LECTURE 11: AXIOMATIC SET THEORY PART 3

1. Natural numbers and the ordinal ω

Recall from last lecture that a set x is an ordinal just in case it is transitive and wellordered by $\in$ (membership). Note that $\varnothing$ is an ordinal, and we often denote this particular ordinal by 0.

Exercise 1. Let x be a set of ordinal numbers. Prove that $\mathcal{U}(x)$ is an ordinal number.

Definition 1.1. The set ω is defined by

$$x \in \omega \iff (\forall y(\varnothing \in y \wedge (\forall z(z \in y \rightarrow \mathcal{S}(z) \in y)) \rightarrow x \in y).$$

The elements of ω are called *natural numbers*.

Note that

$$\varnothing \in \omega \wedge \forall z(z \in \omega \rightarrow \mathcal{S}(z) \in \omega).$$

So the definition of ω makes it the smallest set which contains $\varnothing$ and is closed on $\mathcal{S}$. This now makes it possible to do proofs by the familiar method of *mathematical induction* on ω: If y is a set such that $\varnothing \in y$ and whenever $z \in y$ then $\mathcal{S}(z) \in y$, then by definition of ω we must have $\omega \subseteq y$.

Theorem 1.2. *(1) Every natural number (i.e., element of ω) is an ordinal.*
(2) ω is an ordinal.

Proof. This is Exercise 3 for Thursday, January 8. □

2. Classes, ON, and definition by recursion

For each formula $\varphi(w_0, \ldots, w_n)$ and sets $a_1, \ldots, a_n$ we could informally consider the collection

$$\{x : \varphi(x, a_1, \ldots, a_n)\}.$$

A collection of this form, i.e., defined by a formula with set parameters, will be called a *class*. We will usually use capital letters for classes, and reserve lowercase letters for sets.

A class may sometimes *not* exist formally in our theory (only sets exist): For example, if we take φ to be $v_1 = v_1$, then

$$V = \{x : x = x\}$$

is the class of all sets. This is *not* a set: Indeed, if it were a set, then by comprehension we would get that $y = \{x : x \notin x\}$ is a set. But y is self-contradictory, because *if* $y \in y$, then $y \notin y$, and *if* $y \notin y$ then $y \in y$.[1]

[1]This is called *Russell's paradox*. When discovered by Bertand Russell in 1901, it temporarily sent the development of set theory and first order logic back to square one. The seemingly restrictive version of Comprehension that set theory has today was adopted as a result of the paradox.

A class which is not a set is called a *proper class*. Apart from V, another example of a proper class is

$$\in = \{(x, y) : x \in y\}.$$

Exercise 2. Prove that $\in$ is a proper class.

The (proper) class of all ordinals is

$$\mathrm{ON} = \{x : x \text{ is an ordinal number}\}.$$

This makes sense because to say that x is an ordinal can be expressed by a formula.

Theorem 2.1. *The class relation $\in \restriction \mathrm{ON}$ is a wellordering of ON. In fact, if $A \subseteq \mathrm{ON}$ is any non-empty* class *then A has a least element.*

Proof. □

We will write $\alpha < \beta$ whenever $\alpha, \beta \in \mathrm{ON}$ and $\alpha < \beta$.

Exercise 3. Let $\alpha \in \mathrm{ON}$.
(1) Prove that $\mathcal{S}(\alpha) \in ON$.
(2) Prove that $\mathcal{S}(\alpha)$ is the immediate successor of α in the ordering $<$.

Theorem 2.2. *If $\alpha \in \mathrm{ON}$ then exactly one of the following holds:*
(1) $\exists\beta(\alpha = \mathcal{S}(\beta)$;
(2) $\alpha = \mathcal{U}(\alpha)$.

Proof. Suppose we are not in case (1). Since α is transitive we have that $\mathcal{U}(\alpha) \subseteq \alpha$, so it suffices to show that $\alpha \subseteq \mathcal{U}(\alpha)$. For this, let $\beta \in \alpha$. Since $\mathcal{S}(\beta) \neq \alpha$, we must have $\mathcal{S}(\beta) < \alpha$. But then $\beta \in \mathcal{S}(\beta) \in \alpha$. □

Remark 2.3. Ordinals satisfying (1) are called *successor ordinals*; Non-zero ordinals satisfying (2) are called *limit ordinals*.

Despite some classes not having a formal existence, it is practical to be able to speak about classes, but one must keep in mind that classes simply are a practical way of talking about formulas. One situation where classes are practical is in stating *theorem schemata*: That is, Theorems that assert that for each formula φ some sentence derived from φ is a theorem (of ZFC).

The notions *relation, function, domain, wellfounded*, etc., and associate notations, are defined for classes just as they are for sets.

Theorem 2.4 (Schema of Definition by Recursion on ω)**.** *Let $F : V \to V$ be a class function.*[2] *Then there is a unique* set *$g : \omega \to V$ such that*

$$\forall n \in \omega(g(n) = F(g \restriction n)).$$

[2]So this means there is a formula $\varphi(x, y, \ldots)$ with set parameters, and $F(x) = y$ iff $\varphi(x, y, \ldots)$.

Proof. We first show that

$$\forall n \exists! g(g : n \to V \wedge \forall m \in n(g(m) = F(g \upharpoonright m)). \tag{1}$$

For $n = \varnothing$, the empty function $g = \varnothing$ works. Suppose $g : n \to V$ is known to be the unique function satisfying $g(m) = F(g \upharpoonright m)$ for all $m \in n$. Then let $g' = g \cup \{(n, F(g))\}$. We leave it to the reader to show that $g' : \mathcal{S}(n) \to V$ is the unique function satisfying $g'(m) = F(g' \upharpoonright m)$ for all $m \in \mathcal{S}(n)$. The induction principle on ω now implies that (1) holds.

By Replacement and Comprehension

$$z = \{y : \exists n \in \omega(y : n \to V \wedge \forall m \in n(y(m) = F(y \upharpoonright m))\}$$

exists as a set. Suppose $y_1, y_2 \in z$, and $y_1 : n_1 \to V$, $y_2 : n_2 \to V$. If $n_1 = n_2$ then the uniqueness assertion of (1) shows that $y_1 = y_2$. If $n_1 \in n_2$ then $y_2 \upharpoonright n_1 = y_1$ by the uniqueness assertion of (1). Similarly $n_2 \in n_1$.

Let $g = \mathcal{U}(z)$. By the previous paragraph, g is a function, and (1) guarantees that $\text{dom}(g) = \omega$. Since $g(n) = y(n)$ for any $y \in z$ with $\text{dom}(g)$, it follows that $g(n) = F(g \upharpoonright n)$. That g is unique follows since if $h : \omega \to V$ was a function satisfying $h(n) = F(h \upharpoonright n)$ for all n, then $(h \upharpoonright m) \in z$ for all m, from which $h = g$ can easily be proved. □

Definition 2.5. For any class A, we let

$$\bigcap A = \{z : \forall y \in A : z \in y\}.$$

Prima facie, if $\varphi(y)$ is the formula defining A, $\bigcap A$ is the class defined by the formula $\forall y(\varphi(y) \wedge z \in y)$ (having z a free variable). However, if A is not empty, then $\bigcap A$ is a set by comprehension (applied to any element of A; notice that elements of classes are always sets). If A is empty then $\bigcap A = V$ by definition (!).

2.1. **Transitive closure.** As a useful example of using Theorem 2.4 we will show the following:

Theorem 2.6. $\forall x \exists y(y \textit{ is transitive} \wedge x \subseteq y)$.

Proof. Informally, we would like to take $y = x \cup \mathcal{U}(x) \cup \mathcal{U}(\mathcal{U}(x)) \cup \cdots \mathcal{U}^n(x) \cup \cdots$, where $\mathcal{U}^n$ is the union operation n times (for $n = 0$ we let $\mathcal{U}^0(x) = x$). Notice that any member z of y is a member of some $\mathcal{U}^n(x)$, and so the members of z are members of $\mathcal{U}^{n+1}(x)$, proving that y is transitive. So the only problem remaining is showing that y formally exists.

Suppose, as an intermediate step, we can find a function $g : \omega \to V$ such that $g(n) = \mathcal{U}^n(x)$. Then $y = \mathcal{U}(\text{ran}(g))$, so we would be done. It turns out to be easier to define $g(n) = x \cup \mathcal{U}(x) \cup \cdots \cup \mathcal{U}^n(x)$, but we can still use $y = \mathcal{U}(\text{ran}(g))$.

For this, define $F : V \to V$ by $F(z) = x$ if z is either empty or z is not a function whose domain is some $n \in \omega$, or else we define $F(z) = x \cup \mathcal{U}(\text{ran}(z))$ (where $\text{ran}(z)$ is the range of z). Let $g : \omega \to V$ be given by the recursion theorem. An easy induction shows that $g(n) = x \cup \mathcal{U}(x) \cup \cdots \cup \mathcal{U}^n(x)$, as required. □

Definition 2.7. The *transitive closure* of a set x, denoted $\text{trcl}(x)$, is the smallest transitive set having x as a subset. I.e.,

$$\text{trcl}(x) = \bigcap\{y : y \text{ is transitive} \wedge x \subseteq y\}.$$

By the previous theorem, $\text{trcl}(x)$ always exists (as a set).

2.2. **Recursion on the ordinals.** Theorem 2.4 has the following powerful generalization:

Theorem 2.8 (Schema of Definition by Transfinite Recursion.)**.** *Let $F : V \to V$ be a class function. Then there is a (unique) $G : \text{ON} \to V$ such that*

$$\forall \alpha \in \text{ON}(G(\alpha) = F(G \restriction \alpha).$$

Proof. The proof of this can be found in the lecture 12 notes. □

Remark 2.9. The proof gives an explicit formula defining G from the formula defining F. So the previous theorem really is a theorem schema, and the use of classes in its statement could be avoided by instead talking only about formulas.

Theorem 2.10. *There is a unique function $\mathbf{V} : \text{ON} \to V$ such that (writing V_α for $\mathbf{V}(\alpha)$):*

(a) $V_0 = \varnothing$.
(b) $V_{\mathcal{S}(\alpha)} = \mathcal{P}(V_\alpha)$.
(c) $V_\lambda = \mathcal{U}(V_\alpha : \alpha < \lambda\})$ when λ is a limit ordinal.

Proof. Let $F(z) = \varnothing$ if either $x = \varnothing$ or x is not a function whose domain is an ordinal number. If α is an ordinal and $x : \mathcal{S}(\alpha) \to V$, then let $F(x) = \mathcal{P}(x(\alpha))$. If λ is a limit ordinal and $x : \lambda \to V$ we let $F(x) = \mathcal{U}(\text{ran}(x))$. The desired function is given by Theorem 2.8. □

Warning: Despite using uppercase letters for the V_α, the V_α are sets, not proper classes.

Exercise 4. Show that $\alpha < \beta \to V_\alpha \subseteq V_\beta$.

As you may have realized on your own by now, the ordinals, ON, are the formal equivalent of the "stages" in the iterative concept of set, and V_α is the set of those sets formed before stage α. The role of the Axiom of Foundation is that it (along with all the other axioms used so far[3]) makes the set-theoretic universe described by our axioms take exactly the form that the iterative concept of set envisions. This is in some sense the content of the next theorem.

Theorem 2.11 (Uses Foundation)**.** $\forall x \exists \alpha (x \in V_\alpha)$.

Proof. The proof is rather peculiar when you look at it first, so I've tried to write it in a semi-informal style.

We consider the set $y = x \cup \text{trcl}\, x$. Since we are assuming Foundation holds, $\in \restriction y$ is wellfounded. So if $x \notin V_\alpha$ for any α, then

$$z = \{u \in y : (\forall \alpha \in \text{ON}) u \notin V_\alpha\}$$

[3]I haven't been keeping track, but I think all the axioms except Choice have been in use by now.

is non-empty. Note also that all $u \in z$ are non-empty (since otherwise u belongs to V_1). The wellfoundedness of $\in \upharpoonright y$ then implies that z has a $\in \upharpoonright y$-least element, which is to say that there is $u \in z$ which has no elements in common with z (which is literally what Foundation gives us). Then for every $w \in u$ there is some unique least $\alpha_w \in \mathrm{ON}$ such that $w \in V_{\alpha_w}$. Replacement then gives us that $a = \{\alpha_w : w \in u\}$ exists as a set. Let $\beta = \mathcal{U}(a)$. Notice that every element of u belongs to $V_{\mathcal{S}(\beta)}$, so $u \subseteq V_{\mathcal{S}(\beta)}$. But then $u \in V_{\mathcal{S}(\mathcal{S}(\beta))}$, a contradiction. $\square$

Here is an alternative way of viewing the proof[4]: Suppose $x \notin V_\alpha$ for any α. Then there must be some $x_1 \in x$ such that $x_1 \notin V_\alpha$ for any α, since otherwise the argument at the end of the above proof shows that $x \in V_\alpha$ for some α. Choose (!) such an $x_1 \in x$. Since $x_1 \notin V_\alpha$ for any α, repeat the argument to get $x_2 \in x_1$ such that $x_2 \notin V_\alpha$ for any α. And continue. This produces a sequence

$$x \ni x_1 \ni x_2 \ni x_3 \ni \cdots$$

which contradict Foundation! (To see this, consider for instance $\{x_i : i \in \omega\}$; One would have to show this exists as a set, but this can be handled using the recursion theorem for ω, say.)

3. Ordinal arithmetic

By transfinite recursion, we define addition, multiplication and exponentiation of ordinal numbers as follows:

Addition:

$$\begin{aligned} \alpha + 0 &= \alpha; \\ \alpha + \mathcal{S}(\beta) &= \mathcal{S}(\alpha + \beta); \\ \alpha + \lambda &= \mathcal{U}(\{\alpha + \beta : \beta < \lambda\} \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Multiplication:

$$\begin{aligned} \alpha \cdot 0 &= 0; \\ \alpha \cdot \mathcal{S}(\beta) &= \alpha \cdot \beta + \alpha; \\ \alpha \cdot \lambda &= \mathcal{U}(\{\alpha \cdot \beta : \beta < \lambda\} \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

Exponentiation:

$$\begin{aligned} \alpha^0 &= 1 = \mathcal{S}(0); \\ \alpha^{\mathcal{S}(\beta)} &= \alpha^\beta \cdot \alpha; \\ \alpha^\lambda &= \mathcal{U}(\{\alpha^\beta : \beta < \lambda\} \text{ if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

[4]The drawback of the alternative view is that it uses the Axiom of Choice, which we would like to avoid when possible. However, it actually only needs a weaker version of Choice called the *Axiom of Dependent Choices.*

Warning: The definition of these arithmetic operations $+$ and $\cdot$ on ON are *highly* asymmetrical with respect to α and β, and so we should have *no* expectation that they are commutative operations. Indeed, they are not, as the next exercise shows.

Exercise 5. Show that $\omega + 1 \neq 1 + \omega$ and that $2 \cdot \omega \neq \omega \cdot 2$.

Asger Törnquist