AST & FORCING: MANDATORY HOMEWORK NO. 3

This is the 3rd and final mandatory homework assignment for the course Axiomatic Set Theory and Forcing. It is due on Friday, March 21, 2014, at the beginning of lecture.

Corrections, March 16: In the definitions of the posets $\mathbb{D}$ and $\mathbb{M}$, there was a $<$ where there should have been a $\leq$. I thank those who pointed this out to me. If you've found other typos or mistakes, email me at: asgert@math.ku.dk .

This assignment introduces various common posets. The assignment is intended to be shorter and easier than the previous assignments.

Below, M generally refers to an inner model.

Exercise 1 (6 points)**.** Let $f, g \in^{\omega} \omega$. We say that f is *dominated* (or *bounded*) by g if

$$\{n \in \omega : f(n) > g(n)\}$$

is finite.

Let M be an inner model and let G be generic for $\mathrm{Fn}(\omega, \omega)$, where as usual $\mathrm{Fn}(I, J)$ denotes the set of all finite functions $p : x \to J$ with $x \subseteq I$. We have seen in class that in this case $f_G = \bigcup G$ is a function $\omega \to \omega$.

(1) Show that $f_G = \bigcup G$ is not dominated by any $g \in {}^{\omega}\omega \cap M$.

(2) Show that if $f \in {}^{\omega}\omega \cap M$, and $f(n) > 0$ for infinitely many n, then f is not dominated by f_G.

Exercise 2 (6 points)**.** Let $\mathbb{D}$ be the poset consisting of all pairs $\langle s, f\rangle$ where $s \in {}^{<\omega}\omega$ and $f \in {}^{\omega}\omega$, $s \subseteq f$, and

$$\langle s, f\rangle \leq_{\mathbb{D}} \langle t, g\rangle$$

just in case $t \subseteq s$, and $f(n) \geq g(n)$ for all $n \geq \mathrm{dom}(t)$.

Warning: It is implicit (here and elsewhere) that the poset is defined in the model M we intend to force over. That is, to define $\mathbb{D}$ in M, we only consider all pairs $\langle s, f\rangle$ *in* M.

(1) Let G be $\mathbb{D}$-generic over M, and let

$$f_G = \bigcup\{s : (\exists f \in {}^{<\omega}\omega)\langle s, f\rangle \in G\}.$$

Briefly argue why f_G is a function $\omega \to \omega$. (We have already essentially seen this in one of the assignment 2 problems.

(2) Show that f_G dominates every $g \in {}^{\omega}\omega \cap M$, and that no $g \in {}^{\omega}\omega \cap M$ dominates f_G.

Remark 0.1. The poset $\mathbb{D}$ in the previous exercise is called *Hechler forcing* after the mathematician Stephen Hechler.

Exercise 3 (6 points)**.** Let $\mathbb{M}$ be the poset consisting of all pairs $\langle s, x\rangle$ where $s \in {}^{<\omega}2$ and $x \subseteq \omega$ is an infinite set, and $x \cap \operatorname{dom}(s) = \emptyset$. We order $\mathbb{M}$ by defining that $\langle s, x\rangle \leq_{\mathbb{M}} \langle t, y\rangle$ if $t \subseteq s$, $x \subseteq y$, and

$$\{n \in \operatorname{dom}(s) : n \geq \operatorname{dom}(t) \wedge s(n) = 1\} \subseteq y.$$

(1) Briefly explain why $\mathbb{M}$ as defined above is a forcing poset.

(2) Let G be an $\mathbb{M}$-generic filter over M. Show that

$$f_G = \bigcup\{s : (\exists x \subseteq \omega)\langle s, x\rangle \in G\}$$

is a total function $\omega \to 2$, and that

$$x_G = \{n \in \omega : f_G(n) = 1\}$$

is an infinite set.

(3) Show that if $z \in \mathcal{P}(\omega) \cap M$ is any infinite set in M, then either $x_G \setminus z$ is finite, or $x_G \cap z$ is finite.

Remark 0.2. The poset $\mathbb{M}$ described above is usually called *Mathias forcing* after the mathematician Adrian Mathias who first considered it. It is a poset intimately connected with infinite Ramsey theory and partition properties of definable sets.

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