ON THE EXISTENCE OF $\bar{p}$-CORE PARTITIONS OF NATURAL NUMBERS.

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Abstract. Extending previous work of J. B. Olsson, cf. [7], [8], and of K. Erdmann and G. O. Michler, cf. [2], on the number of p-spin blocks of defect zero (p prime) of a double covering group of the symmetric group S_n, we prove that this number is positive for all n whenever $p \geq 7$. More precisely, it is shown that $s_p(n) > 0$ if $p \geq 7$, where $s_p(n)$ denotes the number of bar partitions of n which are $\bar{p}$-cores.

1. Introduction.

Everywhere in this article the following notation is used: p denotes an odd prime number, n a natural number, and we put: $t := (p-1)/2$.

If G is a finite group, there is some interest in the question whether G has a p-block of defect zero, since the existence of such a block means the existence of an irreducible, projective G-module in characteristic p. In general, it is a difficult problem to decide whether G has a p-block of defect zero.

It was recently proved, cf. [3] (see also [4] for an alternative proof), that if $p \geq 5$ then for every n the symmetric group S_n (and the alternating group A_n) has a p-block of defect zero.

On the other hand, the representation theory of 'double covering groups' $\hat{S}_n$ of S_n has also been studied intensively, cf. [5], [6], [7], [8]; by a 'double covering group' of S_n we shall understand any of the groups R_n and T_n given by generators $a_1, \ldots, a_{n-1}, z$ and defining relations:

$$z^2 = 1,\ a_i^2 = (a_i a_{i+1})^3 = z,\ \text{and} \quad [a_i, a_j] = z \quad \text{for } |i-j| \geq 2,$$

for R_n, and:

$$z^2 = a_i^2 = (a_i a_{i+1})^3 = [a_i, z] = 1,\ \text{and} \quad [a_i, a_j] = z \quad \text{for } |i-j| \geq 2,$$

for T_n. Thus, if $n \geq 4$, $n \neq 6$, R_n and T_n are the 2 representation groups of S_n, whereas for $n = 6$, R_n and T_n are both isomorphic to the unique representation group of S_6; cf. [9], pp. 355–357.

Denoting by $\hat{S}_n$ any of the groups R_n and T_n, it is natural to ask similarly whether $\hat{S}_n$ has a spin character, i.e. a faithful, irreducible character, of p-defect zero. This is known (A. O. Morris) to be the case if and only if n has a bar partition which is a $\bar{p}$-core, cf. [8], p. 190; see [7] or below for the definition of '$\bar{p}$-core partition'. Hence, if we denote by $s_p(n)$ the number of $\bar{p}$-core partitions of n, the question of interest is to determine the pairs (p, n) for which $s_p(n) > 0$.

The first study of the numbers $s_p(n)$ was by K. Erdmann and G. O. Michler, cf. [2], who studied the situation for $p = 5$ and $p = 7$: They proved that $s_7(n) > 0$ for all n and gave an explicit criterion for $s_5(n) > 0$ to hold.

We show in the theorem below that $s_p(n) > 0$ for all n if $p \geq 7$.

The structure of the proof is this: For 'small' n the assertion is proved more or less directly, and for 'large' n the problem is by use of a certain trick reduced to the question of representing integers by the quadratic form:

$$2x^2 + 2y^2 + z^2.$$

2.

We shall now recall from [7], pp. 233–237, some facts concerning $\bar{p}$-core partitions. Recall that a *bar partition* of n is a partition $\lambda = (a_1, \ldots, a_m)$ with $a_1 > \ldots > a_m > 0$. Such a partition is represented on the 'p-abacus' which has p runners numbered $0, 1, \ldots, p-1$ going from north to south and has its rows numbered by the non-negative integers: In the representation of λ on the p-abacus there is a bead in the i'th runner and j'th row if and only if $j \in X_i(\lambda)$ where:

$$X_i(\lambda) := \{a \in \mathbb{N}_0 |\ \exists k \in \{1, \ldots, m\} :\ a_k = ap + i\}.$$

Then λ is a $\bar{p}$-core if and only if there are no beads on the 0'th runner, and for each i with $1 \leq i \leq p-1$ the i'th runner contains

$$\ell_i := max(|X_i(\lambda)| - |X_{p-i}(\lambda)|, 0)$$

beads in the first ℓ_i rows.

Consequently, if λ is a $\bar{p}$-core then:

$$n = \sum_{i=1}^{p-1} (p \cdot \frac{1}{2} \ell_i(\ell_i - 1) + i\ell_i), \quad \text{where} \quad \ell_i \ell_{p-i} = 0 \quad \text{for } i = 1, \ldots, p-1. \tag{1}$$

Conversely, a $(p-1)$-tuple $(\ell_1, \ldots, \ell_{p-1})$ of non-negative integers with (1) gives rise to a unique $\bar{p}$-core partition of n. Thus, $s_p(n)$ is the number of such $(p-1)$-tuples, – a fact which was also noted in [2].

From this we conclude that $s_p(n)$ is the number of t-tuples $(y_1, \ldots, y_t) \in \mathbb{Z}^t$ with:

$$n = \sum_{i=1}^{t} (p \cdot \frac{1}{2} y_i(y_i - 1) + iy_i) \ ; \tag{2}$$

in fact the maps: $(\ell_1, \ldots, \ell_{p-1}) \mapsto (y_1, \ldots, y_t)$ given by:

$$y_i = \ell_i - \ell_{p-i} \qquad \text{for} \quad i = 1, \ldots, t \ ,$$

and $(y_1, \ldots, y_t) \mapsto (\ell_1, \ldots, \ell_{p-1})$ given by:

$$(\ell_i, \ell_{p-i}) = (y_i, 0) \quad \text{if} \quad y_i \geq 0, \qquad \text{and} \quad (\ell_i, \ell_{p-i}) = (0, -y_i) \quad \text{if} \quad y_i < 0 \ ,$$

are seen to give mutually inverse bijections between the set of $(\ell_1, \ldots, \ell_{p-1}) \in \mathbb{N}_0^{p-1}$ with (1) , and the set of $(y_1, \ldots, y_t) \in \mathbb{Z}^t$ with (2).

Using

$$\sum_{i=1}^{t} (2i - p)^2 = \frac{1}{6} p(p-1)(p-2) \ ,$$

and writing $x_i = (2i-p) + 2py_i$, we then also conclude that $s_p(n)$ is the number of t-tuples $(x_1, \ldots, x_t) \in \mathbb{Z}^t$ such that:

$$n = \frac{1}{8p}\sum_{i=1}^{t} x_i^2 - \frac{(p-1)(p-2)}{48}, \quad \text{where} \quad x_i \equiv 2i - p \ \ (2p) \tag{3}$$

for $i = 1, \ldots, t$.

Let us now briefly indicate a proof of one of the results in [2], namely that this last equation is solvable for all $n \in \mathbb{N}$, if $p = 7$:

Putting $N := 7 \cdot (8n+5)$, we have that $s_7(n)$ is the number of integral solutions to:

$$N = x^2 + y^2 + z^2 \ , \tag{$*$}$$

with $(x, y, z) \equiv (-5, -3, -1)$ (14). Now one notices that any solution to $(*)$ satisfies either $(x, y, z) \equiv (0, 0, 0)$ (7) or $(x, y, z) \equiv (\pm 5, \pm 3, \pm 1)$ (7) up to permutation of x, y, z. In the latter case one has in fact $(x, y, z) \equiv (\pm 5, \pm 3, \pm 1)$ (14) up to permutation, since x, y, z must all be odd since $N \equiv 3$ (4). From this, one easily concludes that $s_7(n)$ equals $a_3(N)/48$ if $n \not\equiv 2$ (7), and equals $(a_3(N) - a_3(N/7^2))/48$ if $n \equiv 2$ (7), where for $M \in \mathbb{N}$, $a_3(M)$ denotes the number of representations of M as a sum of 3 squares. Now, from the classical formulas for the numbers $a_3(M)$, one then easily derives even explicit formulas for $s_7(n)$. Suppose for example $n \not\equiv 2$ (7), and write $N = p_1^{2a_1} \ldots p_v^{2a_v} \cdot r$, where $p_1, \ldots, p_v$ are distinct odd primes and r is square free and $\equiv 3$ (8); then if h denotes the class number of $\mathbb{Q}(\sqrt{-r})$ and χ_{-r} the Dirichlet character belonging to this field, so that

$$\chi_{-r}(x) = \left(\frac{-r}{x}\right) \text{ for } (x, r) = 1,$$

one has:

$$s_7(n) = \frac{1}{2} h \cdot \Pi_{i=1}^{v} \left(1 + (p_i - \chi_{-r}(p_i)) \cdot \frac{p_i^{a_i} - 1}{p_i - 1}\right).$$

Similar formulas are obtained for $n \equiv 2$ (7). In particular, one has $s_7(n) > 0$ for all $n \in \mathbb{N}$.

The interpretation of $s_p(n)$ as the number of solutions to (3) above suggests a connection to modular forms. In fact the numbers $s_p(n)$ for fixed p are related to the Fourier coefficients of a certain product of classical theta series. One can use this to obtain asymptotic formulas for $s_p(n)$ for fixed p and $n \to \infty$, at least for $p \equiv 1$ (4). We shall report on this elsewhere.

3.

Let us define the functions f_i for $i = 1, \ldots, t (= \frac{p-1}{2})$:

$$f_i(y) = p \cdot \frac{1}{2} y(y-1) + iy.$$

Hence, in order to show that $s_p(n) > 0$ we must show that the equation:

$$n = \sum_{i=1}^{t} f_i(y_i),$$

has a solution in integers $y_1, \ldots, y_t$.

Lemma 1. *Suppose that $p \geq 11$ and that $n \in \mathbb{N}$ with $n \leq p \cdot \frac{1}{2}(p-1)^2 + (p-1)$. Then there exist $y_i \in \mathbb{Z}$, $i = 1, \ldots, t$ such that:*

$$n = \sum_{i=1}^{t} f_i(y_i).$$

Proof. Put $n_0 := p(p-1)^2/2 + p - 1$.

Notice first that if m, $s \in \mathbb{N}$ then there are $y_i \in \{0,1\}$ for $i = 1, \ldots, s$ such that:

$$m = \sum_{i=1}^{s} i y_i,$$

if and only if $m \leq \frac{1}{2}s(s+1)$ (use induction on s). Since $f_i(0) = 0$, $f_i(1) = i$, we see that is suffices to find $s \in \{0, \ldots, t\}$ and $y_i \in \mathbb{Z}$ for $s < i \leq t$ such that:

$$n - \sum_{s<i\leq t} f_i(y_i) \leq \frac{1}{2}s(s+1).$$

So, suppose that $s \in \{0, \ldots, t-1\}$, put $c_0(n) := n$, and define the integers $y_{t-i}(n)$ and $c_{i+1}(n)$ for $i = 0, \ldots, t-s-1$ successively:

$$y_{t-i}(n) := \left[\left((2i+1) + \sqrt{(2i+1)^2 + 8pc_i(n)} \right) /2p \right],$$

and $c_{i+1}(n) := c_i(n) - f_{t-i}(y_{t-i}(n))$, so that $c_i(n) \geq 0$ for $i = 0, \ldots, t-s$. Then we have

$$c_{t-s}(n) = n - \sum_{s<i\leq t} f_i(y_i).$$

Hence the proof is finished if

$$c_{t-s}(n) \leq \frac{1}{2}s(s+1) \qquad \text{for all} \quad n \leq n_0. \tag{1}$$

Now, since $y_{t-i}(n) \in \mathbb{Z}$ is largest possible such that $f_{t-i}(y_{t-i}(n)) \leq c_i(n)$ we have:

$$c_{i+1}(n) \leq py_{t-i}(n) + t - i \leq \frac{1}{2}\left(p + \sqrt{(2i+1)^2 + 8pc_i(n)}\right) \ ;$$

hence, if $i \in \{1, \ldots, t-s\}$ and $G_i(x)$ is a polynomial, the condition

$$(8p)^{2^{t-s-i}-1} c_i(n) \leq G_i(p) \qquad \text{for all} \quad n \leq n_0$$

is seen to be implied by

$$(8p)^{2^{t-s-i+1}-1} c_{i-1}(n) \leq G_{i-1}(p) \qquad \text{for all} \quad n \leq n_0,$$

where

$$G_{i-1}(x) := (2G_i(x) - x \cdot (8x)^{2^{t-s-i}-1})^2 - (2i-1)^2 \cdot (8x)^{2^{t-s-i+1}-2}, \tag{2}$$

provided that $H_i(p) \geq 0$ where

$$H_i(x) := 2G_i(x) - x \cdot (8x)^{2^{t-s-i}-1}. \tag{3}$$

Hence, if we define $G_4(x) := \frac{1}{2}((x-1)/2-4)((x-1)/2-3)$, $H_4(x) := 2G_4(x) - x$, and $G_i(x), H_i(x)$ for $i = 3, 2, 1, 0$ in accordance with (2) and (3), we have that (1) holds for the case $s = t-4$, if

$$H_1(p), \ldots, H_4(p) \geq 0 \quad \text{and} \quad (\ (8p)^{15} c_0(n) \leq G_0(p) \quad \text{for all} \quad n \leq n_0\ ). \tag{4}$$

But as $c_0(n) = n \le p(p-1)^2/2 + p - 1$, (4) holds if

$$H_1(p), \ldots, H_4(p), G(p) \ge 0, \tag{5}$$

where

$$G(x) := G_0(x) - (8x)^{15}(x(x-1)^2/2 + x - 1)\ .$$

Now, $H_1(x), \ldots, H_4(x), G(x)$ are polynomials with rational coefficients which can be computed by using (for example) MAPLE. For example, one finds:

$$G(x) := 2^{-18}(x^{32} - 320x^{31} + 48496x^{30} - \ldots + 5^8 7^{16} 13^8).$$

Using MAPLE again, we can compute approximations to the real roots of these polynomials and thus verify that (5) holds if $p > 43$, i.e. (1) holds for the case $s = t - 4$ if $p > 43$.

In the remaining cases $11 \le p \le 43$ one may verify the lemma by direct computation on a machine. We leave this to the reader indicating only that one finds in each of these cases solutions with $y_i = 0$ for $5 < i < t - 5$. $\square$

Theorem 1. *Suppose that $p \ge 7$ and that $n \in \mathbb{N}$. Then there exists a $\bar{p}$-core partition of n.*

Hence, a double covering group $\hat{S}_n$ of S_n has a spin character of p-defect zero.

Proof. The last statement follows from the first as explained in the introduction.

By [2], or by the remarks made above, we may assume that $p \ge 11$.

We must show the existence of integers $y_1, \ldots, y_t$ such that:

$$n = \sum_{i=1}^{t} f_i(y_i).$$

Because of the lemma, we may and will assume:

$$n \ge p \cdot \frac{1}{2}(p-1)^2 + (p-1).$$

Hence, if we write

$$n = pm + r \qquad \text{with} \quad m \in \mathbb{N}_0, \quad |r| \le p - 1 \quad \text{and} \quad r \equiv 0\ (2),$$

then $m \ge r^2/2$ so that the number $N := 48m - 24r^2 + 5$ is a natural number.

Now, we know, confer [1], that the quadratic form $2x^2 + 2y^2 + z^2$ represents every natural number not of the form $4^s(8l + 7)$. Thus there exist $u, v, w \in \mathbb{Z}$ such that:

$$N = 2u^2 + 2v^2 + w^2\ ; \tag{$*$}$$

in fact, the solvability of $(*)$ in integers follows easily from the fact that N is a sum of 3 squares.

Since r is even, we have $N \equiv 5\ (16)$; a consideration of $(*)$ mod 16 then shows that u and v are odd and $w \equiv \pm 1, \pm 7\ (16)$. We may then assume

$$w \equiv -1\ (8).$$

Considering $(*)$ mod 3 and exchanging u and v if necessary, we find

$$(\ u^2 \equiv 1\ (3),\ v^2 \equiv w^2 \equiv 0\ (3)\ ) \quad \text{or} \quad (\ u^2 \equiv v^2 \equiv w^2 \equiv 1\ (3)\ ).$$

Since u, v, w are all odd, we may then assume

$$u \equiv -1\ (6) \quad \text{and} \quad v \equiv w\ (6).$$

Then the number $v+1-(w+1)/4$ is divisible by 6, and we define the integers:

$$a := \frac{u+1}{6},\ b := -\frac{r}{2} + \frac{1}{6}\left(v+1-\frac{w+1}{4}\right),\ c := \frac{r}{2} + \frac{w+1}{8},$$

and

$$y_1 := a+b,\ y_2 := a+c,\ y_3 := -a+b+c,\ y_4 := -r-b,\ y_5 := r-c,$$

and $y_i = 0$ for $i \geq 6$. Then we have:

$$\sum_{i=1}^{t} f_i(y_i) = \sum_{i=1}^{5}\left(p \cdot \frac{1}{2} y_i(y_i-1) + iy_i\right) = p \cdot \frac{1}{48}(2u^2+2v^2+w^2+24r^2-5)+r$$
$$= pm + r = n.$$

$\square$

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References

[1] L. E. Dickson: 'Integers represented by positive ternary quadratic forms.' Bull. Amer. Math. Soc. **33** (1927), 63–70.

[2] K. Erdmann, G. O. Michler: 'Blocks for symmetric groups and their covering groups and quadratic forms.' Beiträge Algebra Geom. **37** (1996), 103–118.

[3] A. Granville, K. Ono: 'Defect zero p-blocks for finite simple groups.' Trans. Amer. Math. Soc. **348** (1996), 331–347.

[4] I. Kiming: 'A note on a theorem of A. Granville and K. Ono.' J. Number Theory **60** (1996), 97–102.

[5] G. O. Michler, J. B. Olsson: 'Weights for covering groups of symmetric and alternating groups, $p \neq 2$.' Canad. J. Math. **43** (4) (1991), 792–813.

[6] A. O. Morris, J. B. Olsson: 'On p-quotients for spin characters.' J. Algebra **119** (1988), 51–82.

[7] J. B. Olsson: 'Frobenius symbols for partitions and degrees of spin characters.' Math. Scand. **61** (1987), 223–247.

[8] J. B. Olsson: 'On the p-blocks of symmetric and alternating groups and their covering groups.' J. Algebra **128** (1990), 188–213.

[9] I. Schur: 'Über die Darstellung der symmetrischen und der alternierenden Gruppen durch gebrochene lineare Substitutionen.'
No. **16** in: I. Schur: 'Gesammelte Abhandlungen', Bd. I., Springer-Verlag 1973.

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