{r=1}^{n} (-1)^{|\boldsymbol{i}|+r+\sigma(n-m)} \partial_r^1 T_{n-m,m}^{\boldsymbol{i}} =: (*).$$

Recall that

$$(\partial_r^1 T_{n-m,m}^{\boldsymbol{i}})^{\boldsymbol{j}} = (T_{n-m,m}^{\boldsymbol{i}})^{s_r^1(\boldsymbol{j})} = \left(\delta^{\mathbf{1}}_{\boldsymbol{i}(s_r^1 \boldsymbol{j})} \operatorname{tr}_{m+s(\boldsymbol{j})+1}^{\partial_{u(\boldsymbol{j})}\partial_r(\boldsymbol{i})}\right)^{\partial_{u(\boldsymbol{j})}(\boldsymbol{j})}.$$

An easy calculation shows that $\delta^{\mathbf{1}}_{\boldsymbol{i}(s_r^1 \boldsymbol{j})} = \delta^1_{\boldsymbol{i}(\boldsymbol{j})} \delta^{\mathbf{1}}_{i_r - r+1}$. Therefore,

$$\partial_r^1 T_{n-m,m}^{\boldsymbol{i}} = \delta^1_{i_r-r+1} T_{n-m,m}^{\partial_r \boldsymbol{i}}$$

and hence,

$$\begin{aligned}
(*) &= \sum_{\boldsymbol{i} \in J_n^{n-m}} \sum_{r=1}^{n} (-1)^{|\boldsymbol{i}|+r+\sigma(n-m)} \partial_r^1 T_{n-m,m}^{\boldsymbol{i}} \\
&= \sum_{\boldsymbol{i} \in J_n^{n-m}} \sum_{r=1}^{n} (-1)^{|\boldsymbol{i}|+r+\sigma(n-m)} \delta^1_{i_r-r+1} T_{n-m,m}^{\partial_r \boldsymbol{i}} \\
&= \sum_{\boldsymbol{i} \in J_n^{n-m-1}} \sum_{r=1}^{n} (-1)^{|\boldsymbol{i}|+r+1+\sigma(n-m)} \delta^1_{r} T_{n-m,m}^{\partial_r \boldsymbol{i}} \\
&= \sum_{\boldsymbol{i} \in J_n^{n-m-1}} \sum_{r=1}^{n} (-1)^{(n-m-1)+|\boldsymbol{i}|+r+\sigma(n-m-1)} \delta^1_{r} T_{n-m,m}^{\partial_r \boldsymbol{i}} = (2),
\end{aligned}$$

and the proposition is proved. □

4.4 Adams operations on rational algebraic K-theory

To sum up, we have defined the following chain morphisms:

- In section 4.3, we defined a morphism

$$\mathbb{Z}C_*(X) \xrightarrow{T} \mathbb{Z}\operatorname{Sp}_*^{\square}(X).$$

  That is, a collection of split cubes on $X \times \square^*$ is assigned to every n-cube.

- In section 4.2, for every $k \geq 1$, we defined a chain morphism

$$\Psi^k : \mathbb{Z}\operatorname{Sp}_*(X) \to \mathbb{Z}C_*(X).$$

Since the maps Ψ^k and T are functorial, they are natural on schemes X in $\mathcal{C}_B$. Therefore, there is an induced map of 2-iterated chain complexes

$$\Psi^k : \mathbb{Z}\,\mathrm{Sp}^{\square}_{*,*}(X) \to \mathbb{Z}C^{\square}_{*,*}(X) \to \widetilde{\mathbb{Z}}C^{\widetilde{\square}}_{*,*}(X),$$

which induces a morphism on the associated simple complexes

$$\Psi^k : \mathbb{Z}\,\mathrm{Sp}^{\square}_{*}(X) \to \mathbb{Z}C^{\square}_{*}(X) \to \widetilde{\mathbb{Z}}C^{\widetilde{\square}}_{*}(X).$$

The composition with the morphism T gives a morphism

$$\Psi^k : \mathbb{Z}C_*(X) \to \mathbb{Z}C^{\square}_{*}(X) \to \widetilde{\mathbb{Z}}C^{\widetilde{\square}}_{*}(X) \cong NC^{\widetilde{\square}}_{*}(X).$$

Finally, considering the normalized complex of cubes of lemma 4.1.6, we define

$$\Psi^k : NC_*(X) \hookrightarrow \mathbb{Z}C_*(X) \to NC^{\widetilde{\square}}_{*}(X).$$

Corollary 4.4.1. *For every $k \geq 1$, there is a well-defined chain morphism*

$$\Psi^k : NC_*(X) \to NC^{\widetilde{\square}}_{*}(X).$$

If X is a regular noetherian scheme, then for every n, there are induced morphisms

$$\Psi^k : K_n(X) \otimes \mathbb{Q} \to K_n(X) \otimes \mathbb{Q}.$$

Proof. It is a consequence of lemmas 4.1.9 and 4.3.2. □

Theorem 4.4.2. *Let X be a regular noetherian scheme of finite Krull dimension. For every $k \geq 1$ and $n \geq 0$, the morphisms*

$$\Psi^k : K_n(X) \otimes \mathbb{Q} \to K_n(X) \otimes \mathbb{Q}$$

of corollary 4.4.1, agree with the Adams operations defined by Gillet and Soulé in [28].

Proof. We have constructed a functorial morphism, at the level of chain complexes, which by definition can be extended to simplicial schemes. Moreover, they induce the usual Adams operations on the K_0-groups, i.e. the Adams operations derived from the lambda structure coming from the exterior product of locally free sheaves. Then, the statement follows from corollary 2.4.4. □

Corollary 4.4.3. *The Adams operations defined here satisfy the usual identities for any finite dimensional regular noetherian scheme.*

□

Chapter 5

Adams operations on higher arithmetic K-theory

Let X be an arithmetic variety over the ring of integers $\mathbb{Z}$. In order to define the arithmetic Chern character on hermitian vector bundles, Gillet and Soulé have introduced in [26] the arithmetic K_0-group, denoted by $\widehat{K}_0(X)$. They endowed $\widehat{K}_0(X)$ with a pre-λ-ring structure, which was shown to be a λ-ring structure by Rössler in [50]. This group fits in an exact sequence

$$K_1(X) \xrightarrow{\rho} \bigoplus_{p\geq 0} \mathcal{D}^{2p-1}(X,p)/\operatorname{im} d_{\mathcal{D}} \to \widehat{K}_0(X) \to K_0(X) \to 0, \tag{5.0.1}$$

with ρ the Beilinson regulator (up to a constant factor).

Two different definitions for higher arithmetic K-theory have been proposed. Initially, it was suggested by Deligne and Soulé (see [54] §III.2.3.4 and [16], Remark 5.4) that these groups should fit in a long exact sequence

$$\cdots \to K_{n+1}(X) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n-1}(X,\mathbb{R}(p)) \to \widehat{K}_n(X) \to K_n(X) \to \dots,$$

extending the exact sequence (5.0.1). Here $H^*_{\mathcal{D}}(X,\mathbb{R}(p))$ is Deligne-Beilinson cohomology and ρ is the Beilinson regulator. This can be achieved by defining $\widehat{K}_n(X)$ to be the homotopy groups of the homotopy fiber of a representative of the Beilinson regulator (for instance, the representative "ch" defined by Burgos and Wang in [15]).

If X is proper, in [57], Takeda has given an alternative definition of the *higher arithmetic K-groups* of X, by means of homotopy groups modified by the representative of the Beilinson regulator "ch". We denote these higher arithmetic K-groups by $\widehat{K}_n^T(X)$. In this case, these groups fit in exact sequences

$$K_{n+1}(X) \xrightarrow{\rho} \bigoplus_{p\geq 0} \mathcal{D}^{2p-n-1}(X,p)/\operatorname{im} d_{\mathcal{D}} \to \widehat{K}_n^T(X) \to K_n(X) \to 0,$$

analogous to (5.0.1).

The two definitions do not agree, but, as proved by Takeda, they are related by a natural isomorphism:

$$\widehat{K}_n(X) \cong \ker(\mathrm{ch} : \widehat{K}_n^T(X) \to \widehat{\mathcal{D}}^{2p-n}(X,p)), \qquad n \geq 0.$$

In this chapter, we give a pre-λ-ring structure on the higher arithmetic K-groups $\widehat{K}_n(X)_{\mathbb{Q}}$ and $\widehat{K}_n^T(X)_{\mathbb{Q}}$. It is compatible with the λ-ring structure on the algebraic K-groups, $K_n(X)$, defined by Gillet and Soulé in [28]. Moreover, for $n = 0$ we recover the λ-ring structure of $\widehat{K}_0(X)$.

The chapter is organized as follows. The first two sections cover the preliminaries. Specifically, in the first section, we recall the construction of the chain morphism "ch" given by Burgos and Wang in [15]. In the second section we give the definition of the higher arithmetic K-groups due to Deligne-Soulé and Takeda. The last two sections are the central work of the chapter. In the third section we give a homological version of the higher arithmetic K-groups of Takeda tensored by $\mathbb{Q}$. The fourth section is devoted to prove the commutativity of the Adams operations of chapter 4 with "ch". We then define the Adams operations on the higher arithmetic K-groups.

Notation. If A is an abelian group, we denote

$$A_{\mathbb{Q}} := A \otimes \mathbb{Q}.$$

The general facts on chain complexes and iterated chain complexes used in this chapter where discussed in section 1.2.

5.1 Higher Bott-Chern forms

We recall here the Burgos-Wang construction of the Beilinson regulator, given in [15]. Using the chain complex of cubes (see section 1.3.3), the transgression morphism (see section 4.3), and the Chern character form of a vector bundle, they obtained a chain morphism whose induced morphism in homology is the Beilinson regulator.

We focus the discussion on the case of smooth proper complex varieties, since this will be the case in our applications. Nevertheless, most of the constructions can be adapted to the non-proper case by using hermitian metrics smooth at infinity. See the original reference for details.

In this section, all schemes are over $\mathbb{C}$.

5.1.1 Chern character form

We recall here the construction of higher Bott-Chern forms, due to Burgos and Wang, in [15]. These forms are the extension to hermitian n-cubes of the Chern character form of a hermitian vector bundle. For details, see the given reference or alternatively see [11], 3.2.

Let X be a smooth proper complex variety. A *hermitian vector bundle* $\overline{E} = (E, h)$ is an algebraic vector bundle E over X together with a smooth hermitian metric on E. The reader is referred to [60] for details.

For every hermitian vector bundle $\overline{E}$, there is a closed differential form

$$\mathrm{ch}(\overline{E}) \in \bigoplus_{p \geq 0} \mathcal{D}^{2p}(X, p),$$

representing the Chern character class $\mathrm{ch}(E) = [\mathrm{ch}(\overline{E})] \in H^*_{dR}(X)$. Let $K_{\overline{E}}$ denote the curvature form of the only connection on E that is compatible with both the metric and the complex structure. Then, the Chern character form is given by

$$\mathrm{ch}(\overline{E}) = \mathrm{Tr}(\exp(-K_{\overline{E}})).$$

Although the class of $\mathrm{ch}(\overline{E})$ is independent of the hermitian metric, the form depends on the particular hermitian metric.

The following properties are satisfied:

- If $\overline{E} \cong \overline{F}$ is an isometry of hermitian vector bundles, then

$$\mathrm{ch}(\overline{E}) = \mathrm{ch}(\overline{F}).$$

- Let $\overline{E}_1$ and $\overline{E}_2$ be two hermitian vector bundles. If $\overline{E}_1 \oplus \overline{E}_2$ and $\overline{E}_1 \otimes \overline{E}_2$ have the hermitian metrics induced by those on $\overline{E}_1$ and $\overline{E}_2$, then

$$\begin{aligned} \mathrm{ch}(\overline{E}_1 \oplus \overline{E}_2) &= \mathrm{ch}(\overline{E}_1) + \mathrm{ch}(\overline{E}_2), \\ \mathrm{ch}(\overline{E}_1 \otimes \overline{E}_2) &= \mathrm{ch}(\overline{E}_1) \wedge \mathrm{ch}(\overline{E}_2). \end{aligned}$$

5.1.2 Hermitian cubes

Let X be a smooth proper complex variety. Let $\mathcal{P}(X)$ be the category of vector bundles over X. Let $\widehat{\mathcal{P}}(X)$ be the category whose objects are the hermitian vector bundles over X, and whose morphisms are

$$\mathrm{Hom}_{\widehat{\mathcal{P}}(X)}((E, h), (E', h')) = \mathrm{Hom}_{\mathcal{P}(X)}(E, E').$$

The category $\widehat{\mathcal{P}}(X)$ inherits an exact category structure from that of $\mathcal{P}(X)$.

We fix a universe $\mathcal{U}$ so that $\widehat{\mathcal{P}}(X)$ is $\mathcal{U}$-small for every smooth proper complex variety X. Every vector bundle admits a smooth hermitian metric. It follows that the forgetful functor

$$\widehat{\mathcal{P}}(X) \to \mathcal{P}(X)$$

is an equivalence of categories. Its quasi-inverse is constructed by choosing a hermitian metric for each vector bundle. Therefore, the algebraic K-groups of X can be computed in terms of the category $\widehat{\mathcal{P}}(X)$.

Denote by $\widehat{S}.(X)$ the Waldhausen simplicial set corresponding to the exact category $\widehat{\mathcal{P}}(X)$ and let $\mathbb{Z}\widehat{C}_*(X) = \mathbb{Z}C_*(\widehat{\mathcal{P}}(X))$. The cubes in the category $\widehat{\mathcal{P}}(X)$ are called *hermitian cubes.*

Let $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$ and $N\widehat{C}_*(X)$ denote the quotient of the complex of cubes by the degenerate cubes and the normalized complex of cubes. Recall that these complexes compute the K-groups of X tensored by $\mathbb{Q}$.

Hermitian cubes with canonical kernels. Let $\overline{E}$ be a hermitian vector bundle and let $F \subset \overline{E}$ be an inclusion of vector bundles. Then F inherits a hermitian metric from the hermitian metric of $\overline{E}$. It follows that there is an induced hermitian metric on the kernel of a morphism of hermitian vector bundles. Hence, it makes sense to extend the definition of cubes with canonical kernels of 4.3.12 in the following sense.

Definition 5.1.1. Let $\overline{E}$ be a hermitian n-cube. We say that $\overline{E}$ has *canonical kernels* if for every $i = 1, \ldots, n$ and $\boldsymbol{j} \in \{0,1,2\}^{n-1}$, there is an inclusion $(\partial_i^0\overline{E})^{\boldsymbol{j}} \subset (\partial_i^1\overline{E})^{\boldsymbol{j}}$ of sets, the morphism

$$f_i^0 : \partial_i^0\overline{E} \to \partial_i^1\overline{E}$$

is the canonical inclusion of cubes and the metric on $\partial_i^0\overline{E}$ is induced by the metric of $\partial_i^1\overline{E}$ by means of f_i^0.

Let $\mathbb{Z}K\widehat{C}_*(X)$ denote the complex of hermitian cubes with canonical kernels. The quotient of the complex of cubes with canonical kernels by the degenerate cubes with canonical kernels is denoted by $\widetilde{\mathbb{Z}}K\widehat{C}_*(X)$.

Remark 5.1.2. In [15], Burgos and Wang defined the notion of *emi-cubes*, in order to define the morphism "ch". With the notation of last definition, the emi-cubes are those for which the metric on $\partial_i^0\overline{E}$ is induced by the metric of $\partial_i^1\overline{E}$, without the need of f_i^0 to be the set inclusion. In loc. cit., the purpose was that the Chern form of the transgression bundle associated to a cube defined a chain morphism. Our more restrictive notion arises because we want the transgression to define a morphism, before being composed with the Chern form.

Lemma 5.1.3. *The morphism $\lambda : \mathbb{Z}C_n(X) \to \mathbb{Z}KC_n(X)$, as defined in section 4.3.3, induces a morphism*

$$\lambda : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \to \widetilde{\mathbb{Z}}K\widehat{C}_*(X).$$

Proof. See [15], lemma 3.7. □

5.1.3 The transgression bundle and the Chern character

Recall that the transgression bundle of an n-cube E, $\mathrm{tr}_n(E)$, was defined in section 4.3.2.

The Fubini-Study metric on $\mathbb{P}^1$ induces a metric on the line bundle $\mathcal{O}(1)$. We denote by $\overline{\mathcal{O}(1)}$ the corresponding hermitian line bundle.

Observe that given a hermitian n-cube $\overline{E}$, the transgression bundle $\overline{\mathrm{tr}_n(\overline{E})}$ has a hermitian metric naturally induced by the metric of $\overline{E}$ and by the metric of $\overline{\mathcal{O}(1)}$.

Proposition 5.1.4 (Burgos-Wang). *For every n-cube with canonical kernels $\overline{E}$, the following identities hold:*

$$\begin{aligned} \delta_i^0 \operatorname{ch}(\operatorname{tr}_n(\overline{E})) &= \operatorname{ch}(\operatorname{tr}_{n-1}(\partial_i^1 \overline{E})), & (5.1.5)\\ \delta_i^1 \operatorname{ch}(\operatorname{tr}_n(\overline{E})) &= \operatorname{ch}(\operatorname{tr}_{n-1}(\partial_i^0 \overline{E})) + \operatorname{ch}(\operatorname{tr}_{n-1}(\partial_i^2 \overline{E})), & (5.1.6)\end{aligned}$$

Proof. It follows from lemma 4.3.5. Just observe that the direct sum decomposition (4.3.7) is orthogonal. □

The cochain complex of differential forms on $X \times (\mathbb{P}^1)^n$, $\widetilde{\mathcal{D}}_{\mathbb{P}}^*(X,p)$, was defined in section 3.5. It is the simple complex associated to the 2-iterated complex

$$\widetilde{\mathcal{D}}_{\mathbb{P}}^{r,-n}(X,p) := \frac{\mathcal{D}^r(X \times (\mathbb{P}^1)^n, p)}{D_n^r + \mathcal{W}_n^r}.$$

By proposition 3.5.7, there is a quasi-isomorphism of complexes

$$\widetilde{\mathcal{D}}_{\mathbb{P}}^*(X,p) \xrightarrow{\varphi} \mathcal{D}^*(X,p),$$

given by

$$\alpha \in \mathcal{D}^r(X \times (\mathbb{P}^1)^n, p) \mapsto \pi_*(\alpha \bullet T_n) = \begin{cases} \frac{1}{(2\pi i)^n} \int_{(\mathbb{P}^1)^n} \alpha \bullet T_n & n > 0,\\ \alpha & n = 0. \end{cases}$$

Recall that $\widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p)$ is the chain complex associated to the cochain complex $\widetilde{\mathcal{D}}_{\mathbb{P}}^*(X,p)[2p]$ and that T_n is the differential form

$$T_n = \frac{1}{2n!} \sum_{i=1}^{n} (-1)^i S_n^i \in \mathcal{D}_{\log}^n((\mathbb{C}^*)^n, n),$$

with

$$S_n^i = \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma \log|z_{\sigma(1)}|^2 \frac{dz_{\sigma(2)}}{z_{\sigma(2)}} \wedge \cdots \wedge \frac{dz_{\sigma(i)}}{z_{\sigma(i)}} \wedge \frac{d\bar{z}_{\sigma(i+1)}}{\bar{z}_{\sigma(i+1)}} \wedge \cdots \wedge \frac{d\bar{z}_{\sigma(n)}}{\bar{z}_{\sigma(n)}}.$$

Let X be a smooth proper complex variety and, for every $n \geq 0$, let

$$\widetilde{\mathbb{Z}}\widehat{C}_n(X) \xrightarrow{\operatorname{ch}} \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-n}(X,p)$$

be the morphism defined by

$$\operatorname{ch}(\overline{E}) = (-1)^n \operatorname{ch}(\operatorname{tr}_n(\lambda(\overline{E}))) \in \mathcal{D}^{2p}(X \times (\mathbb{P}^1)^n, p)$$

for every hermitian n-cube $\overline{E}$.

Theorem 5.1.7 (Burgos-Wang). *Let X be a smooth proper complex variety. The morphism* ch *is a chain morphism. The induced morphism*

$$K_n(X) \xrightarrow{\text{Cub}} H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X)) \xrightarrow{\text{ch}} \bigoplus_{p\geq 0} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p))$$

agrees with the Beilinson regulator.

Theorem 5.1.8 (Burgos-Wang). *Let X be a smooth proper complex variety.*

(1) There is a chain morphism

$$\begin{array}{rcl} \widetilde{\mathbb{Z}}\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p), \\ \overline{E} & \mapsto & \mathrm{ch}_n(\overline{E}) := \dfrac{(-1)^n}{(2\pi i)^n}\displaystyle\int_{(\mathbb{P}^1)^n} \mathrm{ch}(\mathrm{tr}_n(\lambda(\overline{E}))) \wedge T_n. \end{array} \tag{5.1.9}$$

(2) The composition

$$K_n(X) \xrightarrow{\text{Cub}} H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X)) \xrightarrow{\text{ch}} \bigoplus_{p\geq 0} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p))$$

is the Beilinson regulator.

The form $\mathrm{ch}_n(\overline{E})$ is called the *Bott-Chern form* of the hermitian n-cube $\overline{E}$.

Remark 5.1.10. Observe that, by means of the isomorphism $N\widehat{C}_*(X) \cong \widetilde{\mathbb{Z}}\widehat{C}_*(X)$ of lemma 1.2.36, the Chern character is also represented by the morphism

$$\begin{array}{rcl} N\widehat{C}_*(X) & \xrightarrow{\text{ch}} & \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p) \\ \overline{E} \in N\widehat{C}_n(X) & \mapsto & \mathrm{ch}_n(\overline{E}). \end{array}$$

The next proposition tells us that the morphism "ch" maps the split exact sequences to zero in the complex $\bigoplus_{p\geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p)$.

Proposition 5.1.11. *Let X be a smooth proper complex variety. Consider a split exact sequence*

$$\overline{E} : 0 \to \overline{E}^0 \to \overline{E}^0 \oplus \overline{E}^1 \to \overline{E}^1 \to 0$$

of hermitian vector bundles over X. Then, in the complex $\bigoplus_{p\geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p)$, it holds $\mathrm{ch}(\overline{E}) = 0$.

Proof. Clearly, the exact sequence $\overline{E}$ already has canonical kernels. Let us compute $\mathrm{ch}(\mathrm{tr}_1(\overline{E}))$. By definition, $\mathrm{tr}_1(\overline{E})$ is the kernel of the morphism

$$\begin{array}{rcl} \overline{E}^0(1) \oplus \overline{E}^1(1) \oplus \overline{E}^1(1) & \to & \overline{E}^1(2) \\ (a,b,c) & \mapsto & b \otimes x - c \otimes y. \end{array}$$

For every locally free sheaf B, there is a short exact sequence

$$0 \to B \xrightarrow{f} B(1) \oplus B(1) \xrightarrow{g} B(2) \to 0$$

where f sends b to $(b \otimes y, b \otimes x)$ and g sends (b, c) to $b \otimes x - c \otimes y$. Moreover, if $\overline{B}$ is a hermitian vector bundle, then the monomorphism f preserves the hermitian metric. It follows that the hermitian vector bundle $\mathrm{tr}_1(\overline{E})$ is $\overline{E}^0(1) \oplus \overline{E}^1$ and therefore

$$\mathrm{ch}(\mathrm{tr}_1(\overline{E})) = \mathrm{ch}(\overline{E}^0(1) \oplus \overline{E}^1) = \mathrm{ch}(\overline{E}^0(1)) + \mathrm{ch}(\overline{E}^1).$$

Since $\mathrm{ch}(\overline{\mathcal{O}(1)}) = 1 + \omega \in D_1^2 + \mathcal{W}_1^2$, the differential form $\mathrm{ch}(\overline{E}^0(1)) + \mathrm{ch}(\overline{E}^1)$ is zero in the complex $\bigoplus_{p \geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X, p)$. □

5.2 Higher arithmetic K-theory

In this section we discuss the extension to arithmetic varieties of the Chern character on complex varieties. Then, we recall the definition of the arithmetic K-group given by Gillet and Soulé in [26]. The last two sections are devoted to review the two definitions of higher arithmetic K-theory.

Recall that the definition of an arithmetic ring and an arithmetic variety were given in 3.6.1.

In this section we restrict ourselves to proper arithmetic varieties over the arithmetic ring $\mathbb{Z}$. Note, however, that most of the results are valid under the less restrictive hypothesis of the variety being proper over $\mathbb{C}$. Moreover, one could extend the definition of higher arithmetic K-groups, $\widehat{K}_n(X)$, to quasi-projective varieties, by considering vector bundles with hermitian metrics smooth at infinity and the complex of differential forms $\mathcal{D}_{\mathbb{P}}^*(X, p)$.

5.2.1 Chern character for arithmetic varieties

If X is an arithmetic variety over $\mathbb{Z}$, let $X(\mathbb{C})$ denote the associated complex variety, consisting of the $\mathbb{C}$-valued points on X. Let F_∞ denote the complex conjugation on $X(\mathbb{C})$ and $X_{\mathbb{R}} = (X(\mathbb{C}), F_\infty)$ the associated real variety.

Recall that the *real Deligne-Beilinson cohomology of* X is defined as the cohomology of $X_{\mathbb{R}}$:

$$H_{\mathcal{D}}^n(X, \mathbb{R}(p)) = H_{\mathcal{D}}^n(X_{\mathbb{R}}, \mathbb{R}(p)) = H_{\mathcal{D}}^n(X(\mathbb{C}), \mathbb{R}(p))^{\overline{F}_\infty^* = id}.$$

It is computed as the cohomology of the real Deligne complex:

$$\mathcal{D}^n(X, p) = \mathcal{D}^n(X_{\mathbb{R}}, p) = \mathcal{D}^n(X(\mathbb{C}), p)^{\overline{F}_\infty^* = id},$$

i.e.,

$$H_{\mathcal{D}}^n(X, \mathbb{R}(p)) \cong H^n(\mathcal{D}^n(X, p), d_{\mathcal{D}}).$$

Definition 5.2.1. Let X be a proper arithmetic variety over $\mathbb{Z}$. A *hermitian vector bundle* $\overline{E}$ over X is a pair (E,h), where E is a locally free sheaf on X and where h is a F_∞^*-invariant hermitian metric on the associated vector bundle $E(\mathbb{C})$ over $X(\mathbb{C})$.

Let $\widehat{\mathcal{P}}(X)$ denote the category of hermitian vector bundles over X. The simplicial set $\widehat{S}.(X)$ and the chain complexes $\mathbb{Z}\widehat{C}_*(X)$, $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$ and $N\widehat{C}_*(X)$ are defined accordingly.

If $\overline{E}$ is a hermitian vector bundle over X, the Chern character form $\mathrm{ch}(\overline{E})$ is F_∞^*-invariant. Therefore

$$\mathrm{ch}(\overline{E}) \in \bigoplus_{p\geq 0} \mathcal{D}^{2p}(X,p).$$

It follows that there is a chain morphism

$$\mathbb{Z}\widehat{S}_*(X)[-1] \xrightarrow{\mathrm{Cub}} \widetilde{\mathbb{Z}}\widehat{C}_*(X) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p).$$

5.2.2 Arithmetic K_0-group

In [26], Gillet and Soulé defined the arithmetic K_0-group of an arithmetic variety, $\widehat{K}_0(X)$. We give here a slightly different presentation using the Deligne complex of differential forms and the differential operator $-2\partial\bar{\partial}$. The comparison of the two definitions is already given by Takeda in [57].

Let X be an arithmetic variety and let $\widetilde{\mathcal{D}}^*(X,p) = \mathcal{D}^*(X,p)/\operatorname{im} d_{\mathcal{D}}$. Consider pairs $(\overline{E},\alpha)$, where $\overline{E}$ is a hermitian vector bundle over X and where $\alpha \in \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-1}(X,p)$ is a differential form. Then, $\widehat{K}_0(X)$ is the quotient of the free abelian group generated by these pairs by the subgroup generated by the sums

$$(\overline{E}^0,\alpha_0) + (\overline{E}^2,\alpha_2) - (\overline{E}^1, \alpha_0+\alpha_2 - \mathrm{ch}(\overline{E})),$$

for every exact sequence of hermitian vector bundles over X,

$$\overline{E} : 0 \to \overline{E}^0 \to \overline{E}^1 \to \overline{E}^2 \to 0,$$

and every $\alpha_0,\alpha_2 \in \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-1}(X,p)$.

Among other properties, this group fits in an exact sequence

$$K_1(X) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-1}(X,p) \xrightarrow{a} \widehat{K}_0(X) \xrightarrow{\zeta} K_0(X) \to 0$$

(see [26] for details).

Gillet and Soulé, together with Rössler (see [50]), showed that there is a λ-ring structure on $\widehat{K}_0(X)$.

5.2.3 Deligne-Soulé higher arithmetic K-theory

Although there is no reference in which the theory is developed, it has been suggested by Deligne and Soulé (see [54] §III.2.3.4 and [16], Remark 5.4) that the higher arithmetic K-theory should be obtained as the homotopy groups of the homotopy fiber of a representative of the Beilinson regulator. We sketch here the construction, in order to show that Adams operations can be defined.

Let $\widehat{\mathcal{D}}^{2p-*}(X,p)$ be the complex with

$$\widehat{\mathcal{D}}^{2p-n}(X,p) = \begin{cases} \mathcal{D}^{2p-n}(X,p) & \text{if } n \neq 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Let

$$\widehat{\mathrm{ch}} : \widetilde{\mathbb{Z}}\widehat{C}_n(X) \to \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-n}(X,p),$$

be the composition of $\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}_n(X) \to \bigoplus_{p\geq 0} \mathcal{D}^{2p-n}(X,p)$ with the natural map

$$\bigoplus_{p\geq 0} \mathcal{D}^{2p-n}(X,p) \to \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-n}(X,p).$$

Let $\mathcal{K}(\cdot)$ be the Dold-Puppe functor from the category of chain complexes of abelian groups to the category of simplicial abelian groups (see [17]). Consider the morphism

$$\mathcal{K}(\widehat{\mathrm{ch}}) : \widehat{S}.(X) \to \mathcal{K}.(\mathbb{Z}\widehat{S}_*(X)) \xrightarrow{\mathrm{Cub}} \mathcal{K}(\widetilde{\mathbb{Z}}\widehat{C}_*(X)) \xrightarrow{\widehat{\mathrm{ch}}} \mathcal{K}\Big(\bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p)\Big),$$

and denote by $|\mathcal{K}(\widehat{\mathrm{ch}})|$ the morphism induced on the realization of the simplicial sets.

Definition 5.2.2. For every $n \geq 0$, the *(Deligne-Soulé) higher arithmetic K-group* of X is defined as

$$\widehat{K}_n(X) = \pi_{n+1}(\text{Homotopy fiber of } |\mathcal{K}(\widehat{\mathrm{ch}})|).$$

Proposition 5.2.3. *Let X be a proper arithmetic variety. Then,*

(i) The group $\widehat{K}_0(X)$ agrees with the arithmetic K-group defined by Gillet and Soulé in [26].

(ii) Let $s(\widehat{\mathrm{ch}})$ denote the simple complex associated to the chain morphism $\widehat{\mathrm{ch}}$. If $n > 0$, there is an isomorphism $\widehat{K}_n(X)_{\mathbb{Q}} \cong H_n(s(\widehat{\mathrm{ch}}),\mathbb{Q})$.

(iii) There is a long exact sequence

$$\cdots \to K_{n+1}(X) \xrightarrow{\mathrm{ch}} H_{\mathcal{D}}^{2p-n-1}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{K}_n(X) \xrightarrow{\zeta} K_n(X) \to \cdots$$

with end

$$\cdots \to K_1(X) \xrightarrow{\mathrm{ch}} \widetilde{\mathcal{D}}^{2p-1}(X,p) \xrightarrow{a} \widehat{K}_0(X) \xrightarrow{\zeta} K_0(X) \to 0.$$

Proof. The first and third statements follow by definition. The second statement follows from proposition 1.2.30 and theorem 1.3.15. □

In section 5.3.2, we will endow $\bigoplus_{n\geq 0}\widehat{K}_n(X)$ with a product structures, induced by the product structure defined by Takeda.

5.2.4 Takeda higher arithmetic K-theory

In this section we recall the definition of higher arithmetic K-groups given by Takeda in [57]. He first develops a theory of homotopy groups modified by a suitable chain morphism ρ. As a particular case, the higher arithmetic K-groups are given by the homotopy groups of $\widehat{S}.(X)$ modified by the Chern character morphism.

We use the theory on simplicial sets recalled in sections 1.1.1 and 1.1.2.

Let T be a pointed CW-complex and fix $* \in T$ a base point. Let $C_*(T)$ be the cellular complex of T given by

$$C_n(T) = H_n\big(\operatorname{sk}_n(T), \operatorname{sk}_{n-1}(T)\big).$$

The differential ∂ is the connecting morphism

$$H_n\big(\operatorname{sk}_n(T), \operatorname{sk}_{n-1}(T)\big) \xrightarrow{\partial} H_{n-1}\big(\operatorname{sk}_{n-1}(T), \operatorname{sk}_{n-2}(T)\big)$$

corresponding to the long exact sequence associated to the triple

$$\big(\operatorname{sk}_n(T), \operatorname{sk}_{n-1}(T), \operatorname{sk}_{n-2}(T)\big).$$

For a reference on cellular homology see, for instance, [35] or any basic book on algebraic topology.

Let (W_*, d) be a chain complex and denote $\widetilde{W}_* = W_*/\operatorname{im} d$. Suppose we are given a chain morphism $\rho : C_*(T) \to W_*$. Consider pairs (f, ω) where

- ▷ $f : S^n \to T$ is a pointed cellular map, and,
- ▷ $\omega \in \widetilde{W}_{n+1}$.

Let I be the closed unit interval $[0, 1]$ with the usual CW-complex structure. Two pairs (f, ω) and (f', ω') are *homotopy equivalent* if there exists a pointed cellular map

$$h : S^n \times I/\{*\} \times I \to T$$

such that the following conditions hold:

(1) h is a topological homotopy between f and f', i.e.

$$h(x, 0) = f(x), \quad \text{and} \quad h(x, 1) = f'(x).$$

(2) Let $[S^n \times I] \in C_{n+1}(S^n \times I)$ denote the fundamental chain of $S^n \times I$. Then,

$$\omega' - \omega = (-1)^{n+1}\rho\big(h_*([S^n \times I])\big).$$

Being homotopy equivalent is an equivalence relation, which we denote by $\sim$. Then, for every n, the *modified homotopy group* $\widehat{\pi}_n(T,\rho)$ is defined to be the set of all homotopy classes of pairs as above. Takeda proves that these are abelian groups.

Let X be a proper arithmetic variety over $\mathbb{Z}$. Let $|\widehat{S}.(X)|$ denote the geometric realization of the simplicial set $\widehat{S}.(X)$. It follows that $|\widehat{S}.(X)|$ is a CW-complex.

Let $\widehat{D}^s_*(X) \subset \mathbb{Z}\widehat{S}_*(X)$ be the complex generated by the degenerate simplices of $\widehat{S}.(X)$. The cellular complex $C_*(|\widehat{S}.(X)|)$ is naturally isomorphic to the complex $\mathbb{Z}\widehat{S}_*(X)/\widehat{D}^s_*(X)$. In the sequel we will identify these two complexes by this isomorphism.

As shown in [57], theorem 4.4, the map $\mathrm{ch}\circ\mathrm{Cub}$ maps the degenerate simplices of $\widehat{S}.(X)$ to zero. It follows that there is a well-defined chain morphism

$$\mathrm{ch}: C_*(|\widehat{S}.(X)|)[-1] \to \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p).$$

Definition 5.2.4 (Takeda)**.** Let X be a proper arithmetic variety over $\mathbb{Z}$. For every $n\geq 0$, the *higher arithmetic K-group* of X, $\widehat{K}^T_n(X)$, is defined by

$$\begin{aligned}\widehat{K}^T_n(X) &= \widehat{\pi}_{n+1}(|\widehat{S}.(X)|,\mathrm{ch})\\ &= \left\{(f: S^{n+1}\to |\widehat{S}.(X)|,\ \omega) \;\middle|\; \omega\in\bigoplus_{p\geq 0}\widetilde{\mathcal{D}}^{2p-n-1}(X,p)\right\}\Big/\sim .\end{aligned}$$

Takeda proves the following results:

(i) For every $n\geq 0$, $\widehat{K}^T_n(X)$ is a group.

(ii) For every $n\geq 0$, there is a short exact sequence

$$K_{n+1}(X) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0}\widetilde{\mathcal{D}}^{2p-n-1}(X,p) \xrightarrow{a} \widehat{K}^T_n(X) \xrightarrow{\zeta} K_n(X)\to 0.$$

The morphisms a,ζ are defined by

$$a(\alpha)=[(0,\alpha)],\qquad \zeta([(f,\alpha)])=[f].$$

(iii) There is a characteristic class

$$\widehat{K}^T_n(X)\xrightarrow{\mathrm{ch}}\bigoplus_{p\geq 0}\mathcal{D}^{2p-n}(X,p),$$

such that

$$\mathrm{ch}([(f,\alpha)])=\mathrm{ch}(f_*(S^n))+d_{\mathcal{D}}\alpha.$$

(iv) $\widehat{K}^T_0(X)$ is isomorphic to the arithmetic K-group defined by Gillet and Soulé in [26].

(v) There is a graded product on $\widehat{K}^T_*(X)$, commutative up to 2-torsion. Therefore, $\widehat{K}^T_*(X)_{\mathbb{Q}}$ is endowed with a graded commutative product.

(vi) There exist pull-back for arbitrary morphisms and push-forward for smooth and projective morphisms. A projection formula is also proved.

Lemma 5.2.5. *Let X be a proper arithmetic variety over $\mathbb{Z}$. Then, for every $n \geq 0$, there is a canonical isomorphism*

$$\widehat{K}_n(X) \cong \ker\big(\widehat{\mathrm{ch}} : \widehat{K}_n^T(X) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \mathcal{D}^{2p-n}(X,p) \to \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-n}(X,p)\big).$$

Proof. This is proved by Takeda in [57]. □

5.3 Rational Takeda higher arithmetic K-groups

By parallelism with the algebraic situation, it is natural to expect that the higher arithmetic K-groups tensored by $\mathbb{Q}$ can be described in homological terms. In proposition 5.2.3, we saw that the Deligne-Soulé higher arithmetic K-groups are isomorphic to the homology groups of the simple complex associated to the Beilinson regulator "ch", after tensoring by $\mathbb{Q}$. In this section we show that rational Takeda higher arithmetic K-groups admit also a homological description. We prove that $\widehat{K}_n^T(X)_{\mathbb{Q}}$ can be obtained considering a variant of the complex of cubes, together with what we call modified homology groups.

5.3.1 Modified homology groups

We briefly describe here the analogue, in a homological context, of the modified homotopy groups given by Takeda in [57].

The modified homology groups are the dual notion of the *truncated relative cohomology groups* defined by Burgos in [13], as one can observe comparing both definitions and the satisfied properties. These groups appear naturally in other contexts. For instance, one can express the description of hermitian-holomorphic Deligne cohomology given by Aldrovandi in [2], §2.2, in terms of modified homology groups.

Moreover, these groups can be seen as a particular case of the modified homology groups of a diagram as introduced in section 3.10.1, now with a diagram consisting of one morphism $A_* \to B_*$. To ease the reading of this chapter, we give the details of the construction applied to this particular case.

Let (A_*, d_A) and (B_*, d_B) be two chain complexes and let $A_* \xrightarrow{\rho} B_*$ be a chain morphism. If $\widetilde{B}_* = B_*/\operatorname{im} d_B$, consider pairs

$$(a,b) \in A_n \oplus \widetilde{B}_{n+1}, \quad \text{with } d_A a = 0.$$

We define an equivalence relation as follows. We say that $(a,b) \sim (a',b')$ if, and only if, there exists $h \in A_{n+1}$, such that

$$d_A h = a - a', \quad \text{and} \quad \rho(h) = b - b'.$$

Let $ZA_* = \ker d_A$ be the group of cycles in A_*.

Definition 5.3.1. Let $(A_*, d_A), (B_*, d_B)$ be two chain complexes and let $\rho : A_* \to B_*$ be a chain morphism. For every n, the *n-th modified homology group of A_* with respect to ρ* is defined as

$$\widehat{H}_n(A_*, \rho) := \{(a,b) \in ZA_n \oplus \widetilde{B}_{n+1}\}/\sim .$$

The group $\widehat{H}_n(A_*, \rho)$ can be rewritten as

$$\widehat{H}_n(A_*, \rho) = \big\{(a,b) \in ZA_n \oplus B_{n+1}\big\}/\big\{(0, d_B b), (d_A a, \rho(a)), a \in A_{n+1}, b \in B_{n+2}\big\}.$$

The class of a pair (a,b) in $\widehat{H}_n(A_*, \rho)$ is denoted by $[(a,b)]$.

These modified homology groups can be seen as the homology groups of the simple of ρ truncated appropriately.

Let $\sigma_{>n}B_*$ be the bête truncation of the chain complex B_*, that is,

$$\sigma_{>n}B_r = \begin{cases} B_r & r > n, \\ 0 & r \leq n. \end{cases}$$

Let $\rho_{>n}$ be the composition of $\rho : A_* \to B_*$ with the canonical morphism $B_* \to \sigma_{>n}B_*$. Then, it follows from the definition that

$$H_r(s(\rho_{>n})) = \begin{cases} H_r(s(\rho)) & r > n, \\ \widehat{H}_n(A_*, \rho) & r = n, \\ H_r(A_*) & r < n. \end{cases}$$

Observe that, for every n, there are well-defined morphisms

$$\begin{array}{rclrcl} \widetilde{B}_{n+1} & \xrightarrow{a} & \widehat{H}_n(A_*, \rho), & b & \mapsto & [(0, -b)], \\ \widehat{H}_n(A_*, \rho) & \xrightarrow{\zeta} & H_n(A_*), & [(a,b)] & \mapsto & [a], \\ \widehat{H}_n(A_*, \rho) & \xrightarrow{\rho} & ZB_n & [(a,b)] & \mapsto & \rho(a) - d_B(b). \end{array}$$

The following proposition is the homological analogue of theorem 3.3 together with proposition 3.9 of [57] and the dual of propositions 4.3 and 4.4 of [13].

Proposition 5.3.2. *(i) Let $\rho : A_* \to B_*$ be a chain morphism. Then, there are exact sequences*

(a) $0 \to H_n(s_*(\rho)) \to \widehat{H}_n(A_*, \rho) \xrightarrow{\rho} ZB_n \to H_{n-1}(s_*(\rho))$.

(b) $H_{n+1}(A_*) \xrightarrow{\rho} \widetilde{B}_{n+1} \xrightarrow{a} \widehat{H}_n(A_*, \rho) \xrightarrow{\zeta} H_n(A_*) \to 0$.

(ii) Assume that there is a commutative square of chain complexes

$$\begin{tikzcd} A_* \arrow[r, "\rho"] \arrow[d, "f_1"'] & B_* \arrow[d, "f_2"] \\ C_* \arrow[r, "\rho'"'] & D_*. \end{tikzcd}$$

Then, for every n, there is an induced morphism

$$\begin{array}{rcl} \widehat{H}_n(A_*,\rho) & \xrightarrow{f} & \widehat{H}_n(C_*,\rho') \\ [(a,b)] & \mapsto & [(f_1(a), f_2(b))]. \end{array}$$

(iii) If f_1 is a quasi-isomorphism and f_2 is an isomorphism, then f is an isomorphism.

Proof. The exact sequences follow from the long exact sequences associated to the following short exact sequences:

$$0 \to B_*/\sigma_{>n}B_*[-1] \to s_*(\rho) \to s_*(\rho_{>n}) \to 0,$$
$$0 \to \sigma_{>n}B_*[-1] \to s_*(\rho_{>n}) \to A_* \to 0.$$

The second and the third statements are left to the reader. □

Corollary 5.3.3. *There is a canonical isomorphism*

$$H_n(s_*(\rho)) \cong_{can} \ker(\widehat{H}_n(A_*,\rho) \xrightarrow{\rho} B_n).$$

□

5.3.2 Definition of the rational Takeda arithmetic K-theory

We want to give a homological description of the rational Takeda arithmetic K-groups. Since these groups fit in the exact sequences

$$K_{n+1}(X)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X,p) \xrightarrow{a} \widehat{K}_n^T(X)_{\mathbb{Q}} \xrightarrow{\zeta} K_n(X)_{\mathbb{Q}} \to 0,$$

it is natural to expect that the modified homology groups associated to the Beilinson regulator ch give the desired description.

Therefore, consider the modified homology groups $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X),\widehat{\mathrm{ch}})$ associated to the chain map

$$\widehat{\mathrm{ch}} : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p) \to \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p)$$

given in (5.1.9). We want to see that there is an isomorphism

$$\widehat{K}_n^T(X)_{\mathbb{Q}} \cong \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X),\widehat{\mathrm{ch}})_{\mathbb{Q}}. \tag{5.3.4}$$

In order to prove this fact, considering the long exact sequences associated to $\widehat{K}_n^T(X)_{\mathbb{Q}}$, to $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X),\widehat{\mathrm{ch}})_{\mathbb{Q}}$ and the five lemma, it would be desirable to have a factorization of the morphism "ch" by Cub in the form

$$\begin{tikzcd} \mathbb{Z}\widehat{S}_*(X) \arrow[rr, "\mathrm{Cub}"] \arrow[dr] & & \widetilde{\mathbb{Z}}\widehat{C}_*(X) \arrow[r, "\widehat{\mathrm{ch}}"] & \bigoplus_{p\geq 0}\widehat{\mathcal{D}}^{2p-*}(X,p) \\ & C_*(|\widehat{S}.(X)|)[-1] \arrow[ur, dotted, "\mathrm{Cub}"] \arrow[urr, "\widehat{\mathrm{ch}}"'] & & \end{tikzcd}$$

Let $\mathcal{P}$ be a small exact category. If $\tau_i \in \mathfrak{S}_n$ is the permutation that interchanges i with $i+1$, then for every $E \in S_n(\mathcal{P})$ one has

$$\begin{array}{rcll} \mathrm{Cub}(s_0 E) & = & s_1^1\,\mathrm{Cub}(E), & \\ \mathrm{Cub}(s_n E) & = & s_n^0\,\mathrm{Cub}(E), & \\ \mathrm{Cub}(s_i E) & = & \tau_i\,\mathrm{Cub}(s_i E), & i=1,\dots,n-1. \end{array} \tag{5.3.5}$$

(See [57], lemma 4.1). It follows that the dotted arrow Cub of last diagram

$$C_*(|\widehat{S}.(X)|)[-1] \cong \mathbb{Z}\widehat{S}_*(X)/\widehat{D}^s_*(X)[-1] \xrightarrow{\mathrm{Cub}} \widetilde{\mathbb{Z}}\widehat{C}_*(X)$$

does not exist, since the image by Cub of a degenerate simplex in $S_n(\mathcal{P})$ is not necessarily a degenerate cube.

Therefore, in order to prove (5.3.4), we should find a new complex, $\widetilde{\mathbb{Z}}\widehat{C}^s_*(X)$, quasi-isomorphic to the complex of hermitian cubes, admitting a factorization of "ch" of the form:

$$\begin{tikzcd} \mathbb{Z}\widehat{S}_*(X) \arrow[r, "\mathrm{Cub}"] \arrow[d] & \widetilde{\mathbb{Z}}\widehat{C}_*(X) \arrow[r, "\widehat{\mathrm{ch}}"] \arrow[d, "\sim"'] & \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p) \\ C_*(|\widehat{S}.(X)|)[-1] \arrow[r, dotted, "\mathrm{Cub}"] & \widetilde{\mathbb{Z}}\widehat{C}^s_*(X) \arrow[ur, "\widehat{\mathrm{ch}}"'] & \end{tikzcd}$$

In this way, we divide the proof in two steps: to prove that there is an isomorphism $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \widehat{\mathrm{ch}}) \cong \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X), \widehat{\mathrm{ch}})$, and then that $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X), \widehat{\mathrm{ch}})_{\mathbb{Q}} \cong \widehat{K}^T_n(X)_{\mathbb{Q}}$. This will be shown in Theorem 5.3.11, once this factorization of Cub is obtained.

Factorization of Cub**.** Consider the complex of cubes $\widehat{T}_n(X) \subseteq \mathbb{Z}\widehat{C}_n(X)$, generated by the n-cubes $\overline{E}$ such that $\tau_i\overline{E} = \overline{E}$ for some index i. In the proof of theorem 4.4 in [57], Takeda shows that if $\overline{E} \in \widehat{T}_n(X)$, then $\mathrm{ch}(\overline{E}) = 0$. Hence the morphism "ch" is zero on the degenerate simplices in $\mathbb{Z}\widehat{S}_*(X)$. It follows that ch factorizes as

$$C_*(|\widehat{S}.(X)|)[-1] \xrightarrow{\cong} \mathbb{Z}\widehat{S}_*(X)/\widehat{D}^s_*(X)[-1] \xrightarrow{\mathrm{Cub}} \widetilde{\mathbb{Z}}\widehat{C}_*(X)/\widehat{T}_*(X) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p).$$

However, the complex $\widetilde{\mathbb{Z}}\widehat{C}_*(X)/\widehat{T}_*(X)$ is not quasi-isomorphic to $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$. Nevertheless, since the complex $\mathbb{Z}\widehat{S}_*(X)/\widehat{D}^s_*(X)$ is quasi-isomorphic to $\mathbb{Z}\widehat{S}_*(X)$ (see 1.2.26), it seems reasonable to think that there exists a complex which is quasi-isomorphic to $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$ and which factors the morphism ch as above. This is done in the sequel.

Let $\mathcal{P}$ be a small exact category. The smallest complex to consider is the following. For every n, let

$$C_n^{deg}(\mathcal{P}) := \{\mathrm{Cub}(s_i E),\ E \in S_n(\mathcal{P}),\ i \in \{1,\dots,n-1\}\}.$$

Let $\mathbb{Z}C_n^{deg}(\mathcal{P})$ be the free abelian group on $C_n^{deg}(\mathcal{P})$ and let

$$\widetilde{\mathbb{Z}}C_n^{deg}(\mathcal{P}) := \frac{\mathbb{Z}C_n^{deg}(\mathcal{P}) + \mathbb{Z}D_n(\mathcal{P})}{\mathbb{Z}D_n(\mathcal{P})}.$$

Lemma 5.3.6. *Let $\overline{E} \in S_n(\mathcal{P})$.*

(i) $d\,\mathrm{Cub}(s_iE) \in \mathbb{Z}C_n^{deg}(\mathcal{P}) + \mathbb{Z}D_n(\mathcal{P})$, *for all* $i = 1, \dots, n-1$.

(ii) For $i = 1, \dots, n-1$, *the equality*

$$d\,\mathrm{Cub}(s_iE) = \sum_{j=0}^{i-1}(-1)^{j+1}\,\mathrm{Cub}(s_{i-1}\partial_jE) + \sum_{j=i+1}^{n}(-1)^j\,\mathrm{Cub}(s_i\partial_jE)$$

holds in $\widetilde{\mathbb{Z}}C_n^{deg}(\mathcal{P})$.

Proof. By definition,

$$d\,\mathrm{Cub}(s_iE) = \sum_{j=0}^{n}\sum_{l=0}^{2}(-1)^{j+l}\partial_j^l\,\mathrm{Cub}(s_iE).$$

Since $\partial_i^l\tau_i = \partial_{i+1}^l$ for all $l = 0, 1, 2$, by (5.3.5) we have

$$\partial_i^l\,\mathrm{Cub}(s_iE) = \partial_{i+1}^l\,\mathrm{Cub}(s_iE).$$

Hence these two terms cancel each other in the sum previous sum. So, assume that $j \neq i, i+1$. If $l = 1$, then, by 1.3.13,

$$\partial_j^1\,\mathrm{Cub}(s_iE) = \mathrm{Cub}(\partial_js_iE) = \begin{cases} \mathrm{Cub}(s_{i-1}\partial_jE) & j < i, \\ \mathrm{Cub}(s_i\partial_{j-1}E) & j > i+1. \end{cases}$$

If $l = 0$ and $j \neq n$, or $l = 2$ and $j \neq 0$, $\partial_j^l\,\mathrm{Cub}(s_iE)$ is a degenerate cube and hence it is zero in the group $\widetilde{\mathbb{Z}}C_n^{deg}(\mathcal{P})$. Finally, we have

$$\begin{aligned} \partial_n^0\,\mathrm{Cub}(s_iE) &= \mathrm{Cub}(s_i\partial_nE), \\ \partial_0^2\,\mathrm{Cub}(s_iE) &= \mathrm{Cub}(s_{i-1}\partial_0E). \end{aligned}$$

The statements of the lemma follow from these calculations. □

It follows from the last lemma that $\widetilde{\mathbb{Z}}C_*^{deg}(\mathcal{P})$ is a chain complex with the differential induced by the differential of $\widetilde{\mathbb{Z}}C_*(\mathcal{P})$.

Proposition 5.3.7. *The complex* $\widetilde{\mathbb{Z}}C_*^{deg}(\mathcal{P})$ *is quasi-isomorphic to zero.*

Proof. We prove the proposition by constructing a chain of chain complexes

$$0 = C_*^0 \subset C_*^1 \subset \cdots \subset C_*^{n-2} \subset C_*^{n-1} = \widetilde{\mathbb{Z}}C_*^{deg}(\mathcal{P}), \tag{5.3.8}$$

such that all the quotients C_*^i/C_*^{i-1} are homotopically trivial, that is, there exists a homotopy

$$h_n : C_n^i/C_n^{i-1} \to C_{n+1}^i/C_{n+1}^{i-1}$$

such that
$$dh_n + h_{n-1}d = id.$$
It means in particular that for every i, the complex C^i_*/C^{i-1}_* is quasi-isomorphic to zero. Then, since $C^0_* = 0$, it follows inductively that C^i_* is quasi-isomorphic to zero for all i and the proposition is proved.

For every $i = 1, \dots, n-1$, let
$$\mathbb{Z}C^{deg,i}_n(\mathcal{P}) = \{\mathrm{Cub}(s_j E),\ E \in S_n(\mathcal{P}),\ j \in \{1, \dots, i\}\},$$
and let
$$C^i_n = \frac{\mathbb{Z}C^{deg,i}_n(\mathcal{P}) + \mathbb{Z}D_n(\mathcal{P})}{\mathbb{Z}D_n(\mathcal{P})}.$$
By lemma 5.3.6, (ii), C^i_* are chain complexes with the differential induced by the differential of $\widetilde{\mathbb{Z}}C_*(\mathcal{P})$. Moreover, for every i there is an inclusion of complexes $C^i_* \subseteq C^{i+1}_*$.

Fix $E \in S_n(\mathcal{P})$ and an index i. Consider an element $\mathrm{Cub}(s_i E) \in C^i_*/C^{i-1}_*$ and define
$$h_n(\mathrm{Cub}(s_i E)) = (-1)^{i+1}\,\mathrm{Cub}(s_i s_i E).$$
Then, by lemma 5.3.6, in the complex C^i_*/C^{i-1}_*,
$$d\,\mathrm{Cub}(s_i E) \;=\; \sum_{j=i+1}^{n+1} (-1)^j\,\mathrm{Cub}(s_i \partial_j E)$$
and
$$\begin{aligned} dh_n(\mathrm{Cub}(s_i E)) &= \sum_{j=i+1}^{n+1} (-1)^{i+j+1}\,\mathrm{Cub}(s_i \partial_j s_i E) \\ &= \mathrm{Cub}(s_i E) + \sum_{j=i+2}^{n+1} (-1)^{i+j+1}\,\mathrm{Cub}(s_i s_i \partial_{j-1} E) \\ &= \mathrm{Cub}(s_i E) + h_{n-1}(d\,\mathrm{Cub}(s_i E)). \end{aligned}$$
Therefore, we have proved that C^i_*/C^{i-1}_* is homotopically trivial. □

Let
$$\widetilde{\mathbb{Z}}C^s_*(\mathcal{P}) := \frac{\mathbb{Z}C_*(\mathcal{P})}{\mathbb{Z}D_*(\mathcal{P}) + \mathbb{Z}C^{deg}_*(\mathcal{P})}.$$

Corollary 5.3.9. *The natural chain morphism*
$$\widetilde{\mathbb{Z}}C_*(\mathcal{P}) \to \widetilde{\mathbb{Z}}C^s_*(\mathcal{P})$$
is a quasi-isomorphism.

□

If $\mathcal{P} = \widehat{\mathcal{P}}(X)$, we simply write $\widetilde{\mathbb{Z}}\widehat{C}^s_*(X) := \widetilde{\mathbb{Z}}C^s_*(\widehat{\mathcal{P}}(X))$ and $\widetilde{\mathbb{Z}}\widehat{C}^{deg}_*(X) := \widetilde{\mathbb{Z}}\widehat{C}^{deg}_*(\widehat{\mathcal{P}}(X))$. Since ch is zero on $\mathbb{Z}\widehat{D}_*(X) + \mathbb{Z}\widehat{C}^{deg}_*(X)$, we have obtained the following corollary.

Corollary 5.3.10. *Let X be a proper arithmetic variety over $\mathbb{Z}$.*

(i) The map ch *admits a factorization as*

$$C_*(|\widehat{S}.(X)|)[-1] \xrightarrow{\text{Cub}} \widetilde{\mathbb{Z}}\widehat{C}^s_*(X) \xrightarrow{\text{ch}} \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p).$$

(ii) The natural morphism

$$\widetilde{\mathbb{Z}}\widehat{C}_*(X) \xrightarrow{\sim} \widetilde{\mathbb{Z}}\widehat{C}^s_*(X)$$

is a quasi-isomorphism.

At this point, we have all the ingredients to prove that there is an isomorphism between $\widehat{K}^T_n(X)_{\mathbb{Q}}$ and $\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \widehat{\text{ch}})_{\mathbb{Q}}$.

For the proof of next theorem recall that the Hurewicz morphism

$$\pi_n(|\widehat{S}.(X)|) \to H_n(|\widehat{S}.(X)|)$$

maps the class of a pointed map $S^n \xrightarrow{f} |\widehat{S}.(X)|$ to $f_*([S^n])$.

Theorem 5.3.11. *Let X be a proper arithmetic variety over $\mathbb{Z}$. Then, for every $n \geq 0$, there is an isomorphism*

$$\widehat{K}^T_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \widehat{\text{ch}})_{\mathbb{Q}}.$$

Moreover, there are commutative diagrams

$$\begin{tikzcd}
\widehat{K}^T_n(X)_{\mathbb{Q}} \arrow[r, "\zeta"] \arrow[d, "\cong"'] & K_n(X)_{\mathbb{Q}} \arrow[d, "\cong"] \\
\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \widehat{\text{ch}})_{\mathbb{Q}} \arrow[r, "\zeta"] & H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \mathbb{Q})
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{K}^T_n(X)_{\mathbb{Q}} \arrow[r, "\widehat{\text{ch}}"] \arrow[d, "\cong"'] & \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p) \arrow[d, "="] \\
\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \widehat{\text{ch}})_{\mathbb{Q}} \arrow[r, "\widehat{\text{ch}}"] & \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p).
\end{tikzcd}$$

Proof. Consider the chain complex $\widetilde{\mathbb{Z}}\widehat{C}^s_*(X) = \frac{\mathbb{Z}\widehat{C}_*(X)}{\mathbb{Z}\widehat{D}_*(X)+\mathbb{Z}\widehat{C}^{deg}_*(X)}$, defined before corollary 5.3.10. Let $\widehat{H}_*(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X), \widehat{\text{ch}})$ denote the modified homology groups with respect to the morphism

$$\widetilde{\mathbb{Z}}\widehat{C}^s_*(X) \xrightarrow{\widehat{\text{ch}}} \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p).$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathbb{Z}}\widehat{C}_*(X) \arrow[r, "\widehat{\text{ch}}"] \arrow[d, "\sim"'] & \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p) \arrow[d, "="] \\
\widetilde{\mathbb{Z}}\widehat{C}^s_*(X) \arrow[r, "\widehat{\text{ch}}"] & \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p).
\end{tikzcd}$$

By lemma 5.3.2, there is an induced isomorphism

$$\widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \widehat{\mathrm{ch}}) \xrightarrow{\pi} \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X), \widehat{\mathrm{ch}}),$$

which commutes with ζ.

It remains to prove that there is an isomorphism

$$\widehat{K}^T_n(X)_{\mathbb{Q}} \cong \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X), \widehat{\mathrm{ch}}),$$

commuting with ζ.

Consider the chain morphism

$$C_*(|\widehat{S}.(X)|)[-1] \xrightarrow{\mathrm{Cub}} \widetilde{\mathbb{Z}}\widehat{C}^s_*(X).$$

By theorem 1.3.15, the isomorphism

$$K_n(X)_{\mathbb{Q}} \xrightarrow{\mathrm{Cub}} H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \mathbb{Q})$$

is given by the composition

$$K_n(X) = \pi_{n+1}(|\widehat{S}.(X)|)_{\mathbb{Q}} \xrightarrow{\text{Hurewicz}} H_n(C_*(|\widehat{S}.(X)|)[-1])_{\mathbb{Q}} \xrightarrow{\mathrm{Cub}} H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X), \mathbb{Q}).$$

which sends the class of a cellular map $[f : S^{n+1} \to |\widehat{S}.(X)|]$ to $\mathrm{Cub}\, f_*([S^{n+1}])$.

If $f, f' : S^{n+1} \to |\widehat{S}.(X)|$ are homotopic with cellular homotopy h, then

$$dh_*[S^{n+1} \times I] = (-1)^{n+1}(f'_*[S^{n+1}] - f_*[S^{n+1}])$$

in $C_*(|\widehat{S}.(X)|)[-1]$.

Let

$$\begin{array}{ccc} \widehat{K}^T_n(X)_{\mathbb{Q}} & \xrightarrow{\mathrm{Cub}^s} & \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X), \widehat{\mathrm{ch}})_{\mathbb{Q}} \\ [(f : S^{n+1} \to |\widehat{S}.(X)|, \omega)] & \mapsto & [(\mathrm{Cub}\, f_*([S^{n+1}]), -\omega)]. \end{array}$$

This morphism is well defined. Indeed, let h be a cellular homotopy between two pairs (f, ω) and (f', ω'). Then, if we denote $\alpha = (-1)^{n+1}\,\mathrm{Cub}\, h_*([S^{n+1} \times I])$, we have

$$d(\alpha) = \mathrm{Cub}\, f'_*([S^{n+1}]) - \mathrm{Cub}\, f_*([S^{n+1}]),$$

and

$$\mathrm{ch}(\alpha) = \omega - \omega'.$$

Finally, consider the diagram

$$\begin{tikzcd} K_{n+1}(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] \arrow[d, "\cong"'] & \widetilde{\mathcal{D}}^{2p-n-1}(X,p) \arrow[r, "a"] \arrow[d, "="'] & \widehat{K}^T_n(X)_{\mathbb{Q}} \arrow[r, "\zeta"] \arrow[d, "\mathrm{Cub}^s"'] & K_n(X)_{\mathbb{Q}} \arrow[r] \arrow[d, "\cong"'] & 0 \\ H_{n+1}(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X))_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] & \widetilde{\mathcal{D}}^{2p-n-1}(X,p) \arrow[r, "a"] & \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X), \widehat{\mathrm{ch}})_{\mathbb{Q}} \arrow[r, "\zeta"] & H_n(\widetilde{\mathbb{Z}}\widehat{C}^s_*(X))_{\mathbb{Q}} \arrow[r] & 0 \end{tikzcd}$$

Since the rows are exact sequences, the statement of the proposition follows from the five lemma. □

Corollary 5.3.12. *Let X be a proper arithmetic variety over $\mathbb{Z}$. Then, for every $n \geq 0$, there is an isomorphism*

$$\widehat{K}_n^T(X)_{\mathbb{Q}} \xrightarrow{\cong} \widehat{H}_n(N\widehat{C}_*(X), \widehat{\mathrm{ch}})_{\mathbb{Q}}.$$

□

Product structure on rational arithmetic K-theory. Takeda, in [57], defines a product structure for $\widehat{K}_n^T(X)$ compatible with the Loday product of algebraic K-theory, and for which the morphism

$$\widehat{\mathrm{ch}} : \widehat{K}_n^T(X) \to \widehat{\mathcal{D}}^{2p-*}(X, p)$$

is a ring morphism (see loc. cit., proposition 6.8). Since there is a natural isomorphism

$$\widehat{K}_n(X) \cong \ker(\widehat{\mathrm{ch}} : \widehat{K}_n^T(X) \to \widehat{\mathcal{D}}^{2p-*}(X, p)),$$

there is an induced Loday product on $\widehat{K}_n(X)$.

In algebraic K-theory, the Adams operations are derived from the lambda operations by a polynomial relation. In order to do that, the product structure for $\bigoplus_{n\geq 0} K_n(X)$ is the one for which $\bigoplus_{n\geq 1} K_n(X)$ is a square zero ideal.

Therefore, we consider the product structure on $\bigoplus_{n\geq 0} \widehat{K}_n(X)$ for which $\bigoplus_{n\geq 1} \widehat{K}_n(X)$ is a square zero ideal and agrees with the Loday product otherwise.

After tensoralizing with $\mathbb{Q}$, and using the description of $\widehat{K}_n(X)_{\mathbb{Q}}$ via the isomorphism

$$\widehat{K}_n(X)_{\mathbb{Q}} \cong H_n(s(\widehat{\mathrm{ch}}), \mathbb{Q}),$$

the product is defined as follows.

Lemma 5.3.13. *Let $(\overline{E}, \alpha) \in \widehat{K}_0(X)_{\mathbb{Q}}$ and let $(\overline{F}, \beta) \in \widehat{K}_n(X)_{\mathbb{Q}}$ with $n \geq 0$. Then, by the product structure on $\widehat{K}_n(X)_{\mathbb{Q}}$ induced by the product structure on $\widehat{K}_n^T(X)_{\mathbb{Q}}$, we have*

$$(\overline{E}, \alpha) \otimes (\overline{F}, \beta) = (\overline{E} \otimes \overline{F}, \alpha \bullet \mathrm{ch}(\overline{F}) + \mathrm{ch}(\overline{E}) \bullet \beta - \alpha \bullet d_{\mathcal{D}}(\beta)) \in \widehat{K}_n(X)_{\mathbb{Q}}.$$

□

Remark 5.3.14. With the notation of the previous lemma, if $n > 0$ then $\overline{F}$ is an n-cube such that $d\overline{F} = 0$ and $\beta \in \bigoplus_{p\geq 0} \mathcal{D}^{2p-n-1}(X, p)$ is a differential form such that $\mathrm{ch}(\overline{F}) = d_{\mathcal{D}}(\beta)$. Hence,

$$\alpha \bullet \mathrm{ch}(\overline{F}) = \alpha \bullet d_{\mathcal{D}}(\beta)$$

and therefore,

$$(\overline{E}, \alpha) \otimes (\overline{F}, \beta) = (\overline{E} \otimes \overline{F}, \mathrm{ch}(\overline{E}) \bullet \beta).$$

5.4 Adams operations on higher arithmetic K-theory

In this section we construct the Adams operations on the higher arithmetic K-groups tensored by the rational numbers. The construction is adapted to both definitions of higher arithmetic K-groups.

5.4.1 Adams operations on hermitian cubes

Let X be a proper arithmetic variety over $\mathbb{Z}$. In this section we study the Adams operations on the complex of hermitian cubes, as defined in chapter 4.

In our current setting, we consider the transgression morphism constructed by means of projective lines. Recall that, for every n, the transgression morphism

$$\mathbb{Z}KC_n(X) \xrightarrow{T} \bigoplus_{m=0}^{n} \mathbb{Z}\,\mathrm{Sp}^{\mathbb{P}}_{n-m,m}(X)$$

was defined in section 4.3.4.

For every $k \geq 1$, the k-th Adams operation on the complex of cubes on X is defined as the composition

$$NC_*(X) \xrightarrow{T} \mathbb{Z}\,\mathrm{Sp}^{\mathbb{P}}_*(X) \xrightarrow{\Psi^k} \widetilde{\mathbb{Z}}C^{\widetilde{\mathbb{P}}}_*(X),$$

where T is the transgression morphism that assigns to every n-cube on X a collection of split cubes on $X \times (\mathbb{P}^1)^*$ and Ψ^k is the k-th Adams operation on split cubes.

When considering hermitian cubes, the concept of split cubes should be modified in order to incorporate the information given by the metric. Recall that the notion of split cubes was introduced in 4.2.6.

Let $\overline{E}$ be a hermitian n-cube on X. Then there is a naturally induced smooth hermitian metric on the cube $\mathrm{Sp}(\overline{E})$.

Definition 5.4.1. Let $\overline{E} \in \widehat{C}_n(X)$ be a hermitian n-cube and assume that there exists an isomorphism $f : \mathrm{Sp}(E) \to E$ making (E, f) a split n-cube. We say that $(\overline{E}, f)$ is *hermitian split*, if the isomorphism $f : \mathrm{Sp}(\overline{E}) \to \overline{E}$ is an isometry.

Denote by $\mathbb{Z}\widehat{\mathrm{Sp}}_*(X)$ the complex of hermitian split cubes on X. Let $\mathbb{Z}\widehat{\mathrm{Sp}}^{\mathbb{P}}_*(X)$ and $\widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X)$ be the chain complexes corresponding to $\mathbb{Z}\,\mathrm{Sp}^{\mathbb{P}}_*(X)$ and $\widetilde{\mathbb{Z}}C^{\widetilde{\mathbb{P}}}_*(X)$, respectively, by considering hermitian cubes on $X \times (\mathbb{P}^1)^*$.

Lemma 5.4.2. *Let $\overline{E} \in \widehat{C}_n(X)$ be a hermitian n-cube with canonical kernels. Then, $T(\overline{E})$ lies in $\mathbb{Z}\widehat{\mathrm{Sp}}^{\mathbb{P}}_*(X)$.*

Proof. It is a consequence of lemmas 4.3.15 and 4.3.5. □

Hence, composing with the morphism λ, there is a well-defined morphism

$$N\widehat{C}_*(X) \xrightarrow{\lambda} \mathbb{Z}K\widehat{C}_*(X) \xrightarrow{T} \mathbb{Z}\widehat{\mathrm{Sp}}^{\mathbb{P}}_*(X).$$

For every $k \geq 1$, the composition with the morphism

$$\Psi^k : \mathbb{Z}\widehat{\mathrm{Sp}}^{\mathbb{P}}_*(X) \to \widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X)$$

gives the k-th Adams operation on the complex of hermitian cubes on X.

5.4.2 Adams operations and the Beilinson regulator

Let X be a proper arithmetic variety over $\mathbb{Z}$.

Let

$$\Psi^k : \mathcal{D}^{2p-*}(X,p) \to \mathcal{D}^{2p-*}(X,p)$$

be the morphism that maps α to $k^p\alpha$. That is, we endow $\bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p)$ with the canonical λ-ring structure of lemma 1.3.28, corresponding to the graduation given by p.

We will define the Adams operations on the higher arithmetic K-groups of X from a commutative diagram of the form

$$\begin{tikzcd} N\widehat{C}_*(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "\Psi^k"'] & \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p) \arrow[d, "\Psi^k"] \\ \widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X) \arrow[r, "\mathrm{ch}"'] & \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p). \end{tikzcd} \tag{5.4.3}$$

We proceed as follows:

(1) We first define the bottom arrow $\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X) \to \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p)$.

(2) We show that there are isomorphisms

$$\begin{aligned} \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X), \widehat{\mathrm{ch}})_{\mathbb{Q}} &\cong \widehat{K}^T_n(X)_{\mathbb{Q}}, \\ H_n(s(\widehat{\mathrm{ch}}), \mathbb{Q}) &\cong \widehat{K}_n(X)_{\mathbb{Q}}, \end{aligned}$$

with $\widehat{\mathrm{ch}}$ the composition

$$\widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p) \to \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p).$$

(3) We prove that the diagram (5.4.3) is commutative.

Let $\overline{E} \in \widehat{C}_n(X \times (\mathbb{P}^1)^m)$ be a hermitian n-cube on $X \times (\mathbb{P}^1)^m$. We define

$$\mathrm{ch}_{n,m}(\overline{E}) := \frac{(-1)^{n(m+1)}}{(2\pi i)^n} \int_{(\mathbb{P}^1)^{n+m}} \mathrm{ch}(\mathrm{tr}_n(\lambda(\overline{E}))) \bullet T_{n+m} \in \bigoplus_{p\geq 0} \mathcal{D}^{2p-n-m}(X,p).$$

Proposition 5.4.4. *There is a chain morphism*

$$\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p),$$

which maps $\overline{E} \in \widehat{C}_n(X \times (\mathbb{P}^1)^m)$ *to* $\mathrm{ch}_{n,m}(\overline{E})$. *The composition*

$$K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} H_n(\widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X),\mathbb{Q}) \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} \mathcal{D}^{2p-n}(X,p),$$

is the Beilinson regulator.

Proof. First of all, observe that ch is well defined. Indeed, if $\overline{E} = p_i^*\overline{\mathcal{O}(1)} \otimes \overline{F}$, then $\mathrm{ch}(\overline{E}) \in \sigma_i \mathcal{D}^{2p-*}(X \times (\mathbb{P}^1)^{n-1}, p-1) + \omega_i \wedge \sigma_i \mathcal{D}^{2p-*-2}(X \times (\mathbb{P}^1)^{n-1}, p-1)$ and hence $\mathrm{ch}(\overline{E}) = 0$.

In order to prove that ch is a chain morphism, observe that ch factors as

$$\widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_{n,m}(X) \xrightarrow{\overline{\mathrm{ch}}} \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-n,m}_{\mathbb{P}}(X,p) \xrightarrow{\varphi} \bigoplus_{p\geq 0} H^{2p-n-m}_{\mathcal{D}}(X,\mathbb{R}(p)),$$

where φ is the quasi-isomorphism of proposition 3.5.7 and $\overline{\mathrm{ch}}(\overline{E}) = \overline{\mathrm{ch}}_{n,m}(\overline{E})$ is defined by

$$\overline{\mathrm{ch}}_{n,m}(\overline{E}) = (-1)^{n(m+1)}\,\mathrm{ch}(\mathrm{tr}_n(\lambda(\overline{E}))) \in \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p}(X \times (\mathbb{P}^1)^{m+n}, p),$$

for any $\overline{E} \in \widehat{C}_n(X \times (\mathbb{P}^1)^m)$. Hence, it is enough to see that $\overline{\mathrm{ch}}$ is a chain morphism.

Let $\overline{E} \in \widetilde{\mathbb{Z}}\widehat{C}_n(X \times (\mathbb{P}^1)^m)$. Since ch is a closed differential form, we have

$$\begin{aligned} d_s(\overline{\mathrm{ch}}_{n,m}(\overline{E})) &= (-1)^{nm}\delta\,\mathrm{ch}(\mathrm{tr}_n(\lambda(\overline{E}))) \\ &= \sum_{i=1}^{m}(-1)^{i+nm}\,\mathrm{ch}(\mathrm{tr}_n(\lambda(\delta_i^1\overline{E} - \delta_i^0\overline{E}))) \\ &\quad + \sum_{i=m+1}^{n+m}\sum_{j=0}^{2}(-1)^{i+j+nm}\,\mathrm{ch}(\mathrm{tr}_n(\lambda(\partial_{i-m}^j\overline{E}))) \\ &= (-1)^n\overline{\mathrm{ch}}_{n,m-1}(\delta\overline{E}) + \overline{\mathrm{ch}}_{n-1,m}(d\overline{E}), \end{aligned}$$

as desired.

Finally, since there is a commutative diagram

$$\begin{tikzcd} \widetilde{\mathbb{Z}}\widehat{C}_*(X) \arrow[dr, "\mathrm{ch}"] \arrow[dd, "i"', "\cong"] & \\ & \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p), \\ \widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X) \arrow[ur, "\mathrm{ch}"'] & \end{tikzcd}$$

the morphism ch induces the Beilinson regulator. $\square$

We have therefore constructed the bottom arrow of diagram (5.4.3).

For the next proposition, let $\widehat{\mathrm{ch}} : \widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X) \to \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p)$ be the composition of the morphism defined in proposition 5.4.4 with the natural projection $\bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p) \to \bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p)$.

Proposition 5.4.5. *There are isomorphisms*

$$\begin{aligned} \widehat{H}_n(\widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X), \widehat{\mathrm{ch}})_{\mathbb{Q}} &\cong \widehat{K}^T_n(X)_{\mathbb{Q}}, \\ H_n(s(\widehat{\mathrm{ch}}), \mathbb{Q}) &\cong \widehat{K}_n(X)_{\mathbb{Q}}, \end{aligned}$$

induced by the isomorphism $H_n(\widetilde{\mathbb{Z}}C^{\widetilde{\mathbb{P}}}_*(X), \mathbb{Q}) \cong K_n(X)_{\mathbb{Q}}$ of proposition 4.3.2.

Proof. Both isomorphisms are a consequence of proposition 4.3.2, and the five lemma using the exact sequences of lemma 5.3.2 and proposition 5.2.3. □

At this point, all that remains to see is that the diagram (5.4.3) is commutative. This will be a consequence of the next series of lemmas on the Koszul complex of hermitian cubes.

Recall that, as defined in section 4.2.3, the k-th Koszul complex of a vector bundle E is the exact sequence

$$0 \to \Psi^k(E)^0 \xrightarrow{\varphi_0} \dots \xrightarrow{\varphi_{k-1}} \Psi^k(E)^k \to 0$$

with

$$\Psi^k(E)^p = E \cdot \overset{p}{\ldots} \cdot E \otimes E \wedge \overset{k-p}{\ldots} \wedge E = S^p E \otimes \bigwedge\nolimits^{k-p} E.$$

Recall also that for any exact sequence of $(m-1)$-cubes

$$0 \to A^0 \xrightarrow{f^0} \cdots \xrightarrow{f^{j-1}} A^j \xrightarrow{f^j} \cdots \xrightarrow{f^{r-1}} A^r \to 0,$$

$\mu^j(A)$ is the short exact sequence of $(m-1)$-cubes defined by

$$\mu^j(A) \quad : \quad 0 \to \ker f^j \to A^j \to \ker f^{j+1} \to 0, \quad j = 0, \dots, r-1.$$

If $\overline{E}$ is a hermitian vector bundle, there is an induced metric on the tensor product $T^p\overline{E} = \overline{E} \otimes \overset{k}{\ldots} \otimes \overline{E}$. Since $S^p\overline{E}$ and $\bigwedge^p \overline{E}$ are subbundles of $T^p\overline{E}$, they inherit a hermitian metric from that on $\overline{E}$. Therefore, $\Psi^k(\overline{E})^p$ is a hermitian vector bundle for every k, p, with metric induced by the one on $\overline{E}$.

The following is a known result. We give a proof for completeness.

Lemma 5.4.6. *For all $k \geq 1$, the k-th Koszul complex of a vector bundle E is split, i.e. for all $0 \leq j \leq k-1$, $\mu^j(\Psi^k(E))$ is a split short exact sequence.*

Proof. It is sufficient to check that the lemma is true in a canonical way for the Koszul complex of a finite dimensional vector space.

Let E be a finite dimensional vector space and let $\mathfrak{S}_p$ be the group of permutations of p elements. Consider the inclusions

$$S^pE \xrightarrow{i_p} T^pE, \quad \text{and} \quad \bigwedge^p E \xrightarrow{j_p} T^pE$$

defined by

$$\begin{aligned} i_p(x_{i_1}\cdot \ldots \cdot x_{i_p}) &= \sum_{\sigma\in\mathfrak{S}_p} x_{\sigma(i_1)}\otimes\ldots\otimes x_{\sigma(i_p)}, \\ j_p(x_{i_1}\wedge\ldots\wedge x_{i_p}) &= \sum_{\tau\in\mathfrak{S}_p} (-1)^{|\tau|} x_{\tau(i_1)}\otimes\ldots\otimes x_{\tau(i_p)}. \end{aligned}$$

Observe that there are natural projections

$$\begin{array}{ccccccc} T^pE & \xrightarrow{\pi_p} & S^pE, & \text{and} & T^pE & \xrightarrow{\rho_p} & \bigwedge^p E \\ x_{i_1}\otimes\ldots\otimes x_{i_p} & \mapsto & x_{i_1}\cdot\ldots\cdot x_{i_p}, & & x_{i_1}\otimes\ldots\otimes x_{i_p} & \mapsto & x_{i_1}\wedge\ldots\wedge x_{i_p}. \end{array}$$

For every p, the morphisms φ_p in the Koszul complex is given by the composition:

$$\varphi_p : S^pE\otimes \bigwedge^{k-p} E \xrightarrow{i_p\otimes j_{k-p}} T^kE \xrightarrow{\pi_{p+1}\otimes\rho_{k-p-1}} S^{p+1}E\otimes\bigwedge^{k-p-1} E.$$

To see that for every p $\mu^p(\Psi^k(E))$ is split, we will find a section for the short exact sequence

$$0\to\ker\varphi_p\to S^pE\otimes\bigwedge^{k-p} E \xrightarrow{\varphi_p} \operatorname{im}\varphi_p\to 0.$$

Define

$$i'_p = \frac{1}{p!}i_p, \quad j'_p = \frac{1}{p!}j_p, \quad \pi'_p = \frac{1}{p!}\pi_p, \quad \text{and} \quad \rho'_p = \frac{1}{p!}\rho_p.$$

Observe that π'_p and ρ'_p are retractions of i_p and j_p respectively, and that π_p and ρ_p are retractions of i'_p and j'_p respectively. Let

$$\psi_p : S^{p+1}E\otimes\bigwedge^{k-p-1} E \xrightarrow{i'_{p+1}\otimes j'_{k-p-1}} T^kE \xrightarrow{\pi'_p\otimes\rho'_{k-p}} S^pE\otimes\bigwedge^{k-p} E.$$

We will see that $\frac{1}{k}\psi_p$ is a section of φ_p, i.e. that for every $z\in\operatorname{im}\varphi_p$, we have $\varphi_p\psi_p(z) = k\cdot z$. That is, writing $z=\varphi_p(x)$, we have to see that for every $x\in S^pE\otimes\bigwedge^{k-p}E$,

$$\varphi_p\psi_p\varphi_p(x) = k\varphi_p(x).$$

Let $x = x_{i_1}\cdot\ldots x_{i_p}\otimes x_{j_1}\wedge\cdots\wedge x_{j_{k-p}}\in S^pE\otimes\bigwedge^{k-p}E$. Then,

$$\varphi_p(x) = \sum_{\substack{\sigma\in\mathfrak{S}_p\\ \tau\in\mathfrak{S}_{k-p}}} (-1)^{|\tau|} x_{\sigma(i_1)}\cdot\ldots\cdot x_{\sigma(i_p)}\cdot x_{\tau(j_1)}\otimes x_{\tau(j_2)}\wedge\ldots\wedge x_{\tau(j_{k-p})}.$$

Observe that if $\tau(j_1)=l$, then there is a decomposition $\tau=\tau'\rho$ with $\tau',\rho\in\mathfrak{S}_p$, $\rho(1,\ldots,k-p)=(l,1,\ldots,\widehat{l},\ldots,k-p)$ and $\tau'(1)=1$. The signature of ρ is $(-1)^{l-1}$. Hence,

$$\begin{aligned}\varphi_p(x) &= \sum_{\tau\in\mathfrak{S}_{k-p}}(-1)^{|\tau|}p!\; x_{i_1}\cdot\ldots\cdot x_{i_p}\cdot x_{\tau(j_1)}\otimes x_{\tau(j_2)}\wedge\ldots\wedge x_{\tau(j_{k-p})}\\ &= \sum_{l=1}^{k-p}(-1)^{l-1}p!(k-p-1)!\; x_{i_1}\cdot\ldots\cdot x_{i_p}\cdot x_{j_l}\otimes x_{j_1}\wedge\ldots\widehat{x_{j_l}}\ldots\wedge x_{j_{k-p}}.\end{aligned}$$

Write $\beta=\frac{(-1)^{l-1}}{(k-p)(p+1)!}$. Then, proceeding in the same way, we obtain:

$$\begin{aligned}\psi_p\varphi_p(x) &= \sum_{l=1}^{k-p}\beta\sum_{\sigma\in\mathfrak{S}_{p+1}}x_{\sigma(i_1)}\cdot\ldots\cdot x_{\sigma(i_p)}\otimes x_{\sigma(j_l)}\wedge x_{j_1}\wedge\ldots\widehat{x_{j_l}}\ldots\wedge x_{j_{k-p}}\\ &= \sum_{l=1}^{k-p}\beta\Big[x_{i_1}\cdot\ldots\cdot x_{i_p}\otimes x_{j_l}\wedge x_{j_1}\wedge\ldots\widehat{x_{j_l}}\ldots\wedge x_{j_{k-p}}\\ &\quad+\sum_{t=1}^{p}x_{i_1}\cdot\ldots\widehat{x_{i_t}}\ldots\cdot x_{i_p}\cdot x_{j_l}\otimes x_{i_t}\wedge x_{j_1}\wedge\ldots\widehat{x_{j_l}}\ldots\wedge x_{j_{k-p}}\Big].\end{aligned}$$

Observe that

$$x=(p+1)\sum_{l=1}^{k-p}\beta\; x_{i_1}\cdot\ldots\cdot x_{i_p}\otimes x_{j_l}\wedge x_{j_1}\wedge\ldots\widehat{x_{j_l}}\ldots\wedge x_{j_{k-p}}.$$

Hence, if we write

$$y=\sum_{l=1}^{k-p}\beta\sum_{t=1}^{p}x_{i_1}\cdot\ldots\widehat{x_{i_t}}\ldots\cdot x_{i_p}\cdot x_{j_l}\otimes x_{i_t}\wedge x_{j_1}\wedge\ldots\widehat{x_{j_l}}\ldots\wedge x_{j_{k-p}},$$

we have

$$\psi_p\varphi_p(x)=\frac{x}{p+1}+y.$$

Finally, we have

$$\varphi_p(y)=(a)+(b),$$

with

$$\begin{aligned}(a) &= \sum_{l=1}^{k-p}\frac{(-1)^{l-1}p!(k-p-1)!}{(k-p)(p+1)}\sum_{t=1}^{p}x_{i_1}\cdot\ldots\widehat{x_{i_t}}\ldots x_{i_p}\cdot x_{j_l}x_{i_t}\otimes x_{j_1}\wedge\ldots\widehat{x_{j_l}}\ldots\wedge x_{j_{k-p}}\\ &= \frac{p}{(p+1)(k-p)}\varphi_p(x),\end{aligned}$$

and, denoting $\alpha = \frac{p!(k-p-1)!}{(k-p)(p+1)}$ and $f = x_{i_1}\cdot \ldots \widehat{x_{i_t}} \ldots \cdot x_{i_p}$,

$$\begin{aligned}(b) &= \sum_{t=1}^{p}\sum_{l=1}^{k-p}\sum_{s=1}^{l-1}(-1)^{l-s-2}\alpha f\cdot x_{j_l}\cdot x_{j_s}\otimes x_{i_t}\wedge x_{j_1}\wedge\ldots\widehat{x_{j_s}}\ldots\widehat{x_{j_l}}\ldots\wedge x_{j_{k-p}} \\ &\quad +\sum_{t=1}^{p}\sum_{l=1}^{k-p}\sum_{s=l+1}^{k-p}(-1)^{l-s-1}\alpha f\cdot x_{j_l}\cdot x_{j_s}\otimes x_{i_t}\wedge x_{j_1}\wedge\ldots\widehat{x_{j_l}}\ldots\widehat{x_{j_s}}\ldots\wedge x_{j_{k-p}} \\ &= 0.\end{aligned}$$

Therefore,

$$\varphi_p\psi_p\varphi_p(x) = \frac{1}{p+1}\varphi_p(x) + \frac{p}{(p+1)(k-p)}\varphi_p(x) = k\varphi_p(x).$$

Hence, the morphism $\psi'_p = \frac{1}{k}\psi_p$, is a section of φ_p on $\operatorname{im}\varphi_p$ and therefore

$$S^pE\otimes\bigwedge^{k-p}E\cong\ker\varphi_p\oplus\operatorname{im}\varphi_p.$$

□

Lemma 5.4.7. *Let $\overline{E}$ be a hermitian vector bundle. Then, for all $k\geq 1$ and for all $0\leq j\leq k-1$,*

$$\operatorname{ch}(\mu^j(\Psi^k(E))) = 0.$$

Proof. In the proof of this lemma we keep the notation introduced in the proof of last lemma. Fix $k\geq 1$ and $0\leq j\leq k-1$. By the previous lemma, $\mu^j(\Psi^k(E))$ is a split short exact sequence, that is, there is an isomorphism

$$S^p\overline{E}\otimes\bigwedge^{k-p}\overline{E}\cong\ker\varphi_p\oplus\operatorname{im}\varphi_p.$$

In order to prove that $\operatorname{ch}(\mu^j(\Psi^k(E))) = 0$, it is enough to check that the hermitian metrics on $\ker\varphi_p$ and $\operatorname{im}\varphi_p$ are induced by the hermitian metric of $S^p\overline{E}\otimes\bigwedge^{k-p}\overline{E}$ up to a constant.

Clearly, the hermitian metric on $\ker\varphi_p$ is induced by the hermitian metric of $S^p\overline{E}\otimes\bigwedge^{k-p}\overline{E}$. The hermitian metric on $\operatorname{im}\varphi_p$ is induced by the metric of $S^{p+1}\overline{E}\otimes\bigwedge^{k-p-1}\overline{E}$, i.e. by the metric of $T^k\overline{E}$. Then, the claim follows from the (up to a constant factor) commutative square of inclusion morphisms

$$\begin{tikzcd} & S^p\overline{E}\otimes\bigwedge^{k-p}\overline{E} \arrow[d, hook, "i_p\otimes j_{k-p}"] \\ \operatorname{im}\varphi_p \arrow[ur, hook, "\psi^k"] \arrow[r, hook, "i_{p+1}\otimes j_{k-p-1}"'] & T^k\overline{E}. \end{tikzcd}$$

□

The next corollary follows from the definition of the Adams operations of a split n-cube and the last lemma.

Corollary 5.4.8. *Let $n > 0$ and let $\overline{E}$ be a hermitian split n-cube on X. Then, $\Psi^k(\overline{E}) \in \widetilde{\mathbb{Z}}\widehat{C}_n^{\widetilde{\mathbb{P}}}(X)$ is a sum of hermitian split cubes.*

□

Lemma 5.4.9. *Let X be a smooth proper complex variety and let $\overline{E}$ be a hermitian vector bundle over X. Then,*

$$\Psi^k \operatorname{ch}(\overline{E}) = \operatorname{ch} \Psi^k(\overline{E})$$

in the group $\bigoplus_{p\geq 0} \mathcal{D}^{2p}(X,p)$.

Proof. In [26], Gillet and Soulé proved that

$$\lambda^k \operatorname{ch} = \operatorname{ch} \lambda^k.$$

Let $\overline{E}$ be a hermitian vector bundle and let

$$\psi^k(\overline{E}) = N_k(\lambda^1(\overline{E}), \dots, \lambda^k(\overline{E})),$$

with N_k being the k-th Newton polynomial. These are the Adams operations associated to the lambda operations λ^k on vector bundles. It follows that

$$\psi^k(\operatorname{ch}(\overline{E})) = \operatorname{ch}(\psi^k(\overline{E})).$$

Observe that by definition,

$$\Psi^k(\operatorname{ch}(\overline{E})) = \psi^k(\operatorname{ch}(\overline{E})).$$

Let $\Psi^k(\overline{E})$ be the secondary Euler characteristic class of the Koszul complex of $\overline{E}$. By remark 1.3.26, in the quotient group $K_0(X)$, we have

$$\psi^k(\overline{E}) = \Psi^k(\overline{E}).$$

This means that there exist short exact sequences $s_1, \dots, s_r$ such that

$$\psi^k(\overline{E}) - \Psi^k(\overline{E}) = \sum_{i=1}^{r} d(s_i).$$

One can see that the short exact sequences s_i can be chosen to be of the form

$$\mu^p(\Psi^{k_i}(\overline{E_i})) \otimes \overline{A}_i,$$

with $\overline{E_i}, \overline{A}_i$ some hermitian vector bundles and some indices k_i, p. By last lemma, $\operatorname{ch}(s_i) = 0$ for all i and hence

$$\operatorname{ch}(\Psi^k(\overline{E})) = \operatorname{ch}(\psi^k(\overline{E})).$$

□

The next lemma gives the key point in the proof of theorem 5.4.11. It states that, if $n > 0$, the Chern character form of a hermitian split n-cube is zero. In this way, all the terms "added" by the transgression morphism in order to be able to define a chain morphism representing the Adams operations, will be cancelled after applying the morphism ch, and only the initial desired term $\Psi^k(\operatorname{tr}_n E)$ will remain.

Lemma 5.4.10. *Let $n > 0$ and let $\overline{E} \in \mathbb{Z}\widehat{C}_n(X \times (\mathbb{P}^1)^m)$ be a hermitian split n-cube. Then,*

$$\operatorname{ch}_{n,m}(\overline{E}) = 0$$

in $\bigoplus_{p\geq 0} \mathcal{D}^{2p-n-m}(X,p)$.

Proof. Recall that if $\overline{E}$ is a hermitian n-cube

$$\operatorname{tr}_n(\overline{E}) = \operatorname{tr}_1 \operatorname{tr}_{n-1}(\overline{E}) = \operatorname{tr}_1\big(\operatorname{tr}_{n-1}(\partial_n^0\overline{E}) \to \operatorname{tr}_{n-1}(\partial_n^1\overline{E}) \to \operatorname{tr}_{n-1}(\partial_n^2\overline{E})\big).$$

Then, if $\overline{E}$ is split, the sequence

$$\operatorname{tr}_{n-1}(\partial_n^0\overline{E}) \to \operatorname{tr}_{n-1}(\partial_n^1\overline{E}) \to \operatorname{tr}_{n-1}(\partial_n^2\overline{E})$$

is orthogonally split. The result follows from 5.1.11. □

Theorem 5.4.11. *Let X be a proper arithmetic variety over $\mathbb{Z}$. The diagram*

$$\begin{tikzcd} N\widehat{C}_*(X) \arrow[r, "\operatorname{ch}"] \arrow[d, "\Psi^k"'] & \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p) \arrow[d, "\Psi^k"] \\ \widetilde{\mathbb{Z}}\widehat{C}^{\widetilde{\mathbb{P}}}_*(X) \arrow[r, "\operatorname{ch}"'] & \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p) \end{tikzcd}$$

is commutative.

Proof. Let $\overline{E}$ be a hermitian n-cube. By lemma 5.4.2, $T(\overline{E})$ is a sum of hermitian split cubes. That is, if $m < n$, $T_{n-m,m}(\overline{E})$ is a hermitian split $(n-m)$-cube. By corollary 5.4.8, lemma 5.4.10, lemma 5.4.9 and the definition of Ψ^k on differential forms, we have

$$\begin{aligned} \operatorname{ch}(\Psi^k(\overline{E})) &= \frac{(-1)^n}{(2\pi i)^n}\int_{(\mathbb{P}^1)^n} \operatorname{ch}(\Psi^k(\operatorname{tr}_n(\lambda(\overline{E})))) \wedge T_n \\ &= \frac{(-1)^n}{(2\pi i)^n}\int_{(\mathbb{P}^1)^n} \Psi^k(\operatorname{ch}(\operatorname{tr}_n(\lambda(\overline{E})))) \wedge T_n \\ &= \Psi^k(\operatorname{ch}(\overline{E})). \end{aligned}$$

□

5.4.3 Adams operations on higher arithmetic K-theory

Let X be a proper arithmetic variety over $\mathbb{Z}$. Proposition 5.4.5 and theorem 5.4.11 enable us to define, for every $k \geq 0$, the *Adams operation on higher arithmetic K-groups*:

- Since the simple complex associated to a morphism is a functorial construction, for every k there is an Adams operation morphism on the Deligne-Soulé higher arithmetic K-groups:
$$\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}}.$$

- By proposition 5.3.2 for every k there is an Adams operation morphism on the Takeda higher arithmetic K-groups:
$$\Psi^k : \widehat{K}^T_n(X)_{\mathbb{Q}} \to \widehat{K}^T_n(X)_{\mathbb{Q}}.$$

We have proved the following theorems.

Theorem 5.4.12 (Adams operations)**.** *Let X be a proper arithmetic variety over $\mathbb{Z}$ and let $\widehat{K}_n(X)$ be the n-th Deligne-Soulé arithmetic K-group. There are Adams operations*

$$\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}},$$

compatible with the Adams operations in $K_n(X)_{\mathbb{Q}}$ and $\bigoplus_{p\geq 0} H^{2p-n}_{\mathcal{D}}(X, \mathbb{R}(p))$, by means of the morphisms a and ζ.

Theorem 5.4.13 (Adams operations)**.** *Let X be a proper arithmetic variety over $\mathbb{Z}$ and let $\widehat{K}^T_n(X)$ be the n-th arithmetic K-group defined by Takeda in [57]. Then, for every $k \geq 0$ there exists an Adams operation morphism $\Psi^k : \widehat{K}^T_n(X)_{\mathbb{Q}} \to \widehat{K}^T_n(X)_{\mathbb{Q}}$ such that the following diagram is commutative:*

$$\begin{tikzcd}
K_{n+1}(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] \arrow[d, "\Psi^k"'] & \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X,p) \arrow[r, "a"] \arrow[d, "\Psi^k"'] & \widehat{K}^T_n(X)_{\mathbb{Q}} \arrow[r, "\zeta"] \arrow[d, "\Psi^k"'] & K_n(X)_{\mathbb{Q}} \arrow[r] \arrow[d, "\Psi^k"'] & 0 \\
K_{n+1}(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] & \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X,p) \arrow[r, "a"] & \widehat{K}^T_n(X)_{\mathbb{Q}} \arrow[r, "\zeta"] & K_n(X)_{\mathbb{Q}} \arrow[r] & 0.
\end{tikzcd}$$

Moreover, the diagram

$$\begin{tikzcd}
\widehat{K}^T_n(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"] \arrow[d, "\Psi^k"'] & \bigoplus_{p\geq 0} \mathcal{D}^{2p-n}(X,p) \arrow[d, "\Psi^k"] \\
\widehat{K}^T_n(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}"'] & \bigoplus_{p\geq 0} \mathcal{D}^{2p-n}(X,p)
\end{tikzcd}$$

is commutative

Lambda operations. Let X be a proper arithmetic variety over $\mathbb{Z}$. Consider the product structure on $\widehat{K}_*(X)_{\mathbb{Q}}$ defined in before lemma 5.3.13. Then, by equation (1.3.24), there are induced lambda operations

$$\lambda^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}}.$$

Corollary 5.4.14 (Pre-λ-ring)**.** *Let X be a proper arithmetic variety over $\mathbb{Z}$. Then, $\widehat{K}_*(X)_{\mathbb{Q}}$ is a pre-λ-ring. Moreover, there is a commutative square*

$$\begin{array}{ccc} \widehat{K}_n(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} \\ \Big\downarrow{\scriptstyle \lambda^k} & & \Big\downarrow{\scriptstyle \lambda^k} \\ \widehat{K}_n(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} \end{array}$$

Proof. The first statement is a consequence of lemma 1.3.27. The diagram is commutative since the Adams and lambda operations in $K_*(X)$ are related under the product structure on $K_*(X)$ which is zero in $\bigoplus_{n\geq 1} K_n(X)$. □

Proposition 5.4.15. *Let X be a proper arithmetic variety over $\mathbb{Z}$. The Adams operations given here for $\widehat{K}_0(X)_{\mathbb{Q}}$ agree with the ones given by Gillet and Soulé in [26].*

Proof. It follows from the definition. □

Consider now the product structure in $\bigoplus_{n\geq 0} \widehat{K}^T_n(X)_{\mathbb{Q}}$ having $\bigoplus_{n\geq 1} \widehat{K}^T_n(X)_{\mathbb{Q}}$ as a zero square ideal and agrees with the product defined by Takeda in [57] otherwise.

Corollary 5.4.16 (Pre-λ-ring)**.** *Let X be a proper arithmetic variety over $\mathbb{Z}$. Then, $\widehat{K}^T_*(X)_{\mathbb{Q}}$ is a pre-λ-ring. Moreover, there is a commutative square*

$$\begin{array}{ccc} \widehat{K}^T_n(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} \\ \Big\downarrow{\scriptstyle \lambda^k} & & \Big\downarrow{\scriptstyle \lambda^k} \\ \widehat{K}^T_n(X)_{\mathbb{Q}} & \xrightarrow{\zeta} & K_n(X)_{\mathbb{Q}} \end{array}$$

Proof. The proof is analogous to the proof of last corollary. □

Remark 5.4.17. In order to prove that the pre-λ-ring structure on $\widehat{K}_*(X)_{\mathbb{Q}}$ given here is actually a λ-ring structure, it is necessary to find precise exact sequences relating, at the level of vector bundles, the equalities in $K_0(X)$

$$\begin{aligned} \Psi^k(E\otimes F) &= \Psi^k(E)\otimes \Psi^k(F), \\ \Psi^k(\Psi^l(E)) &= \Psi^{kl}(E). \end{aligned}$$

This implies finding formulas for

$$\bigwedge^k(E\otimes F), \qquad \bigwedge^k\left(\bigwedge^l(E)\right),$$

in terms of tensor and exterior products. The theory of Schur functors, gives a formula for the first term. However, the second formula is an open problem. Nevertheless, even for the first equality, when we try to apply the formulas to our concrete situation, the combinatorics become really complicated.

Apèndix A

Resum en català

La conjectura de Mordell assegura que hi ha un nombre finit de punts racionals en una corba no singular sobre $\mathbb{Q}$ i de gènere més gran que 1. La versió geometrica d'aquesta conjectura fou demostrada per Manin l'any 1963 (vegeu [44]), usant la connexió de Gauss-Manin. Això suggeria que les teories geomètriques estaven més desenvolupades que les aritmètiques. La teoria d'Arakelov fou introduïda per Arakelov a [3], per tal de donar anàlegs aritmètics als resultats de geometria algebraica. Arakelov va donar una nova definició de la classe d'un divisor sobre un model no singular d'una corba algebraica definida sobre un cos de nombres algebraics. Després, va definir una teoria d'intersecció per aquestes classes de divisors, tot seguint la teoria d'intersecció de divisors en geometria algebraica.

La idea és que es pot compactificar una corba definida sobre l'anell d'enters d'un cos de nombres considerant funcions de Green en la corba complexa associada. Aquest treball inicial en superfícies aritmètiques va ser millorat, entre altres, per Deligne [16], Szpiro [56] i Faltings [19]. Aquests treballs donaren resultats en superfícies aritmètiques com la fórmula d'adjunció, el teorema de l'índex de Hodge i el teorema de Riemann-Roch. Faltings, a [18], va ser el primer de demostrar la conjectura de Mordell. Vojta, a [58], donà una demostració basada en les eines de la teoria d'Arakelov.

Aquests treballs foren generalitzats a dimensions superiors per Gillet i Soulé, a [24], on definiren una teoria d'intersecció per varietats aritmètiques. Aquest article és el punt de partida d'un programa amb la finalitat d'obtenir una teoria d'intersecció aritmètica, resseguint la teoria d'intersecció algebraica, però adequada per varietats aritmètiques. En un inici, aquest programa incloïa la definició de grups de Chow aritmètics dotats d'un producte d'intersecció, la definició d'un grup K_0 aritmètic i la definició de classes característiques amb els corresponents teoremes de Riemann-Roch.

El programa hauria de prosseguir amb el desenvolupament d'una teoria d'intersecció aritmètica superior, que hauria d'incloure la definició de grups de Chow aritmètics superiors equipats amb un producte, la definició de K-teoria aritmètica superior, la definició de classes característiques i teoremes de Riemann-Roch superiors.

Tot seguit fem un breu repàs al programa d'Arakelov i expliquem la contribució d'aquesta tesi en la seva completació. Comencem recordant els anàlegs algebraics.

Teoria d'intersecció algebraica. Sigui X una varietat algebraica equidimensional i sigui $CH^p(X)$ el *grup de Chow* de cicles algebraic de codimensió p. Hi ha vàries estratègies per equipar-lo amb un producte

$$CH^p(X) \otimes CH^q(X) \xrightarrow{\cdot} CH^{p+q}(X).$$

La primera teoria es basa en el "moving lemma". Donada la classe de dos subvarietats irreductibles, el mètode consisteix en trobar representants que intersequen pròpiament. Aquesta estratègia és vàlida per a esquemes quasi-projectius sobre un cos. Una altra tècnica, desenvolupada per Fulton i MacPherson, es basa en la deformació al con normal. En aquest cas, no cal que l'esquema sigui quasi-projectiu i a més, la teoria és vàlida per esquemes definits sobre l'espectre d'un domini de Dedekind.

Alternativament, a [23], Gillet i Soulé mostraren que la teoria d'intersecció es pot desenvolupar transferint el producte dels grups de K-teoria algebraica d'un esquema regular noeterià X als grups de Chow. Aquesta aproximació es basa en l'isomorfisme graduat:

$$\bigoplus_{p\geq 0} K_0(X)^{(p)}_{\mathbb{Q}} \cong K_0(X)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} CH^p(X)_{\mathbb{Q}}.$$

Aquí, les peces $K_0(X)^{(p)}_{\mathbb{Q}}$ són els espais de vectors propis de les operacions d'Adams Ψ^k a $K_0(X)_{\mathbb{Q}}$ i "ch" és el caràcter de Chern.

La relació de commutació del caràcter de Chern amb el push-forward ve donada pel teorema de Grothendieck-Riemann-Roch. Sigui Td la classe de Todd del fibrat tangent sobre una varietat algebraica. Siguin X, Y esquemes regulars quasi-projectius i plans sobre l'espectre d'un anell de Dedekind S, i sigui $f : X \to Y$ un S-morfisme pla i projectiu. Aleshores, pel teorema de Grothendieck-Riemann-Roch, hi ha un diagrama commutatiu

$$\begin{tikzcd} K_0(X) \arrow[r, "\mathrm{Td}(X)\cdot \mathrm{ch}(\cdot)"] \arrow[d, "f_*"'] & \bigoplus_{p\geq 0} CH^p(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{Td}(Y)\cdot \mathrm{ch}(\cdot)"] & \bigoplus_{p\geq 0} CH^p(Y)_{\mathbb{Q}}. \end{tikzcd}$$

A [7], Bloch va desenvolupar una teoria de *grups de Chow algebraics superiors* per varietats algebraiques X llises sobre un cos. Aquests grups es denoten per $CH^p(X, n)$, per $n, p \geq 0$. Bloch demostrà que hi ha un isomorfisme

$$\bigoplus_{p\geq 0} K_n(X)^{(p)}_{\mathbb{Q}} \cong K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X, n)_{\mathbb{Q}},$$

i va donar una estructura multiplicativa a $CH^*(X, *)$, que es basa en el "moving lemma".

Aquesta teoria s'establí com a candidata a la cohomologia motívica. Més tard, s'han proposat altres candidats a cohomologia motívica, que són vàlids per a classes d'esquemes més grans. Sota certes condicions, les noves definicions coincideixen amb els grups de Chow algebraics superiors. Per aquesta raó, els grups de Chow de Bloch s'han mantingut com una descripció bàsica i senzilla de la cohomologica motívica per varietats llises sobre certs cossos.

Grups de Chow aritmètics i teoria d'intersecció aritmètica. Tal i com hem mencionat més amunt, l'aconteixement de la teoria d'intersecció aritmètica es deu a Gillet i Soulé a [24]. A loc. cit., una *varietat aritmètica* és un esquema regular, quasi-projectiu i pla sobre un anell aritmètic. Sigui X una varietat aritmètica. Un *cicle aritmètic* en X és una parella (Z, g), amb Z un cicle algebraic i g una corrent de Green per Z, això és, una corrent en la varietat complexa associada a X que satisfà la relació

$$dd^c g + \delta_Z = [\omega].$$

Aquí, ω és una forma diferencial llisa i δ_Z la current associada a X. Aleshores, el *grup de Chow aritmètic* $\widehat{CH}^*(X)$ es defineix com el grup abelià lliure generat pels cicles aritmètics quocient una certa relació d'equivalència.

Sigui X una varietat aritmètica i $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ la conjugació complexa. Denotem per $E^{p,q}(X)$ l'espai vectorial de les formes diferencials ω en $X(\mathbb{C})$, a valors complexos de tipus (p, q), que satisfan la relació $F_\infty^* \omega = (-1)^p \omega$. Denotem per $\widetilde{E}^{p,p}(X)$ el quocient de $E^{p,p}(X)$ per $(\operatorname{im} \partial + \operatorname{im} \bar{\partial})$.

Gillet i Soulé van provar les següents propietats:

(i) Hi ha una successió exacta:

$$CH^{p-1,p}(X) \xrightarrow{\rho} \widetilde{E}^{p-1,p-1}(X) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0, \tag{1}$$

on $CH^{p-1,p}(X)$ és el terme $E_2^{p-1,-p}(X)$ de la successió espectral de Quillen (vegeu [48], §7) i ρ és el regulador de Beilinson (llevat un factor constant).

(ii) Hi ha un producte

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \xrightarrow{\cdot} \widehat{CH}^{p+q}(X)_{\mathbb{Q}}$$

amb el qual $\bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$ és una $\mathbb{Q}$-àlgebra graduada commutativa i unitària.

(iii) Si X, Y són projectius i $f : X \to Y$ és un morfisme, existeix un pull-back

$$f^* : \widehat{CH}^p(Y) \to \widehat{CH}^p(X).$$

Si f és propi, X, Y són equidimensionals i $f_{\mathbb{Q}} : X_Q \to Y_{\mathbb{Q}}$ és llisa, hi ha un push-forward

$$f_* : \widehat{CH}^p(X) \to \widehat{CH}^{p-\delta}(Y),$$

on $\delta = \dim X - \dim Y$. A més, la fórmula de la projecció se satisfà.

A [25] i [26], Gillet i Soulé continuaren el projecte tot definint *classes característiques* per un fibrat vectorial hermític sobre una varietat aritmètica X. Per tal de definir el *caràcter de Chern aritmètic* "ch", van introduir el *grup* K_0 *aritmètic*, $\widehat{K}_0(X)$, i van demostrar que "ch" indueix un isomorfisme entre $\widehat{K}_0(X)_{\mathbb{Q}}$ i $\bigoplus_{p\geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$. Tot seguit repassem breument la definició de $\widehat{K}_0(X)$ i "ch".

Sigui X una varietat aritmètica. Un *fibrat vectorial hermític* $\overline{E} = (E, h)$ sobre X és un feix localment lliure de rang finit en X juntament amb una mètrica hermítica en el fibrat holomòrfic associat.

Sigui $\overline{E}$ un fibrat vectorial hermític sobre X. Aleshores, existeix un caràcter de Chern aritmètic

$$\widehat{\mathrm{ch}}(\overline{E}) \in \bigoplus_{p\geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$$

caracteritzat per cinc propietats. En concret, per les propietats de functorialitat, additivitat, multiplicativitat, compatibilitat amb les formes de Chern i per una condició de normalització. A més, per a tota successió exacta de fibrats vectorials hermítics $\epsilon : 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$, el caràcter de Chern satisfà:

$$\widehat{\mathrm{ch}}(\overline{E}) = \widehat{\mathrm{ch}}(\overline{S}) + \widehat{\mathrm{ch}}(\overline{Q}) - (0, \widetilde{\mathrm{ch}}(\epsilon)),$$

on $\widetilde{\mathrm{ch}}(\epsilon)$ és la forma de Bott-Chern secondària de ϵ. Aquest fet porta a la següent definició de $\widehat{K}_0(X)$. Sigui $\widehat{K}_0(X)$ el grup generat per parelles $(\overline{E}, \alpha)$, amb $\alpha \in \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-1}(X,p)$, quocient la relació

$$(\overline{S}, \alpha') + (\overline{Q}, \alpha'') = (\overline{E}, \alpha' + \alpha'' + \widetilde{\mathrm{ch}}(\epsilon)),$$

per cada successió exacta ϵ com més amunt. Aquest grup forma part de la successió exacta

$$K_1(X) \xrightarrow{\rho} \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-1}(X,p) \to \widehat{K}_0(X) \to K_0(X) \to 0, \tag{2}$$

amb ρ el regulador de Beilinson (llevat un factor constant).

Aleshores, el caràcter de Chern indueix un isomorfisme

$$\widehat{\mathrm{ch}} : \widehat{K}_0(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}.$$

Tal i com és el cas en la situació algebraica, aquest isomorfisme té en compte la descomposició graduada de $\widehat{K}_0(X)$ donada per les operacions d'Adams. Això és, $\widehat{K}_0(X)$ té una estructura de pre-λ-anell de manera que $\widehat{\mathrm{ch}}$ indueix un isomorfisme entre els espais de vectors propis de $\widehat{K}_0(X)_{\mathbb{Q}}$ donats per les operacions d'Adams i els grups de Chow aritmètics:

$$\widehat{\mathrm{ch}} : \widehat{K}_0(X)^{(p)}_{\mathbb{Q}} \xrightarrow{\cong} \widehat{CH}^p(X)_{\mathbb{Q}}.$$

Gillet i Soulé, juntament amb els resultats de Bismut i els seus col.laboradors, van demostrar un *teorema de Grothendieck-Riemann-Roch aritmètic* (vegeu [27] i [22]). Faltings (vegeu [19] i [20]) va donar una altra aproximació al teorema de Grothendieck-Riemann-Roch.

Sigui $\widehat{\mathrm{Td}}$ la classe de Todd aritmètica del fibrat tangent d'una varietat aritmètica, siguin X, Y varietats aritmètiques i sigui $f : X \to Y$ un morfisme projectiu i pla de varietats aritmètiques, llis sobre els nombres racionals. Aleshores, el *teorema de*

Grothendieck-Riemann-Roch aritmètic afirma que hi ha un diagrama commutatiu:

$$\begin{tikzcd}
\widehat{K}_0(X)_{\mathbb{Q}} \arrow[r, "\widehat{\mathrm{Td}}(X)\cdot\widehat{\mathrm{ch}}(\cdot)"] \arrow[d, "f_*"'] & \bigoplus_{p\geq 0}\widehat{CH}^p(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\
\widehat{K}_0(Y)_{\mathbb{Q}} \arrow[r, "\widehat{\mathrm{Td}}(Y)\cdot\widehat{\mathrm{ch}}(\cdot)"] & \bigoplus_{p\geq 0}\widehat{CH}^p(Y)_{\mathbb{Q}}.
\end{tikzcd}$$

A [13], Burgos va donar una definició alternativa dels grups de Chow aritmètics, que consisteix en considerar un espai diferent de formes de Green associades a un cicle algebraic, amb l'ús de la *cohomologia de Deligne-Beilinson.* Per esquemes projectius, la definició de Burgos coincideix amb la definició de Gillet i Soulé.

Tot seguit descrivim breument la seva definició. Sigui X una varietat aritmètica i sigui $(\mathcal{D}^*_{\log}(X,p), d_{\mathcal{D}})$ el complex de Deligne de formes diferencials en la varietat real associada $X_{\mathbb{R}}$, amb singularitats logarítmiques al llarg de l'infinit (vegeu [16] o [13]). La cohomologia d'aquest complex dóna els grups de cohomologia de Deligne-Beilinson de $X_{\mathbb{R}}$, $H^*_{\mathcal{D}}(X,\mathbb{R}(p))$. Per a una subvarietat de X irreductible i de codimensió p Z, considerem la cohomologia de Deligne-Beilinson amb supports en Z:

$$H^*_{\mathcal{D},Z}(X,\mathbb{R}(p)) = H^*(s(\mathcal{D}^*_{\log}(X,p) \to \mathcal{D}^*_{\log}(X\setminus Z,p))).$$

Es té un isomorfisme

$$cl : \mathbb{R}[Z] \xrightarrow{\cong} H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)).$$

Sigui $\widetilde{\mathcal{D}}^*_{\log}(X,p)$ el quocient de $\mathcal{D}^*_{\log}(X,p)$ per la imatge de la diferencial $d_{\mathcal{D}}$. Cal puntualitzar aquí que, al grau $2p-1$, la diferencial $d_{\mathcal{D}}$ és $-2\partial\bar{\partial} = (4\pi i)dd^c$. Una *forma de Green* per a una subvarietat irreductible de codimensió p Z, és un element $(\omega,\tilde{g}) \in \mathcal{D}^{2p}_{\log}(X,p)\oplus\widetilde{\mathcal{D}}^{2p-1}_{\log}(X\setminus Z,p)$, tal que $\omega = d_{\mathcal{D}}\tilde{g}$ i

$$cl(Z) = [(\omega,\tilde{g})] \in H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)).$$

Aleshores, un *cicle aritmètic* és ara una parella $(Z,(\omega,\tilde{g}))$, amb $(\omega,\tilde{g})$ una forma de Green per a Z. El *grup de Chow aritmètic de* X, $\widehat{CH}^p(X)$, es defineix com el grup abelià lliure generat pels cicles aritmètics quocient una relació d'equivalència donada pel grup de cicles aritmètics racionals.

Els grups de Chow aritmètics definits per Burgos satisfan les propietats anàlogues *(i)-(iii)* que hem mencionat més amunt pels grups de Chow aritmètics definits per Gillet i Soulé. En particular, la successió exacta (1) s'escriu com:

$$CH^{p-1,p}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0. \tag{3}$$

En aquest treball hem adoptat la definició de grups de Chow aritmètics donada per Burgos.

Més tard, a [14], Burgos, Kramer i Kühn van desenvolupar una teoria formal d'anells de Chow aritmètics, on diferent complexos de grups abelians que calculen teories de cohomologies adequades juguen el paper de les fibres a l'infinit. Això és, l'espai de les formes de Green es pot canviar per complexos amb propietats diferents, per tal d'obtenir teories d'intersecció adequades que satisfacin diferents propietats.

Teoria d'intersecció aritmètica superior. En certa manera, podríem considerar que el programa d'Arakelov en grau zero s'ha completat. Per tal d'anar més enllà en l'objectiu de trobar anàlegs aritmètics per a les teories algebraiques, voldríem trobar el formalisme d'una teoria d'intersecció superior per a varietats aritmètiques. Aquesta hauria d'incloure una teoria de grups de Chow aritmètics superiors equipats amb un producte d'intersecció, la definició d'uns grups de K-teoria aritmètics superiors, classes característiques i teoremes de Riemann-Roch.

Deligne i Soulé (vegeu [16], Remark 5.4 i [54] §III.2.3.4) van suggerir que l'extensió del grup K_0 aritmètic a grau superior hauria de suposar l'extensió de la successió exacta (2) per tal d'obtenir una successió exacta llarga

$$\cdots \to K_{n+1}(X) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n-1}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{K}_n(X) \xrightarrow{\zeta} K_n(X) \to \cdots$$

$$\cdots \to K_1(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}_{\log}^{2p-1}(X,p) \xrightarrow{a} \widehat{K}_0(X) \xrightarrow{\zeta} K_0(X) \to 0.$$

El morfisme ρ és el regulador de Beilinson, això és, el caràcter de Chern prenent valors en la cohomologia de Deligne-Beilinson real. Per tant, la component arquimediana dels K-grups aritmètics superiors vindria donada pel regulador de Beilinson:

$$\rho : K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}} \to \bigoplus_{p\geq 0} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)).$$

Anàlogament, els grups de Chow aritmètics superiors podrien definir-se de manera que extenguessin la successió exacta (3) en una successió exacta llarga:

$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$

$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \widetilde{\mathcal{D}}_{\log}^{2p-1}(X,p) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0.$$

Aquestes successions exactes llarges es poden obtenir considerant els grups d'homotopia de la fibra homotòpica d'un representant simplicial del regulador de Beilinson.

Grups de Chow aritmètics superiors. Si X és propi, Goncharov, a [30], va definir grups de Chow aritmètics superiors usant aquestes idees.

Sigui ${}'\mathcal{D}^{2p-*}(X,p)$ el complex de Deligne de corrents en X i sigui $E^{2p}(X)(p)$ el grup de formes diferencials de grau $2p$ amb twist p. Denotem per ${}'\widetilde{\mathcal{D}}^{2p-*}(X,p)$ el quocient de ${}'\mathcal{D}^{2p-*}(X,p)$ pel complex

$$\cdots \to 0 \to \cdots \to 0 \to E^{2p}(X)(p) \to 0.$$

Sigui $Z^p(X,*)$ el complex de cadenes que defineix $CH^p(X,*)$. Goncharov va definir un regulador explícit

$$Z^p(X,*) \xrightarrow{\mathcal{P}} {}'\widetilde{\mathcal{D}}^{2p-*}(X,p).$$

Els *grups de Chow aritmètics superiors* d'una varietat complexa regular X vénen donats per la homologia del simple del morfisme $\mathcal{P}$:

$$\widehat{CH}^p(X,n) := H_n(s(\mathcal{P})).$$

Per $n = 0$, aquests grups coincideixen amb els donats per Gillet i Soulé. De totes maneres, aquesta construcció deixa obertes les següents qüestions:

(1) El morfisme induït per $\mathcal{P}$ és el regulador de Beilinson?

(2) Es pot definir una estructura multiplicativa a $\bigoplus_{p,n} \widehat{CH}^p(X,n)$?

(3) Es poden definir pull-backs?

La principal obstrucció a l'hora de respondre aquestes preguntes és que el complex de corrents no té bones propietats respecte pull-back o producte. A més, les tècniques de comparació de reguladors generalment s'apliquen a morfismes definits per la classe de varietats quasi-projectives, i aquest no és el cas del morfisme $\mathcal{P}$.

K-teoria aritmètica superior. La primera contribució en la direcció d'obtenir un definició explícita dels K-grups aritmètics superiors és la descripció simplicial del regulador de Beilinson donada per Burgos i Wang a [15]. Sigui X una varietat complexa. Sigui $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$ el complex de cubs de fibrats vectorials hermítics en X. Els seus grups d'homologia amb coeficients racionals són els K-grups algebraics tensorialitzats amb $\mathbb{Q}$, i.e. hi ha un isomorfisme $H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X),\mathbb{Q}) \cong K_n(X)_{\mathbb{Q}}$ (vegeu [47]). A [15], Burgos i Wang van definir un morfisme de cadenes

$$\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \to \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p).$$

Aquí, $\widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p)$ és un complex construït a partir de formes diferencials a $X \times (\mathbb{P}^1)^{\cdot}$, que és quasi-isomorf al complex de Deligne de formes diferencials en X amb singularitats logarítmiques al llarg de l'infinit, $\mathcal{D}_{\log}^{2p-*}(X,p)$. A més a més, si X és compacta, aleshores existeix un quasi-isomorfisme invers explícit $\widetilde{\mathcal{D}}_{\mathbb{P}}^{*}(X,p) \to \mathcal{D}^*(X,p)$ que dóna lloc a un morfisme

$$\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \to \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p).$$

Burgos i Wang demostraren que aquest morfisme indueix el regulador de Beilinson en cohomologia a coeficients racionals.

La idea de la construcció és la següent. Un associa a cada n-cub E en X un feix localment lliure, $\mathrm{tr}_n(E)$, a $X \times (\mathbb{P}^1)^n$, que dóna una deformació del cub inicial E per cubs escindits. Aleshores, si "ch" és la forma de Chern donada per les fórmules de Weil, $\mathrm{ch}(\mathrm{tr}_n(E))$ és una forma diferencial a $\mathcal{D}_{\log}^{2p-n}(X \times (\mathbb{P}^1)^n, p)$. Si X és compacta, es pot integrar aquesta forma al llarg de $(\mathbb{P}^1)^n$ contra unes certes formes diferencials T_n, obtenint una forma diferencial a X.

Sigui $\widehat{S}.(X)$ el conjunt simplicial de Waldhausen per K-teoria algebraica associat a la categoria de fibrats vectorials hermítics en X, i sigui $\mathcal{K}.(\cdot)$ el functor de Dold-Puppe de complexos de cadenes a grups simplicials abelians. Aleshores, la composició

$$\widehat{S}.(X) \xrightarrow{Hurewicz} \mathcal{K}.(\mathbb{Z}\widehat{S}_*(X)) \xrightarrow{\mathcal{K}(\mathrm{Cub})} \mathcal{K}.(\widetilde{\mathbb{Z}}\widehat{C}_*(X)) \xrightarrow{\mathrm{ch}} \mathcal{K}.\Big(\bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p)\Big)$$

és un representant simplicial del regulador de Beilinson.

Sigui $\widehat{\mathcal{D}}^*(X,p)$ la "bête"-truncació del complex $\mathcal{D}^*(X,p)$ a grau més gran o igual que $2p$, i sigui

$$\widehat{\mathrm{ch}} : \widehat{S}.(X) \xrightarrow{\widehat{\mathrm{ch}}} \mathcal{K}.\Big(\bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p)\Big),$$

el morfisme induït per "ch". Aleshores, seguint les idees de Deligne i Soulé, un defineix els *K-grups aritmètics superiors* com

$$\widehat{K}_n(X) = \pi_{n+1}(\text{Fibra homotopica de } |\mathcal{K}(\widehat{\mathrm{ch}})|).$$

D'aquesta manera, s'obté la successió exacta llarga extenent (2).

Observeu que aquesta definició de K-grups aritmètics superiors tracta de manera diferent el cas de grau zero de la resta. És a dir, el paper de les formes diferencials en el cas de grau diferent de zero el juguen formes diferencials en el nucli de la diferencial $d_{\mathcal{D}}$, mentres que no s'imposa cap restricció a les formes diferencials del grup de grau zero.

Per tal d'evitar aquesta diferència, Takeda, a [57], va donar una definició alternativa dels *K-grups aritmètics superiors* de X, mitjançant grups d'homotopia modificats pel representant del regulador de Beilinson "ch". Denotem aquests K-grups aritmètics superiors per $\widehat{K}_n^T(X)$. La principal característica d'aquests grups és que en lloc d'extendre la successió exacta (2) a una successió exacta llarga, per a cada n tenim una successió exacta

$$K_{n+1}(X) \xrightarrow{\rho} \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X,p) \xrightarrow{a} \widehat{K}_n^T(X) \xrightarrow{\zeta} K_n(X) \to 0,$$

anàloga a la successió exacta per $\widehat{K}_0(X)$.

Les dues definicions no coincideixen, però tal i com va provar Takeda a [57], estan relacionades per la classe característica "ch":

$$\widehat{K}_n(X) \cong_{can} \ker(\mathrm{ch} : \widehat{K}_n^T(X) \to \widehat{\mathcal{D}}^{2p-n}(X,p)).$$

Resum dels resultats

Els resultats d'aquesta tesi contribueixen al programa de desenvolupar una teoria d'intersecció aritmètica superior. Aquests són els resultats que constitueixen els capítols 3 i 5. Els capítols 2 i 4 consisteixen en resultats preliminars que es necessiten pels capítols 3 i 5, en l'àrea de teoria homotopica de feixos simplicials i K-teoria algebraica.

En el capítol 3, hem desenvolupat una teoria d'intersecció superior en varietats aritmètiques, "à la" Bloch. És a dir, hem modificat els grups de Chow superiors definits per Bloch via una construcció explícita del regulador de Beilinson en termes de cicles algebraics.

Hem construït un representant del regulador de Beilinson usant el complex de Deligne de formes diferencials, en lloc del complex de Deligne de corrents. El regulador que hem obtingut resulta ser una lleugera modificació del regulador descrit per Bloch a [8].

Tot seguit, hem desenvolupat una teoria de grups de Chow aritmètics superiors, $\widehat{CH}^p(X,n)$, per a qualsevol varietat aritmètica X sobre un cos. Aquests grups són els grups d'homologia del simple d'un diagrama de complexos que representa el regulador de Beilinson. Demostrem que hi ha un producte associatiu i commutatiu en $\widehat{CH}^*(X,*) = \bigoplus_{p,n} \widehat{CH}^p(X,n)$, compatible amb el producte d'intersecció algebraic. Per tant, donem un producte d'intersecció aritmètic per varietats aritmètiques sobre un cos.

Les avantatges de la nostra definició sobre la definició de Goncharov són les següents: la construcció és vàlida per varietats aritmètiques quasi-projectives sobre un cos, i no només per varietats projectives; podem provar que el nostre regulador és el regulador de Beilinson; els grups obtinguts són contravariants respecte un morfisme de varietats qualsevol; podem dotar-los d'una estructura multiplicativa. Aquestes millores es deuen principalment al fet que evitem usar el complex de corrents.

Els grups de Chow algebraics definits per Bloch són una descripció simple de la cohomologia motívica per varietats algebraiques llises sobre un cos. Els grups de Chow aritmètics superiors introduïts aquí s'haurien de veure com una descripció simple d'una teoria de cohomologia motívica aritmètica encara per definir.

Tot seguit ens vam centrar en la relació entre els grups de Chow aritmètics superiors definits i els K-grups aritmètics superiors. Per tal de seguir l'esquema algebraic, hauríem de tenir una descomposició dels grups $\widehat{K}_n(X)_{\mathbb{Q}}$ donada pels espais de vectors propis de les operacions Adams $\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}}$. Per la naturalesa de la definició de $\widehat{K}_n(X)$, tant considerant la fibra homotòpica com els grups d'homotopia modificats de Takeda, és aparentment necessari tenir una descripció de les operacions d'Adams en K-teoria algebraica en termes d'un morfisme de cadenes, compatible amb el representant del regulador de Beilinson "ch".

En el capítol 4, obtenim un morfisme de cadenes que indueix les operacions d'Adams en K-teoria algebraica superior, sobre el cos dels nombres racionals. Aquesta definició és de naturalesa combinatòrica. A més, el morfisme està construït amb la idea en ment que hauria de commutar amb el regulador de Beilinson "ch" donat per Burgos i Wang. Per tant, es pot apreciar que ha estat altament inspirat per la definició del regulador de Beilinson i segueix el mateix patró lògic.

En el capítol 5 demostrem que aquest morfisme de cadenes commuta amb "ch" i usem aquest fet per definir operacions d'Adams en els K-grups aritmètics superiors tensorialitzats amb $\mathbb{Q}$. D'aquí es dedueix una estructura de pre-λ-anell per $\widehat{K}_n(X)_{\mathbb{Q}}$ i $\widehat{K}_n^T(X)_{\mathbb{Q}}$.

Futurs estudis en aquesta direcció es centraran en determinar si les operacions d'Adams indueixen una descomposició graduada

$$\widehat{K}_n(X)_{\mathbb{Q}} = \bigoplus_{p\geq 0} \widehat{K}_n(X)_{\mathbb{Q}}^{(p)}$$

de manera que hi hagi un isomorfisme $\widehat{CH}^p(X,n)_{\mathbb{Q}} \cong \widehat{K}_n(X)_{\mathbb{Q}}^{(p)}$, tal i com es dóna en el context algebraic. Observeu que els anàlegs aritmètics de les teories algebraiques discutides aquí, es basen en una construcció explícita d'un cert morfisme en el context algebraic. Aquest és el cas del regulador de Beilinson, per tal de definir anells de Chow aritmètics superiors o K-grups aritmètics superiors, i per les operacions d'Adams en els K-grups aritmètics superiors. Des d'aquest punt de vista, la principal dificultat per demostrar que hi ha un isomorfisme

$$\widehat{CH}^p(X,n)_{\mathbb{Q}} \cong \widehat{K}_n(X)_{\mathbb{Q}}^{(p)}$$

és que, per ara, no hi ha cap representant explícit de l'isomorfisme anàleg algebraic.

El desenvolupament d'aquest treball requeria eines per comparar morfismes dels K-grups algebraics superiors a grups de cohomologia adequats o als mateixos K-grups. Efectivament, hem construït un morfisme de cadenes que hem demostrat que indueix el regulador de Beilinson, i hem construït un morfisme de cadenes que hem provat que indueix les operacions d'Adams en K-teoria algebraica superior. En el capítol 2, estudiem aquestes comparacions a un nivell general, donant teoremes que detallen condicions suficients per tal que dos morfismes coincideixin. La teoria en què es recolzen les demostracions és la teoria homotòpica de feixos simplicials.

Aquests teoremes donen una demostració alternativa que el regulador definit per Burgos i Wang a [15] indueix el regulador de Beilinson. A més a més, demostrem que les operacions d'Adams definides per Grayson a [31] coincideixen, per a tot esquema noeterià regular de dimensió de Krull finita, amb les operacions d'Adams definides per Gillet i Soulé a [28]. En particular, se segueix que les operacions d'Adams definides per Grayson satisfan les identitats usuals d'un λ-anell, fet que no quedava demostrat en l'article de Grayson.

Resultats

Tot seguit expliquem l'estructura d'aquest manuscript i en detallem els resultats principals.

El **capítol 1** és de caire preliminar. En ell es donen breument els conceptes previs necessaris per a la comprensió del treball central d'aquesta tesi. També està escrit amb el propòsit de fixar la notació i les definicions que s'usaran sovint en els capítols següents. En la primera secció discutim les categories de models simplicials, centrant-nos en la categoria de conjunts simplicials i en els grups abelians cúbics. En la segona secció fixem la notació en multi-índexos i fem un repàs als principals fets en complexos de (co)cadenes. A més, discutim la relació entre grups abelians cúbics o simplicials i els complexos de cadenes. En la tercera secció donem la definió de K-teoria algebraica en termes de la Q-construcció de Quillen i la construcció de Waldhausen. Introduim també el complex de cadenes de cubs, que calcula la K-teoria algebraica a coefficents racionals, i que juga un paper central en la nostra definició de les operacions d'Adams. Finalment, en l'última secció, recordem la definició de la cohomologia de Deligne-Beilinson i n'enunciem les principals propietats usades en aquest treball.

En el **capítol 2** donem teoremes per a la comparació de classes característiques en K-teoria algebraica. Per a una classe d'aplicacions, anomenades *feblement additives*, donem un criteri per decidir quan dues d'elles coincideixen. Aquesta classe inclou tots els morfismes de grups induïts per un morfisme de feixos simplicials, però aquests no són els únics.

Tal i com ja hem mencionat, a [15], Burgos i Wang van definir una variant del caràcter de Chern dels grups de K-teoria algebraica a la cohomologia absoluta de Hodge real,

$$\mathrm{ch} : K_n(X) \to \bigoplus_{p\geq 0} H_{\mathcal{H}}^{2p-n}(X, \mathbb{R}(p)),$$

per a tota varietat complexa llisa X. A loc. cit., van demostrar que aquest morfisme coincideix amb el regulador de Beilinson. La demostració es basa solament en propietats satisfetes pels morfismes i en propietats de la cohomologia absoluta de Hodge real, però no en la seva precisa definició. Per tant, és raonable pensar en l'existència d'un teorema axiomàtic per classes característiques en K-teoria algebraica superior. La demostració de Burgos i Wang usa l'esquema bisimplicial $B.P.$, introduït per Schechtman a [51]. Això requereix un "delooping" en K-teoria i, per tant, el mètode s'aplica només a aplicacions que indueixen un morfisme de grups.

En aquest treball usem tècniques de la teoria de cohomologia generalitzada descrita per Gillet i Soulé a [28]. La idea és que tota aplicació prou bona de K-teoria a K-teoria o a una teoria de cohomologia, està caracteritzada pel seu comportament en els K-grups de l'esquema simplicial $B.GL_N$.

Aquí donem varis teoremes de caracterització. Les principals conseqüències són una caracterització de les *operacions lambda i d'Adams* en K-teoria algebraica superior i una caracterització del *caràcter de Chern* i de les *classes de Chern* en les teories de cohomologia adequades.

Explícitament, sigui $\mathbf{C}$ el "site" gran de Zariski sobre un esquema noeterià de dimensió finita S. Sigui $B.GL_{N/S}$ l'esquema simplicial $B.GL_N \times_{\mathbb{Z}} S$ i $Gr(N, k)$ l'esquema de Grassmanianes sobre S. Sigui $S.\mathcal{P}$ el feix simplicial de Waldhausen que calcula la K-teoria algebraica i sigui $\mathbb{F}.$ un feix simplicial. Observeu que $S.\mathcal{P}$ és un H-espai. Denotem per Ψ_{GS}^k les operacions d'Adams en K-teoria algebraica superior definides per Gillet i Soulé a [28]. Les dues principals conseqüències del nostre teorema d'unicitat són les següents:

Teorema 1 (Corollary 2.4.4)**.** *Sigui $\rho : S.\mathcal{P} \to S.\mathcal{P}$ una aplicació d'H-espais en la categoria homotòpica de feixos simplicials en $\mathbf{C}$. Si, per algun $k \geq 1$, tenim un diagrama commutatiu*

$$\begin{tikzcd} K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"] \arrow[d, "\Psi^k"'] & K_0(B.GL_{N/S}) \arrow[d, "\rho"] \\ K_0(\mathrm{Vect}(B.GL_{N/S})) \arrow[r, "\psi"'] & K_0(B.GL_{N/S}), \end{tikzcd}$$

aleshores, ρ coincideix amb l'operació d'Adams Ψ_{GS}^k, per a tot esquema X sobre S.

Teorema 2 (Theorem 2.5.5). *Sigui $\mathcal{F}^*$ un complex de cocadenes de feixos de grups abelians en* $\mathbf{C}$. *Sigui*

$$S.\mathcal{P} \longrightarrow \prod_{j\in\mathbb{Z}} \mathcal{K}.(\mathcal{F}(j)[2j])$$

una aplicació d'H-espais en la categoria homotòpica de feixos simplicials en $\mathbf{C}$. *Aleshores, els morfismes induïts*

$$K_m(X) \to \prod_{j\in\mathbb{Z}} H^{2j-m}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j))$$

coincideixen amb el caràcter de Chern definit per Gillet a [21] per a tot esquema X, si, i només si, l'aplicació induïda

$$K_0(X) \to \prod_{j\in\mathbb{Z}} H^{2j}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j))$$

és el caràcter de Chern per l'esquema $X = Gr(N,k)$, per a tot N,k.

En particular:

- ▶ Demostrem que les operacions d'Adams definides per Grayson a [31], coincideixen amb les definides per Gillet i Soulé a [28], per a tot esquema noeterià de dimensió de Krull finita. Es dedueix que, per aquesta classe d'esquemes, les operacions definides per Grayson satisfan les identitats usuals de λ-anell.
- ▶ Demostrem que les operacions d'Adams definides en el capítol 4, coincideixen amb les operacions definides per Gillet i Soulé a [28], per a tot esquema noeterià de dimensió de Krull finita.
- ▶ Donem una demostració alternativa al fet que el morfisme definit per Burgos i Wang a [15] coincideix amb el regulador de Beilinson.

El **capítol 3** està dedicat al desenvolupament de la teoria de grups de Chow aritmètics superiors, per a varietats aritmètiques. Com que la teoria de grups de Chow algebraics superiors definida per Bloch, $CH^p(X,n)$, està completament assentada només per esquemes sobre un cos, ens hem de restringir a varietats aritmètiques sobre un cos.

Sigui X una varietat algebraica complexa i sigui $H^*_{\mathcal{D}}(X,\mathbb{R}(p))$ els grups de cohomologia de Deligne-Beilinson a coeficients reals. Per a tot $p \geq 0$, definim dos complexos de cadenes, $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ i $\mathcal{D}^*_{\mathbb{A}}(X,p)_0$, construïts a partir de formes diferencials a $X \times (\mathbb{A}^1)^n$ amb singularitats logarítmiques al llarg de l'infinit. Tenim els següents isomorfismes:

$$H^{2p-n}(\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0) \cong CH^p(X,n)_{\mathbb{R}},$$

i

$$H^r(\mathcal{D}^*_{\mathbb{A}}(X,p)_0) \cong H^r_{\mathcal{D}}(X,\mathbb{R}(p)), \qquad \text{per } r \leq 2p.$$

Demostrem que el complex $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ satisfà les mateixes propietats que el complex $Z^p(X,n)_0$ definit per Bloch a [7]. De fet, en aquest treball usem el seu anàleg cúbic, definit per Levine a [41], doncs és més adequat a l'hora de descriure l'estructura multiplicativa per $CH^*(X,*)$. El subíndex 0 es refereix al complex de cadenes normalitzat associat a un grup abelià cúbic.

A més, es té un morfisme de cadenes natural

$$\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0$$

que indueix, després de composar-lo amb l'isomorfisme

$$K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}$$

descrit per Bloch a [7], el *regulador de Beilinson* (**Theorem 3.4.5**):

$$K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}} \xrightarrow{\rho} \bigoplus_{p\geq 0} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Aquesta construcció usant rectes afins $\mathbb{A}^1$, es pot desenvolupar també usant rectes projectives $\mathbb{P}^1$. En aquest capítol definim també un complex de cadenes, $\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p)$, anàleg al complex de cadenes $\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$, i un complex de cadenes $\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P}}(X,p)$, anàleg al complex $\mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0$. Definim també un morfisme de cadenes

$$\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{P}}(X,p).$$

En aquest cas, si X és un esquema propi, seguint els mètodes de Burgos i Wang a [15], secció 6, la integració al llarg de les rectes projectives indueix un morfisme de cadenes

$$\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P}}(X,p) \to \mathcal{D}^{2p-*}(X,p).$$

D'aquesta manera, obtenim un morfisme de cadenes

$$\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \xrightarrow{\rho} \mathcal{D}^{2p-*}(X,p)$$

que representa el regulador de Beilinson. Observeu que quan X és propi, aquest representant té l'avantatge que el complex de cadenes que conté la imatge de ρ és exactament el complex de Deligne de formes diferencials en X, i no un complex de cadenes de formes diferencials a $X \times (\mathbb{A}^1)^n$. Aquest morfisme és útil per tal de desenvolupar una teoria de grups de Chow aritmètics a l'estil de la K-teoria aritmètica superior de Takeda a [57].

En la segona part d'aquest capítol usem el morfisme ρ per tal de definir els grups de Chow aritmètics superiors $\widehat{CH}^p(X,n)$, per a tota varietat aritmètica X sobre un cos. La definició es basa en el formalisme de la teoria de diagrames i els seus complexos simples associats, desenvolupada per Beilinson a [5]. En concret, considerem el diagrama de complexos de cadenes

$$\widehat{\mathcal{Z}}^p(X,*)_0 = \left(\begin{array}{ccccc} & H^{2p}_{\mathcal{D}}(X\times\mathbb{A}^*,\mathbb{R}(p))_0 & & \widehat{\mathcal{D}}^{2p-*}_{\mathbb{A}}(X,p)_0 & \\ \nearrow^{f_1} & & \nwarrow^{g_1}_{\sim} & & \nearrow^{\rho} \\ Z^p(X,*)_0 & & & \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 & \end{array} \right).$$

Aleshores, els *grups de Chow aritmètics superiors* de X són els grups d'homologia del simple d'aquest diagrama

$$\widehat{CH}^p(X,n) := H_n(s(\widehat{\mathcal{Z}}^p(X,*)_0)).$$

Demostrem les següents propietats:

▶ **Theorem 3.6.11**: Sigui $\widehat{CH}^p(X)$ el grup de Chow aritmètic definit per Burgos. Aleshores, hi ha un isomorfisme natural

$$\widehat{CH}^p(X) \xrightarrow{\cong} \widehat{CH}^p(X,0).$$

▶ **Proposition 3.6.7**: Hi ha una successió exacta llarga

$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$

$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \mathcal{D}_{\log}^{2p-1}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0.$$

▶ **Proposition 3.6.15** (*Pull-back*): Sigui $f : X \to Y$ un morfisme de varietats aritmètiques sobre un cos. Aleshores, es pot definir un pull-back

$$\widehat{CH}^p(Y,n) \xrightarrow{f^*} \widehat{CH}^p(X,n),$$

per a tot p i n, compatible amb el pull-back en els grups $CH^p(X,n)$ i $H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p))$.

▶ **Corollary 3.6.19** (*Invariància homotòpica*): Sigui $\pi : X \times \mathbb{A}^m \to X$ la projecció en X. Aleshores, el pull-back

$$\pi^* : \widehat{CH}^p(X,n) \to \widehat{CH}^p(X \times \mathbb{A}^m, n), \quad n \geq 1$$

és un isomorfisme.

▶ **Theorem 3.9.7** (*Producte*): Existeix un producte en

$$\widehat{CH}^*(X,*) := \bigoplus_{p\geq 0, n\geq 0} \widehat{CH}^p(X,n),$$

que és associatiu, commutatiu graduat respecte el grau donat per n i commutatiu respecte el grau donat per p.

Finalment, donem una breu descripció d'una possible teoria de grups de Chow aritmètics superiors seguint les idees de Takeda a [57], per a la definició dels K-grups aritmètics superiors d'una varietat aritmètica pròpia. En aquesta construcció, usem la definició del regulador de Beilinson via les rectes projectives, i per tant ens hem de restringir a varietats aritmètiques pròpies sobre un cos.

Les dues següents preguntes queden obertes en aquesta tesi:

- ▷ Els grups de Chow aritmètics superiors construïts aquí coincideixen amb els definits per Goncharov?
- ▷ Es pot extendre la definició de $\widehat{CH}^*(X,*)$ a varietats aritmètiques sobre un anell aritmètic?

En el **capítol 4**, construim un representant de les operacions d'Adams en K-teoria algebraica superior. Sigui X un esquema i sigui $\mathcal{P}(X)$ la categoria exacta de feixos localment lliures de rang finit en X. Els K-grups algebraics de X, $K_n(X)$, es defineixen com els K-grups de Quillen de la categoria $\mathcal{P}(X)$.

Aquests grups es poden equipar amb una estructura de λ-anell. Aleshores, les *operacions d'Adams* en cada $K_n(X)$ s'obtenen a partir de les operacions λ per mitjà d'una fórmula polinomial. En la literatura, es troben diferents definicions directes de les operacions d'Adams en els K-grups algebraics d'un esquema X. Usant la teoria homotòpica de feixos simplicials (revisada en el capítol 2), Gillet i Soulé definiren operacions d'Adams per a tot esquema noeterià de dimensió de Krull finita. Grayson, a [31], va construir una aplicació simplicial induint operacions d'Adams en els K-grups d'una categoria exacta proveïda d'una noció de producte tensorial, producte simètric i producte exterior. En particular, Grayson va construir operacions d'Adams pels K-grups algebraics d'un esquema X. Seguint els mètodes de Schechtman a [51], Lecomte, a [40], va definir operacions d'Adams pels grups de K-teoria algebraica racional d'un esquema X equipat amb una família ampla de feixos invertibles.

El nostre objectiu era construir un morfisme de cadenes explícit induint les operacions d'Adams en K-teoria algebraica racional. Aquesta construcció podria servir per entendre millor els espais de vectors propis de les operacions d'Adams.

Considerem el complex de cadenes de cubs associat a la categoria $\mathcal{P}(X)$. McCarthy a [47], demostrà que els grups d'homologia a coeficients racionals d'aquest complex són isomorfs als K-grups algebraics tensorialitzats amb $\mathbb{Q}$ de X (vegeu la secció 1.3.3).

En primer lloc, vam intentar trobar una versió homològica de la construcció simplicial de Grayson, però sembla particularment difícil des del punt de vista combinatòric.

L'aproximació actual es basa en una simplificació obtinguda usant les *transgressions de cubs via rectes afins o projectives*, però té el cost d'haver-nos de restringir a esquemes regulars noeterians. Aquesta va ser la idea de Burgos i Wang a [15], per tal de definir un morfisme de cadenes representant el regulador de Beilinson.

El morfisme buscat havia de commutar amb el representant del regulador de Beilinson "ch". Amb aquesta fi, el morfisme hauria de ser de la forma

$$E \mapsto \Psi^k(\mathrm{tr}_n(E)),$$

amb Ψ^k una descripció de la operació d'Adams k-èssima al nivell de fibrats vectorials. Malauradament, per les eleccions conegudes de Ψ^k, aquesta aplicació no defineix un morfisme de cadenes. La principal obstrucció és que, mentres que per a tota parella de fibrats vectorials hermítics $\overline{E},\overline{F}$, es té una igualtat

$$\mathrm{ch}(\overline{E}\oplus\overline{F}) = \mathrm{ch}(\overline{E}) + \mathrm{ch}(\overline{F}),$$

no és veritat que per a tota parella de fibrats vectorials E, F, tinguem una igualtat

$$\Psi^k(E \oplus F) = \Psi^k(E) \oplus \Psi^k(F).$$

De totes maneres, aquesta igualtat es té al nivell de $K_0(X)$.

A l'arrel del problema trobem que l'aplicació

$$E \mapsto \mathrm{tr}_n(E)$$

no és un morfisme de cadenes. De totes maneres, afegint a aquesta aplicació una col.lecció de cubs amb la propietat de ser escindits en totes les direccions, podem obtenir un morfisme de cadenes, que anomenem *el morfisme de transgressió*. El fet que els cubs afegits són escindits en totes les direccions, implica que es cancelen quan apliquem “ch”. Per tant, tenim encara una commutació de Ψ^k amb “ch”.

Amb aquest truc, a cada cub a X primer li assignem una col.lecció de cubs definits a $X \times (\mathbb{P}^1)^*$ o bé a $X \times (\mathbb{A}^1)^*$, escindits en totes les direccions (**Proposition 4.3.17**). Aquests cubs s'anomenen *cubs escindits*.

Després, via una fórmula purament combinatòrica en les operacions d'Adams d'un feix localment lliure, donem una fórmula per les operacions d'Adams d'un cub escindit (**Corollary 4.2.39**). El punt clau és l'ús de la classe característica secundària d'Euler del complex de Koszul associat a un feix localment lliure de rang finit, seguint les idees de Grayson.

La composició del morfisme de transgressió amb les operacions d'Adams per cubs escindits dóna lloc a un morfisme de cadenes representant les operacions d'Adams per a tot esquema regular noeterià de dimensió de Krull finita (**Theorem 4.4.2**).

Ambdues construccions, amb rectes projectives o amb rectes afins, són completament anàlogues. Un pot escollir la més adequada en cada cas particular. Per exemple, per a definir operacions d'Adams en els K-grups d'un anell regular R, un podria escollir la construcció amb rectes afins, per tal de quedar-se sempre en la categoria d'esquemes afins. Per altra banda, si la nostra categoria d'esquemes és la categoria d'esquemes regulars projectius, la construcció més adequada probablement seria amb rectes projectives.

La principal aplicació de la nostra construcció és la definició d'una estructura de pre-λ-anell pels K-grups aritmètics superiors a coeficients racionals de X.

En el **capítol 5**, donem una estructura de pre-λ-anell a les dues definicions de K-teoria aritmètica superior tensorialitzada amb $\mathbb{Q}$, $\widehat{K}_n(X)_{\mathbb{Q}}$ i $\widehat{K}_n^T(X)_{\mathbb{Q}}$. És compatible amb l'estructura de λ-anell dels K-grups algebraics, $K_n(X)$, definida per Gillet i Soulé a [28], i amb l'estructura canònica de λ-anell a $\bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-*}(X,p)$, via la graduació donada per p (vegeu lema 1.3.28). A més, per $n = 0$ recuperem l'estructura de λ-anell de $\widehat{K}_0(X) \otimes \mathbb{Q}$.

Concretament, construim operacions d'Adams

$$\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}}, \qquad k \geq 0,$$

que, com que hem tensorialitzat per $\mathbb{Q}$, indueixen operacions λ a $\widehat{K}_n(X)_{\mathbb{Q}}$.

Per tal de treballar amb els grups $\widehat{K}_n^T(X)_{\mathbb{Q}}$, introduim els *grups d'homologia modificats*, que són l'anàleg dels grups d'homotopia modificats. Llavors tenim que els grups d'homologia modificats per "ch" donen una descripció homològica de $\widehat{K}_n(X)_{\mathbb{Q}}$ (**Theorem 5.3.11**).

En aquest capítol demostrem que la construcció d'operacions d'Adams del capítol 4 commuta estrictament amb "ch" (**Theorem 5.4.11**), i d'aquí deduim una estructura de pre-λ-anell per $\widehat{K}_n(X)_{\mathbb{Q}}$ i $\widehat{K}_n^T(X)_{\mathbb{Q}}$ (**Corollary 5.4.14 i Corollary 5.4.16**).

De moment, no hem pogut demostrar que aquestes operacions defineixen una estructura de λ-anell.

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Symbol list

In this symbol list, the following conventions are taken: X, Y are schemes, $S.$ is a simplicial set, A^*, A_* are (co)chain complexes, $C.$ is a cubical group, $\boldsymbol{i}, \boldsymbol{j}$ are multi-indices, $n, l, p, r, \ldots$ are indices, $\mathcal{P}$ is a category.

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