On the size distribution of sand

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Abstract

A model is presented of the development of the size distribution of sand while it is transported from a source to a deposit. The model provides a possible explanation of the log-hyperbolic shape that is frequently found in unimodal grain size distributions in natural sand deposits, as pointed out by Bagnold. It implies that the size distribution of a sand deposit is a logarithmic normal-inverse Gaussian (NIG) distribution, which is one of the generalized hyperbolic distributions. The model extends previous models by taking into account that individual grains do not have the same travel time from the source to the deposit. The travel time is assumed to be random so that the wear on the individual grains vary randomly. The model provides an interpretation of the parameters the NIG-distribution, and relates the mean, variance and skewness of the log-size distribution to the physical parameters of the model. This might be useful when comparing empirical size-distributions from different deposits.

1 Introduction

In his very influential book, Bagnold (1941) pointed out that the logarithm of the density function of distribution of the logarithm of the grain size in natural sand deposits looks more like a hyperbola than like a parabola. This indicated that the size distribution is not log-normal, but rather a distribution with (approximately) exponentially decreasing tails. This motivated Barndorff-Nielsen (1977) to introduce the hyperbolic distributions, for which the logarithm of the density function is a hyperbola. In a series of papers the hyperbolic distributions have been shown empirically to give a very good description of the log-size distribution of natural sand deposits, aeolian as well as alluvial, and to be a powerful tool for analyzing the spatial and temporal variation of the size distribution of sediments, see e.g. Vincent (1986), McArthur (1987), Hartmann (1991), and the review paper by Bagnold & Barndorff-Nielsen (1980). Barndorff-Nielsen (1977) actually introduced a more general class of probability distributions, the generalized hyperbolic distributions, that are similar to the hyperbolic distributions. One subclass of these is the class of normal-inverse Gaussian (NIG) distributions that gives an even better fit to the size distributions of sand than the hyperbolic distributions. This is because a NIG-distribution has slightly heavier tails than the original hyperbolic distribution, which is quite often the case for natural sand deposits. Another advantage of the NIG-distributions over the original hyperbolic distributions is that they have much nicer mathematical properties. For instance, simple expressions for the mean, the variance and all other moments can be calculated explicitly. A recent review of the properties of the generalized hyperbolic distributions and related dynamical models, with a view to applications in finance, can be found in Bibby & Sørensen (2003).

In this paper we present a model that provides a possible explanation why sand deposits are log-NIG distributed and gives an interpretation of the parameters of the distribution in terms of physical quantities. This might be useful when comparing the size distributions of sand from different deposits. The model presented here extends a model by Kolmogoroff (1941) by taking into account the fact that the grains found in a particular deposit may not have taken the same time to arrive at the deposit from their source and therefore have not been subject to the same amount of attrition. Thus the wear on the individual grains vary randomly.

The paper is organized as follows. In Section 2 the model of the development of the size-distribution is presented, and the resulting size distribution is derived by simple, but somewhat informal, arguments. In Section 3 some basic properties of the normal-inverse Gaussian distribution are reviewed, and the mean value, variance and other parameters of the log-size distribution are related to the physical parameters of the model. Various further topics are discussed in Section 4. In particular it is pointed out that a size-distributions with the same general shape is obtained even if some of the model assumptions are changed. In the appendix a formally correct derivation of the distribution result in Section 2 can be found.

2 The model and the size-distribution

The basic assumption of the model is that a grain in transit from its source to its present position has experienced a random number of breakage events, for instance forceful collisions or extreme weather conditions, that cause a random fraction of the mass of the grain to break off. Denote the number of such breakage events by N_t, where t denotes the time since the grain started from its source. If we denote the original grain size by s_0 and the fraction of the grain that is broken off in the ith collision by D_i, then the grain size at time t is

$$S_t = s_0 \prod_{i=1}^{N_t} (1 - D_i)$$

and the logarithm of the grain size is

$$\ln S_t = \mu + \sum_{i=1}^{N_t} B_i,$$

where $\mu = \ln(s_0)$ and $B_i = \ln(1 - D_i)$. We assume that the random variables B_i are independent and identically distributed with mean b_1 and with the second moment (the expectation of B_i^2) denoted by b_2. Let further N_t be a Poisson process with parameter λ that is independent of the random variables $B_1, B_2, \ldots$. This means that the number of breakage events that cause a part of the grain to break off in a time interval of length s is Poisson distributed with mean value λs, and that the numbers of such events in disjoint time intervals are independent random variables.

Under these assumptions the expectation of $\sum_{i=1}^{N_t} B_i$ is $\lambda t b_1$, and by a version of the central limit theorem, the distribution of

$$U_\lambda(t) = \frac{1}{\sqrt{\lambda}} \left(\sum_{i=1}^{N_t} B_i - \lambda t b_1 \right) \tag{2.1}$$

is approximately a normal distribution with mean zero and variance tb_2 when λ (or λt) is large. The mathematical details of the derivation of this and the following results on asymptotic distributions are given in the Appendix. We can summarize the result as follows

$$\ln S_t \dot{\sim} N(\mu + \lambda t b_1, \lambda t b_2), \tag{2.2}$$

for large λ, where $\dot{\sim}$ denotes approximate distribution.

All grains do not arrive at the sand deposit from the source at the same time. Some may have been moved back and forth for a long while, whereas other grains may have been moved directly and therefore more quickly to the sand deposit. Obviously the wear on a grain depends on how long it has been transported around. To model this effect we assume that the motion of a grain from its source is given by a Brownian motion with drift ν and infinitesimal variance (diffusion coefficient) σ^2. This means that the distance traveled in a time interval of length s in normal distributed with mean νs and variance $\sigma^2 s$, and that the distances traveled in disjoint time intervals are independent random variables. We assume that the Brownian motion of a grain is independent of the breakage process, i.e. independent of the Poisson process N_t and the random variables $B_1, B_2, \ldots$. The model is one-dimensional: we consider only the projection of the position of the grain onto on a straight line through the source and through the sand deposit. If there is a lot of motion in the lateral direction too, this will cause extra wear and can be build into the model by increasing λ, the intensity of the Poisson process of breakage events. Let τ denote the time a particular grain has taken to arrive at the deposit from the source. This is a random variable (a so-called first hitting time). It is well known, see e.g. p. 363 in Karlin & Taylor (1975) that the distribution of τ is the inverse Gaussian distribution $IG(a/\sigma, \nu/\sigma)$, where a denotes the distance between the source and the deposit. The probability density of the inverse Gaussian distribution $IG(\delta, \gamma)$ is

$$\frac{\delta}{\sqrt{2\pi x^3}} \exp\left(-\tfrac{1}{2}\gamma^2(x-\delta/\gamma)^2/x\right), \quad x > 0.$$

This result goes back to Schrödinger (1915).

The logarithm of the size of a grain that has arrived at the sand deposit at time τ is $\ln S_\tau$. Since the Brownian motion is independent of the breakage process, so is τ. Therefore the conditional distribution of the log-size given τ is, by (2.2), approximately

$$\ln S_t \mathrel{\dot\sim} N(\mu + \beta\zeta, \zeta), \tag{2.3}$$

where $\zeta = \lambda b_2 \tau$ is given, and

$$\beta = b_1/b_2.$$

The distribution of ζ is

$$\zeta \sim IG\left(a\sqrt{\lambda b_2}/\sigma,\ \nu/(\sigma\sqrt{\lambda b_2})\right). \tag{2.4}$$

The distribution of the logarithm of the grain size in the whole population of grains in the deposit is the unconditional distribution $\ln S_\tau$. This is approximately equal to the normal variance-mean mixture given by (2.3) and (2.4), which is by definition the normal-inverse Gaussian distribution $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$ with

$$\begin{aligned} \alpha &= \sqrt{b_1^2/b_2^2 + \nu^2/(\sigma^2\lambda b_2)} \\ \delta &= \frac{a}{\sigma}\sqrt{\lambda b_2} \end{aligned}$$

and with β and μ as above. This distribution has a number of nice properties, some of which will be presented in the next section. The derivation of the approximate population distribution of the logarithm of the grain size given above is not mathematically rigorous, but a formally correct derivation is given in the Appendix.

3 Interpretation of the size distribution parameters

First we review some basic properties of the normal-inverse Gaussian distributions. The probability density function of the $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$-distribution is

$$\frac{\alpha\delta}{\pi} e^{\delta\gamma} \cdot \frac{K_1\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\sqrt{\delta^2 + (x-\mu)^2}} \cdot e^{\beta(x-\mu)}$$

for any real number x. Here K_1 is a modified Bessel function of the third kind. The possible values of the parameters are $\alpha > 0, \delta > 0$, $\beta < |\alpha|$, while μ can be any real number. The expression for the probability density function is not terribly transparent, but the shape of the density is similar to that of the original hyperbolic distribution, except that the tails are slightly heavier. Examples of the shape of the logarithm of the density function are given in Figure 3.1.

If a random variable X is $\mathrm{NIG}(\alpha, \beta, \delta, \mu)$-distributed, then the mean and variance are

$$\mathrm{E}(X) = \mu + \frac{\delta\beta}{\gamma} \quad \text{and} \quad \mathrm{Var}(X) = \frac{\delta\alpha^2}{\gamma^3},$$

where $\gamma = \sqrt{\alpha^2 - \beta^2}$. The skewness is $3\beta/(\alpha\sqrt{\delta\gamma})$, and the kurtosis is $3(1 + 4\beta^2/\alpha^2)/(\delta\gamma)$. Note that when $\beta < 0$, the NIG distribution is negatively skewed.

For the original hyperbolic distribution the shape triangle has turned out to be a very useful tool for detecting trends in the distributional shape when studying samples from several sand deposits. For NIG-distributions a completely analogous shape triangle can be plotted, see e.g. Rydberg (1997). The shape triangle is defined by

$$\{(\chi, \xi) \mid 0 \leq |\chi| < \xi < 1\}, \tag{3.1}$$

where

$$\chi = \frac{\beta/\alpha}{\sqrt{1 + \delta\gamma}} \quad \text{and} \quad \xi = \frac{1}{\sqrt{1 + \delta\gamma}}.$$

These quantities are simple natural measures of asymmetry and heavy-tailedness ("kurtosis") of the normal-inverse Gaussian distributions. This follows by arguments similar to those given by Barndorff-Nielsen et al. (1985) for the hyperbolic distributions. The quantities χ and ξ are invariant under location and scale transformations, and are clearly closely related to the classical skewness and kurtosis given above. When ξ goes to zero the NIG-distribution approaches the normal distribution. In Figure 3.1 NIG log-density functions are plotted for different values of χ and ξ in the shape triangle. Note the parabola of the normal distribution for $\xi = 0$.

A nice property of the NIG distribution it that if X_1 and X_2 are independent so that $X_i \sim \mathrm{NIG}(\alpha, \beta, \delta_i, \mu_i)$, $i = 1, 2$, then we have that

$$X_1 + X_2 \sim \mathrm{NIG}(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2).$$

The normal distribution has a similar property, whereas the sum of two independent hyperbolic distributed random variables is not hyperbolic distributed.

Because there are simple expressions for the first four moments, it is relatively easy to obtain initial moment estimators of the parameters. However better estimates are obtained by maximum likelihood estimation. If an empirical size-distribution has been determined by sieving, estimators can be obtained by the multinomial pseudo-likelihood procedure outlined in Barndorff-Nielsen (1977). When comparing the estimated parameters of NIG-distributions fitted to size-distributions from different sand deposits, the model might be helpful when interpreting the empirical results. Note however, that the model is probably not useful when comparing size-distributions from different locations within the same deposit, because the differences between such size-distributions are most likely due to local sorting and not to the random breakage process that course the NIG-distribution of the model. If an empirical size distribution is poly-modal, it does not make sense to use the model. Sand with a poly-modal size distribution is probably a mixture of two or more populations of grains, each of which may (or may not) be described by the model.

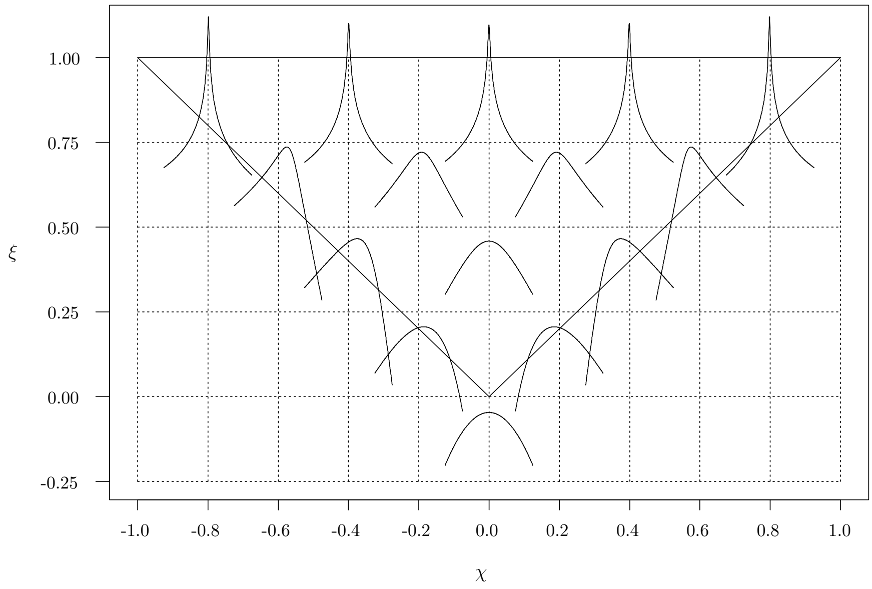

Figure 3.1: Normal-inverse Gaussian log densities with mean 0 and variance 1 for different values of the shape parameters χ and ξ. The log-densities are located in the shape triangle according to their values of χ and ξ.

The mean log-size is

$$\mu + a\lambda b_1/\nu = \mu + \lambda \bar{t} b_1 = \mu + \theta,$$

where $\bar{t} = a/\nu$ is the average travel time of a grain from the source to the deposit (the expected value of τ), so that $\lambda\bar{t}$ is the mean number of breakage events that cause a part of the grain to break off. The quantity $\theta = \lambda\bar{t}b_1$ is the average effect of the breakage process during transport from the source to the deposit. Since b_1 is the mean value of $B_i = \ln(1-D_i)$, it is negative (because $0 < D_i < 1$). Therefore the mean log-size decreases as the distance to the source a increases and as the number of collisions the cause breakage increases, while it increases when the mean speed of a grain ν increases, as one would expect.

The variance of the log-size is

$$\frac{a\lambda}{\nu}\left(b_2 + \frac{\lambda\sigma^2 b_1^2}{\nu^2}\right) = \lambda\bar{t}b_1^2\left(c_0^2 + \lambda\bar{t}\omega^2\right) = \theta^2\omega^2(1+\eta),$$

where $\omega = \sigma/(\sqrt{\bar{t}}\nu) = \sqrt{\bar{t}\sigma^2}/(\bar{t}\nu)$ is the coefficient of variation of the grain position relative to the source at time $\bar{t}$. This ratio measures the random variability of the grain motion relative to the mean distance traveled. The quantity $c_0 = \sqrt{b_2}/b_1$ is closely related to the coefficient of variation of the distribution of $B_i = \ln(1-D_i)$, which equals $\sqrt{c_0^2 - 1}$. Thus c_0 is a measure of the variability of a breakage event relative to a typical value of B_i. Finally, $\eta = c_0^2/(\lambda\bar{t}\omega^2)$ measures the variability of a breakage event relative to the variability of the grain motion and the mean number of breakage events. The variance of the log-size is an increasing function of a, λ, σ^2, b_1^2 and b_2, while it is a decreasing function of ν.

The log-size distribution is always negatively skewed because b_1 is negative. Thus positively skewed log-size distributions cannot be explained by the present model. In such cases alternative or additional processes that influence the size distribution must be sought for. The model predicts that the asymmetry parameter χ is

$$\chi = -\frac{1}{\sqrt{\left(1+\frac{\nu^2 b_2}{\sigma^2 \lambda b_1^2}\right)(1+\nu a/\sigma^2)}} = -\frac{1}{\sqrt{(1+\eta)(1+\omega^{-2})}}$$

and that the parameter ξ is

$$\xi = \frac{1}{\sqrt{1+\nu a/\sigma^2}} = \frac{1}{\sqrt{1+\omega^{-2}}}.$$

We see that the log-size distribution becomes less normal as the coefficient of variation of the grain motion increases. Note also that for a fixed value of the parameter ξ, the asymmetry is determined by the quantity η. Thus if the mean number of breakage events or the variability of the grain motion increases, the log-size distribution becomes more skewed, while the asymmetry deceases if the variability of the breakage process increases.

It is not easy to make informed guesses concerning the values of b_1 and b_2, and hence of θ and η, but the ratio of the empirical values from two deposits of each of θ and η make good sense, provided that b_1 and b_2 can be assumed to have the same values for both deposits. A very simple possible assumption about the breakage process is that probability density function of the fraction of the grain that breaks off is

$$p(x) = \rho(1-x)^{\rho-1}, \quad 0 < x < 1. \tag{3.2}$$

If $\rho > 1$ small fractions are most likely, and if $\rho = 1$ all fractions are equally likely. In any case, the distribution of $-B_i$ is an exponential distribution with $b_1 = -\rho^{-1}$ and $b_2 = 2\rho^{-2}$, so $c_0 = \sqrt{2}$, irrespective of the value of ρ. However, the breakage process can be much more complex. To illustrate this, consider another possible tractable probability density function of D_i:

$$p(x) = [\ln(1/(1-x))]^{\kappa-1}(1-x)^{\rho-1}\rho^\kappa/\Gamma(\kappa), \quad 0 < x < 1,$$

where Γ denotes the gamma function. If $\rho > 1$ and $\kappa > 1$, this density function goes to zero for both small and large fractions, and the most likely fractions are somewhere in the interior of the interval between zero and one. If $\rho > 1$ and $\kappa \leq 1$, small fractions are most likely. For $\kappa = 1$ we recover (3.2). In this case $-B_i$ is gamma distributed with $b_1 = -\kappa/\rho$ and $b_2 = (\kappa + \kappa^2)/\rho^2$, so $c_0^2 = 1 + \kappa^{-1}$. Thus any real number larger than 1 is a possible value of c_0.

4 Discussion

The probability density function of $-\sum_{i=1}^{N_t} B_i$ is explicitly known in the particular case where the probability density of D_i is given by (3.2), so that $-B_i$ is exponentially distributed with mean ρ^{-1}. Apart from a small probability that $N_t = 0$, the density function is

$$f(x) = \sqrt{\rho\lambda t/x}\, e^{-(\lambda t + \rho x)} I_1(2\sqrt{\rho\lambda t x}), \quad x > 0,$$

where I_1 is a modified Bessel function of the first kind. This result might be used as a staring point in a study of how well a NIG-distribution fits the log-size distribution. Incidentally, this probability density has previously been derived by Einstein (1937) as the density function of the position of an individual particle in alluvial sediment transport.

The assumption that the grains move according to a Brownian motion is probably a good approximation under homogeneous conditions. The effect of using a different model for the grain motion is that the distribution of the random arrival time τ is changed. This again implies that the approximate distribution of the log-size is another variance-mean mixture of normal distributions. For many other distributions of τ, the general shape of the density function of the logarithm of the grain size will be similar to that of a NIG-distribution and a hyperbolic distribution. Conditions on the mixing distribution of a normal variance-mean mixture that ensure a tail behaviour similar to that of these distributions were given by Barndorff-Nielsen, Kent & Sørensen (1982). If, for instance, the motion of a grain is modelled by a Brownian motion without drift, and if the source is near a reflecting barrier (e.g. a steep cliff), then the approximate log-size distribution is a hyperbolic cosine distribution, which has exponentially decreasing tails and is very similar to a hyperbolic distribution. It is, however, not one of the generalized hyperbolic distributions, but belongs to the class of generalized logistic distributions (a.k.a. the z-distributions). The logistic distribution itself can also be obtained in a model involving an absorbing barrier (e.g. a ravine). For details see Barndorff-Nielsen, Kent & Sørensen (1982). Diffusion models for which certain first hitting times are generalized inverse Gaussian distributions can be found in Barndorff-Nielsen & Halgreen (1977). Mixing distributions of this type give rise to generalized hyperbolic distributions.

Another possible modification of the model is to assume that N_t is an inhomogeneous Poisson process with intensity λ_t, so that the mean number of breakage events per time unit changes with time. In this case λt should be replaced by $\Lambda(t) = \int_0^t \lambda_s ds$ in (2.2). This implies another mixing distribution, namely the distribution of $\Lambda(\tau)$, but the log-size distribution is still a normal variance-mean mixture.

5 Conclusion

A model was presented that gives a possible explanation of the "hyperbolic" shape of the log-size distribution found in many natural sand deposits. The distribution is a variance-mean mixture of normal distributions because it is a mixture of the size distributions of grains that have spent different amounts of time in transit between their source and the deposit, and therefore have been subject to a randomly varying amount of attrition. Also a possible interpretation of differences between the estimated parameters of size distributions from different deposits was provided. For instance, the mean, variance and an asymmetry parameter was related to the physical parameters of the model. The model can only be used to interpret unimodal and negatively skewed log-size distributions.

Acknowledgement

I am grateful to Bo Martin Bibby for preparing Figure 3.1.

Appendix: Mathematical details

In this section a formally correct derivation of the distributional results in Section 2 is given. The notation is as in Section 2. Let

$$\varphi(u) = \mathrm{E}(e^{iuB_1})$$

denote the characteristic function of the distribution of B_1. Then the characteristic function of $U_\lambda(t)$, given by (2.1), is

$$\mathrm{E}(e^{isU_\lambda(t)}) = \exp\left(\lambda t\left[\varphi(s/\sqrt{\lambda}) - 1 - ib_1 s/\sqrt{\lambda}\right]\right).$$

By a Taylor expansion of φ, it follows that $U_\lambda(t)$ is asymptotically normal as λ goes to infinity, which is a well-known result for compound Poisson processes.

The moment generating function of the distribution of τ (the inverse Gaussian distribution $IG(a/\sigma, \nu/\sigma)$) is

$$\psi(z) = \mathrm{E}(e^{z\tau}) = \exp\left(\frac{a\nu}{\sigma^2}\left[1 - \sqrt{1 - 2z\sigma^2/\nu^2}\right]\right),$$

which is defined for all complex numbers z for which $\Re(z) \leq \nu^2/(2\sigma^2)$. Thus the joint characteristic function of $(U_\lambda(\tau), \tau)$ is

$$\begin{aligned}
\mathrm{E}\left(\exp\left(is_1 U_\lambda(\tau) + is_2\tau\right)\right) &= \mathrm{E}\left[\mathrm{E}\left(\exp\left(is_1 U_\lambda(\tau)\right) \mid \tau\right)\exp(is_2\tau)\right] \\
&= \mathrm{E}\left(\exp\left(\tau\left(\lambda\left[\varphi(s_1/\sqrt{\lambda}) - 1 - ib_1 s_1/\sqrt{\lambda}\right] + is_2\right)\right)\right) \\
&= \psi\left(\lambda\left[\varphi(s_1/\sqrt{\lambda}) - 1 - ib_1 s_1/\sqrt{\lambda}\right] + is_2\right) \\
&= \psi\left(-\tfrac{1}{2}b_2 s_1^2 + is_2 + O(\lambda^{-1/2})\right) \\
&\to \psi\left(-\tfrac{1}{2}b_2 s_1^2 + is_2\right)
\end{aligned}$$

as $\lambda \to \infty$. For λ sufficiently large, the joint distribution of $(U_\lambda(\tau), \tau)$ is approximated by the distribution with characteristic function $\psi\left(-\frac{1}{2}b_2 s_1^2 + is_2\right)$. Therefore the distribution of

$$\ln(S_\tau) = \sqrt{\lambda}U_\lambda(\tau) + \lambda b_1\tau + \mu$$

is approximated by the distribution with characteristic function

$$\psi\left(-\tfrac{1}{2}\lambda b_2 s^2 + i\lambda b_1 s\right)\exp(is\mu) = \exp\left(\frac{a\nu}{\sigma^2}\left[1 - \sqrt{1 + (\lambda b_2 s^2 - is2\lambda b_1)\sigma^2/\nu^2}\right] + is\mu\right).$$

The characteristic function of the NIG$(\alpha, \beta, \delta, \mu)$-distribution is

$$\exp\left(\delta\sqrt{\alpha^2 - \beta^2}\left[1 - \sqrt{1 + (s^2 - is2\beta)/(\alpha^2 - \beta^2)}\right] + is\mu\right),$$

so we see that $\ln(S_\tau)$ is approximately NIG$(\alpha, \beta, \delta, \mu)$-distributed with

$$\begin{aligned}
\alpha &= \sqrt{b_1^2/b_2^2 + \nu^2/(\sigma^2\lambda b_2)} \\
\beta &= b_1/b_2 \\
\delta &= \frac{a}{\sigma}\sqrt{\lambda b_2},
\end{aligned}$$

when λ is sufficiently large.

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