THE COMPLETENESS THEOREM: A GUIDED TOUR

Theorem 1 (Completeness of first order logic, first version; Gödel, 1930)**.** *Let* Γ *is a set of formulas in a language* $\mathcal{L}$ *and* φ *a formula in* $\mathcal{L}$*. Suppose* $\Gamma \models \varphi$*. Then* $\Gamma \vdash \varphi$*.*

In other words, if in all possible models of Γ it is the case that φ also holds (under the same assignment of the variables), then there is a formal proof (i.e., a deduction) of φ from Γ. This is quite impressive, and also somewhat reassuring. It also means that we can dispense with $\vdash$ and only work with $\models$ from now on. Since most human beings (of the mathematical persuasion anyhow) vastly prefer to work with semantical notions rather than syntactical ones, this should be a relief.

This note contains a guided tour of the proof of the completeness theorem. The details can be filled in from reading the relevant parts of our textbook, or taking notes in lecture.

Recall that a set of formulas Γ is *satisfiable* (or *has a model*) if there is some model (i.e. structure) $\mathfrak{A}$ of $\mathcal{L}$ and some $s : V \to |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \Gamma[s]$. Recall also that the set Γ is *consistent* (sometimes called *deductively consistent* for clarity) if there is no formula β such that $\Gamma \vdash \beta$ and $\Gamma \vdash (\neg\beta)$.

Theorem 2 (Completeness of first order logic, second version)**.** *Let* Γ *is a set of formulas in a language* $\mathcal{L}$*. If* Γ *is consistent, then* Γ *is satisfiable.*

The fact that version 1 and 2 of the completeness theorem are equivalent is an exercise (2.5.2 in the book).

We will prove version 2 of the completeness theorem. The approach we took in lecture isn't truly different than that of our textbook, except that it breaks the proof into two natural parts. The proof (either one) essentially is due to Leon Henkin (1949). It has become the standard proof of the completeness theorem.

Definition 3. Let Δ be a set of formulas in some language $\mathcal{L}$. We say that Δ has the *Henkin witness property* if for any formula φ and any variable x, there is a constant symbol c of $\mathcal{L}$ such that $\exists x\varphi \to \varphi^x_c \in \Delta$. (Here we have, as usual, used $\exists x$ to abbreviate $\neg\forall x\neg$.)

Lemma 4 ("Henkin models")**.** *Let* $\mathcal{L}$ *be a language. Suppose* Δ *is a set of formulas in* $\mathcal{L}$ *such that*

(*i*) Δ *is consistent;*
(*ii*) *for each formula* α *of* $\mathcal{L}$*, either* $\alpha \in \Delta$ *or* $(\neg\alpha) \in \Delta$*;*
(*iii*) Δ *has the Henkin witness property.*

Then Δ *is satisfiable.*

This lemma should be viewed in the following way: It's conclusion is the same as that of the completeness theorem (v. 2), but it has stronger assumptions (namely (*ii*) and (*iii*), which do not appear in the completeness theorem). The stronger assumptions (hopefully) make it easier to prove the lemma. Eventually, the completeness theorem is proved by arranging that the situation described in the previous lemma happens.

Exercise 1.

(1) Prove that a set of formulas Δ which satisfy (i) and (ii) above is deductively closed. Is this still true if we only require (ii) to hold?

(2) Let Δ be as in Lemma 4. Show that for any formula φ and variable x, there is constant c such that $\varphi^x_c \to \forall x\ \varphi \in \Delta$.

The proof of this lemma (which in the lecture was called the "main lemma") relies on a few other facts and Lemmas, which we now list:

Lemma 5 (Properties of $=$)**.** *Let $\mathcal{L}$ be a language with $=$ included. Let R and f denote n-place relation and function symbols in $\mathcal{L}$, respectively, if such exist in $\mathcal{L}$. Then*

Eq1: $\vdash \forall x\ x = x$

Eq2: $\vdash \forall x \forall y (x = y \to y = x)$

Eq3: $\vdash \forall x \forall y \forall z (x = y \to (y = z \to z = x))$

Eq4: $\vdash \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n (x_1 = y_1 \to (x_2 = y_2 \to (\cdots (x_n = y_n \to (Px_1 \cdots x_n \to Py_1 \cdots y_n)) \cdots)))$

Eq5: $\vdash \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n (x_1 = y_1 \to (x_2 = y_2 \to (\cdots (x_n = y_n \to (fx_1 \cdots x_n = fy_1 \cdots y_n)) \cdots)))$.

Exercise 2. Prove the previous lemma (properties of $=$). Parts of it are proved on pp. 127–128 in the textbook, so you can use these pages as an extended hint.

We need a lemma to deal with situations where substitutions can't safely be made in a formula because of an unfortunate choice of variables. Roughly speaking, we may want to substitute some term t into for some variable x in a formula φ, but t is not substitutable in for x in φ because a variable that appears in t is quantified over in φ. However, since the variables we use to quantify over are "dummy variables", the specific choice of which is unimportant, it should always be possible to sidestep this problem by changing the variables we quantify over in φ so that none of them occur in t, and the resulting formula – call it φ' – would express the same fact as the old formula. The next lemma is a formal expression of what this fact.

Lemma 6 (Existence of alphabetic variants, Theorem 24I, pp. 126–127)**.** *Fix a language $\mathcal{L}$. Let φ be a formula, t a term, and x a variable symbol. Then there is a formula φ', which differs only from φ by the variables which are quantified over in φ', and such that*

(a) *$\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$;*

(b) *t is substitutable for x in φ'.*

The proof of this lemma is not suited for going through on a blackboard, so you should read it on your own. That is, if you care to do so, because you may (should?) be convinced that the lemma is true by what was said in the paragraph right before it, above. However, the point of the Lemma (and of proving it) is of course to show that our deductive system is strong enough to *prove* that what was said above (and which may seem at first obvious) is true.

Proof of Lemma 4. Follow steps 4–6 in our textbook's proof of the completeness theorem. These steps prove *exactly* Lemma 4. The language used in step 4 (where we remove equality) is what in lecture was defined as $\mathcal{L}' = \mathcal{L} \cup \{E\} \setminus \{=\}$, where E is some new two-place relation symbol. $\square$

Exercise 3. When defining the initial structure $\mathfrak{A}$ in step 4 in the book, which will be a model of $\mathcal{L}'$, we also define an assignment of variables by letting $s : V \to |\mathfrak{A}|$ be the identity. As was pointed out in lecture, this is a bit sloppy (though essentially correct): What we really do is that we let $s : V \to |\mathfrak{A}|$ be the map $v_i \mapsto \langle v_i \rangle$.

Prove that for any term t in $\mathcal{L}'$ we have $\bar{s}(t) = t$. (This fact is stated without proof on lines 4, 7, and 8 on page 138 in the textbook; read this if you need a hint.)

The next step on the path to completeness is:

Lemma 7. *Suppose* Γ *is a consistent set of formulas in some* countable *language* $\mathcal{L}$. *Then there is a language* $\bar{\mathcal{L}} \supseteq \mathcal{L}$, *obtained by adding countably many distinct, new constant symbols to* $\mathcal{L}$, *and a set of formulas* $\Delta \supseteq \Gamma$ *such that*

(*i*) Δ *is consistent;*
(*ii*) *for all formulas* α *in* $\bar{\mathcal{L}}$, *either* $\alpha \in \Delta$ *or* $(\neg\alpha) \in \Delta$;
(*iii*) Δ *has the Henkin witness property.*

Let us assume without proof for the moment that this lemma holds, and finish the proof of the completeness theorem, at least in the case where the language $\mathcal{L}$ is countable:

Proof of completeness (v.2), given the above. Let Γ be a consistent set of formulas in the countable language $\mathcal{L}$. Apply Lemma 7 above to obtain $\bar{\mathcal{L}}$ and $\Delta \supseteq \Gamma$ as described there. Apply Lemma 4 to obtain a model $\mathfrak{B}$ of $\bar{\mathcal{L}}$ and $s : V \to |\mathcal{B}|$ such that $\models_{\mathfrak{B}} \Delta[s]$. Note that $\models_{\mathfrak{B}} \Gamma[s]$. Let $\mathfrak{B}_{|\mathcal{L}}$ denote the *reduct* of $\mathfrak{B}$ to $\mathcal{L}$, that is, $\mathfrak{B}_{|\mathcal{L}}$ is the model we obtain by throwing away the interpretations of symbols not in $\mathcal{L}$. Then $\mathfrak{B}_{|\mathcal{L}} \models \Gamma[s]$, and so Γ is satisfiable, as required. □

The proof of Lemma 7 requires two other, prefatory lemmata. Bellow, φ^c_y denotes the result of replacing the constant c with the variable y wherever c appears in φ.

Lemma 8 (Generalization on constants, Theorem 24F)**.** *Assume* $\Gamma \vdash \varphi$ *and that* c *is a constant symbol that does not occur in* Γ. *Then there is a variable* y, *not occurring in* φ, *such that* $\Gamma \vdash \forall y\ \varphi^c_y$. *Moreover, there is a deduction of* $\forall y\ \varphi^c_y$ *from* Γ *in which* c *does not appear (occur).*

Proof. pp. 123–124 in the textbook. □

Lemma 9 (Corollary 24G)**.** *Assume* $\Gamma \vdash \varphi^x_c$, *where* c *is a constant symbol not occurring in* φ *or any formula in* Γ. *Then* $\Gamma \vdash \forall x\varphi$, *and there is a deduction of* $\forall x\ \varphi$ *from* Γ *in which* c *does not appear.*

Proof. p. 124 in the textbook. □

Proof of Lemma 7. Follow steps 1–3 in our textbook's proof of the completeness theorem; these steps prove *exactly* what Lemma 7 claims. □

Having thus accounted for all the lemmas above, the proof of the completeness theorem is now done in the case when $\mathcal{L}$ is countable. However, with just a small amount of set-theoretic sophistication, we can also prove Lemma 7 for *uncountable* languages; see the last paragraph on p. 141 in the textbook. Thus the proof of the completeness theorem is now complete.

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