Lecture 9: Axiomatic Set Theory

December 16, 2014

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- The iterative concept of set.
- The language of set theory (LOST).
- The axioms of Zermelo-Fraenkel set theory (ZFC).
- Justification of the axioms based on the iterative concept of set.

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for the negation of $x = y$ and $x \in y$, respectively.

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We will allow ourselves to use the standard abbreviations: $\vee, \wedge, \leftrightarrow$ and $\exists$ for "or", "and", "if and only if", and "there exists".

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(Sets that have the same members are identical.)

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(Every non-empty set has a member which has no members in common with it.)

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(For any set z and relation R (which *must* be expressible by some first order formula φ of LOST), if each member x of z bears the relation R to exactly one set y_x, then there is a set to which all these y_x belong.)

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(There is a set that has the empty set as a member and is closed under the operation $\mathcal{S}$.)

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(For any set, there is a set to which all subsets of that set belong.)

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