LECTURE 10: AXIOMATIC SET THEORY PART 2

Before we begin. A very good question came up after the previous lecture...

> **Question:** In the beginning of the course, we used naïve set theory when defining some basic notions of logic, for instance, we used it to define the set Formula($\mathcal{L}$) for a language.
>
> Now that we are trying to formalize set theory using our notions from first order logic, are we not engaging in a form of circular reasoning?

The answer is "yes": We are indeed risking that our definitions become circular. Our axioms of set theory are syntactical entities from first order logic, which in their own definition seemingly rely on principles of (naïve) set theory. Because of this, we **must** insist that the definition of terms and formulas in $\{\in\}$, the Language Of Set Theory, is done using our book's original definition, which does not appeal to set theory. This definition doesn't use set theory, only some basic ideas about being able to form strings of symbols. We also need to accept the induction principle for terms and formulas, and some version of definition by recursion on terms and formulas (so that we can define what it means for a variable to be free), as part of our "metatheory". However, once that is done, then the circularity is avoided. (A lot more could be said about this concern, but since this is a math course and not a philosophy course we won't.)

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1.1. Justifications of the remaining axioms.

4. Pairing: Let x and y be sets. There is some stage s at which both x and y have been formed. Since V_s is formed into a set at stage $s+1$, and $x, y \in V_s$, there is a set to which both x and y belong, namely V_s.

6. Replacement: Let φ be a formula as in the statement of the Axiom Schema of Replacement. For each $x \in z$, let s_x be the stage at which the unique y satisfying $\varphi(x, y, z, w_1, \ldots, w_n)$ is formed. The collection of all these s_x is no larger than z, and so "absolute infinity" demands the existence of a stage s later than all the s_x. Then V_s works for u in the statement of the axiom.

8. Power Set: If x is formed at stage s and $z \subseteq x$, then $z \subseteq V_s$, and so $z \in V_{s+1}$. Since V_{s+1} becomes a set at stage $s+2$, we can use V_{s+1} as a witness to the Power Set Axiom.

9. Choice: If x is formed at stage s and x consists of non- pairwise disjoint sets, then we are looking for a z which may as well be a subcollection of $\mathcal{U}(x) \subseteq V_s$. What we have to do is convince ourselves that such a subcollection exists...

Remark 1.1. (1) Arguing for (or against) Choice on the basis of the iterative concept of set is probably hopeless. If we believe choice to be true for the informal notion of collections, then the above justification succeeds. As we will see in the coming lectures, the picture of the set theoretic universe suggested by the iterative concept turns out to be an accurate description (on a formal level) of the universe that just axioms 0–8 describe. Furthermore, assuming that the axiom system 0–8 is consistent, not only can it be shown that so is the system consisting of all the axioms 0–9, but it can also be shown that axioms 0–8 plus the *negation* of Choice is consistent. (The proof of these "relative consistency" results is quite difficult and would be the topic of a separate course.)

(2) The formal version of the notion of a "stage" will be that of an *ordinal number*, which we introduce below. It turns out (in lecture 11) that there is a first infinite ordinal s (which will be called ω in the formal theory), and so it would not make sense to talk about stage $s-1$, since any stage previous to s is a finite stage (ordinal), and the successor stage of a finite stage is finite. This is the reason we avoided using $s-1$ in our justifications of the axioms.

1.2. **Cartesian products, functions, relations.** The *ordered pair* (x,y) of sets x and y is $\{\{x\},\{x,y\}\}$. Note that

$$(x,y)=(z,w) \iff (x=z \wedge y=w).$$

There is a formula $\varphi(z,x,y)$ in the formal language that expresses the fact that $z=(x,y)$. (This is part of the mandatory homework assignment 2.)

The *Cartesian product* of $u\times v$ of sets u and v is $\{(x,y): x\in u \wedge y\in v\}$, but we don't know yet if this exists as a set. The next theorem says it does.

Theorem 1.2. *$u\times v$ always exists.*

We give two proofs: One uses Power Set, but not Replacement, and the other uses Replacement but not Power Set.

Proof 1. Let u and v be given, and let $x\in u$ and $y\in v$. Note that $\{x\}$ and $\{x,y\}$ are subsets of $u\cup v$, and so $(x,y)=\{\{x\},\{x,y\}\}$ is a subset of $\mathcal{P}(u\cup v)$. So every pair (x,y) with $x\in u$ and $y\in v$ is an element of $\mathcal{P}(\mathcal{P}(u\cup v))$. Finally,

$$u\times v=\{z\in\mathcal{P}(\mathcal{P}(u\cup v)) : \exists x\in u\exists y\in v\ \varphi\},$$

where φ is the formula expressing $z=(x,y)$ (see 11 lines above), exists by Comprehension. □

Proof 2. Let $\varphi(z,x,y)$ be a formula expressing that $z=(x,y)$ (see 12 lines above). Fix $x\in u$. Then for each $y\in v$ there is a unique z such that $\varphi(z,x,y)$, and so the set

$$w_x=\{z:(\exists y)\varphi(x,y,z)\}$$

exists by Replacement, for each x. There is a formula $\psi(w,x,y)$ expressing the fact that $w=w_x$ (see exercise below), and so by another use of Replacement, the set

$$a=\{w_x : x\in u\}$$

exists. Now $u\times v=\mathcal{U}(a)$. □

Exercise 1. Write down the formula ψ needed in the previous proof (feel free to use abbreviations).

Definition 1.3. 0. A (binary) *relation* is a set of ordered pairs. We often write xry instead of $(x, y) \in r$.

1. A relation r is *wellfounded* if

$$\forall x(x \neq \varnothing \rightarrow \exists y \in x \forall z \in x(z, y) \notin r.$$

2. A relation r is a *linear ordering* of a set x just in case $r \subseteq x \times x$, and r linearly orders x in the strict sense. That is, r is transitive, irreflexive, and satisfied the trichotomy

$$\forall x \forall y(xry \vee x = y \vee yrx).$$

3. A wellfounded linear ordering of a set x is called a *wellordering* of x. More generally, we say that r wellorders x if $r \cap x \times x$ is a wellordering of x.

4. A *function* is a relation f such that

$$\forall x \forall y_1 \forall y_2(((x, y_1) \in f \wedge (x, y_2) \in f) \rightarrow y_1 = y_2).$$

5. The notions injective, surjective, bijective, domain and range are defined as usual. The notation $f : x \rightarrow y$ means that f is a function with domain x and range $\subseteq y$.

Example 1.4. Let u be a set, and let

$$\in \restriction u = \{(z, y) \in u \times u : z \in y\}.$$

Then Foundation says that $\in \restriction u$ is wellfounded (for any u).

Definition 1.5. 0. A set x is *transitive* if $\mathcal{U}(x) \subseteq x$. In other words, x is transitive if every element of an element of x is an element of x.

1. A set x is an *ordinal number* if
   (a) x is transitive; and
   (b) $\in \restriction x$ wellorders x.

Remark 1.6. Foundation implies that (b) above is equivalent with the assertion that $\in \restriction x$ linearly orders x.

Theorem 1.7. *Let x and y be ordinal numbers. Then*

$$x \in y \vee x = y \vee y \in x.$$

Proof. This is Problem 2 of the second mandatory assignment. □

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