Proof

Choose $(x_n) \subseteq X$ with $H(x_n) \to H_{max}$. For n, m put $\overline{x}_{nm} = \frac{1}{2}(x_n + x_m)$. Then

$$H_{max} \geq H(\overline{x}_{nm} \geq \frac{1}{2}H(x_n) + \frac{1}{2}H(x_m) + \frac{1}{2}\mathrm{D}(x_n, \overline{x}_{nm}) +$$

Thus (x_n) is a Cauchy sequence w.r.t. $\|\cdot\|$, say with limit element x^*, clearly independent of the sequence we started out with. Put $y^* = \widehat{x^*}$.

Essential to prove: $R(y^*) \leq H_{max}$, equivalently that, for $x \in X$, $H(x) + \mathrm{D}(x, y^*) \leq H_{max}$. Consider $x \in X$ and $(x_n) \subseteq X$ such that $n(H_{max} - H(x_n)) \to 0$. Put $\xi_n = (1 - \frac{1}{n})x_n + \frac{1}{n}x$. Then

$$H_{max} \geq H(\xi_n) \geq (1 - \frac{1}{n})H(x_n) + \frac{1}{n}H(x) + \frac{1}{n}\mathrm{D}(x, \widehat{\xi_n})$$

and $H(x) + \mathrm{D}(x, \widehat{\xi_n}) \leq n\big(H_{max} - H(x_n)\big) + H(x_n)$ follows. By assumptions made, by convergence $\xi_n \to x^*$ and by assumptions of lower semi-continuity, the desired inequality $H(x) + \mathrm{D}(x, y^*) \leq H_{max}$ follows.
□