# Matsumoto algebras for substitutional shift spaces

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## *Substitutions*

Let $\mathfrak{a}$ be a finite set of symbols, and let $\mathfrak{a}^\sharp$ denote the set of finite non-empty words with letters from $\mathfrak{a}$. A *substitution* is a map

$$\tau : \mathfrak{a} \longrightarrow \mathfrak{a}^\sharp$$

We write "$w \dashv v$" when $w, v \in \mathfrak{a}^\sharp$ and $w$ is a subword of $v$.

Define

$$\underline{\mathsf{X}}_\tau = \{(x_i) \in \mathfrak{a}^{\mathbb{Z}} \mid \forall i < j \exists n, a : x_{[i,j]} \dashv \tau^n(a)\}.$$

and equip with

$$\sigma(x_n) = (x_{n+1})$$

## *Shift dynamics*

Using the product topology, $(\underline{\mathrm{X}}_{\mathcal{F}}, \sigma)$ is a topological dynamical system.

Applications among others:

- Automata theory
- Quasicrystals
- Recurrent sets
- Transcendence in $\mathbb{R}$
- Diophantine approximation

'The book': `http://iml.univ-mrs.fr/editions/preprint00/book/prebookdac.html`

## *Some substitutions*

$$\tau_1(\text{א}) = \text{אבא} \qquad \tau_1(\text{ב}) = \text{באאב}$$

$$\tau_2(\alpha) = \alpha\beta \qquad \tau_2(\beta) = \alpha\beta\gamma\delta\epsilon \qquad \tau_2(\gamma) = \alpha\beta$$

$$\tau_2(\delta) = \gamma\delta\epsilon \qquad \tau_2(\epsilon) = \alpha\beta\gamma\delta\epsilon$$

$$\tau_3(1) = 1212345$$

$$\tau_3(2) = 12123451234512345$$

$$\tau_3(3) = 1212345 \qquad \tau_3(4) = 1234512345$$

$$\tau_3(5) = 12123451234512345$$

$$\tau_4(a) = ababacb \qquad \tau_4(b) = ababacbabacbabacb$$

$$\tau_4(c) = abacbabacb$$

## *Substitution properties*

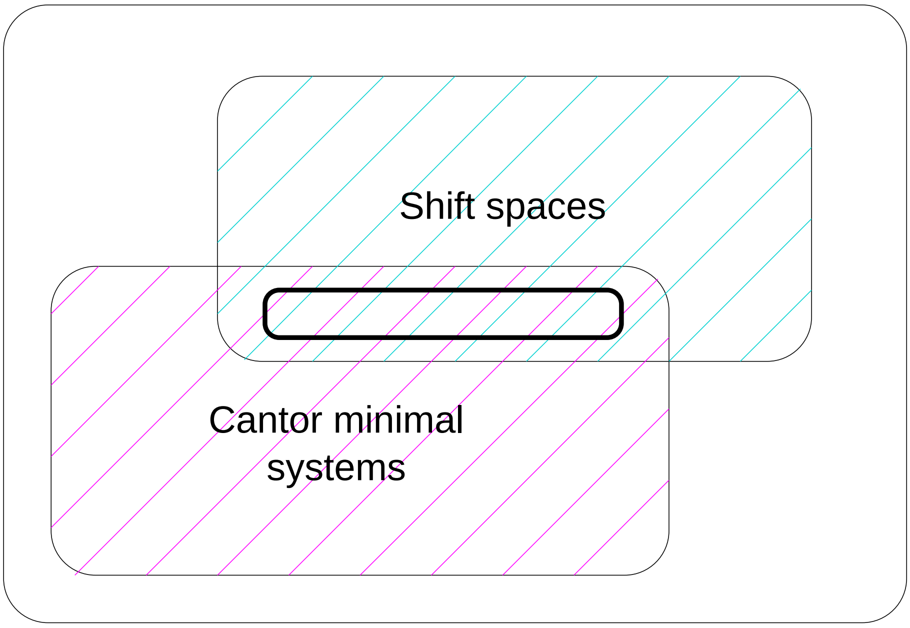
**Definition** $\tau$ is *primitive* if

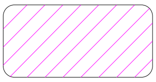
$$\exists N \forall a, b : b \dashv \tau^N(a)$$
$$\forall a : |\tau^N(a)| \longrightarrow \infty$$

**Definition** $\tau$ is *aperiodic* if $|\underline{\mathsf{X}}_\tau| = \infty$.

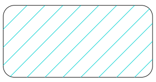
**Observation** If $\tau$ is primitive and aperiodic, then $(\underline{\mathsf{X}}_\tau, \sigma)$ [with product topology] is a Cantor minimal system.

# *$C^*$-algebra invariants*

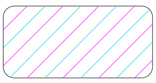



-  *Cantor minimal* crossed product

$$\tau \mapsto C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z}.$$

-  *Matsumoto algebra*

$$\tau \mapsto \mathcal{O}_\tau \otimes \mathbb{K}$$

-  Both!

## *The Matsumoto algebra*

Several equivalent constructions

(i) Generators and relations

(ii) Groupoid algebra

(iii) Cuntz-Pimsner algebra

which − WARNING! − will sometimes differ from the original

(iv) Fock space algebra

cf. Carlsen/Matsumoto.

## *Incidence matrix*

To a substitution $\tau$ one associates the $|\mathfrak{a}| \times |\mathfrak{a}|$-matrix $\mathbf{A}_\tau$ given by

$$(\mathbf{A}_\tau)_{a,b} = \#\text{ of occurrences of } b \text{ in } \tau(a)$$

**Theorem** [Giordano/Putnam/Skau[2]/Durand/Host] When $\tau$ is aperiodic, primitive and proper*,

$$K_0(C(\underline{X}_\tau) \rtimes_\sigma \mathbb{Z}) = \varinjlim(\mathbb{Z}^{|\mathfrak{a}|}, \mathbf{A}_\tau)$$

as ordered groups.

*No loss of generality

## *$C^*$-qualities of $\mathcal{O}_\mathcal{T}$*

$\mathcal{O}_\mathcal{T}$ is nonsimple, and has a maximal ideal isomorphic to $\mathbb{K}^{\mathsf{n}_\mathcal{T}}$ for suitable $\mathsf{n}_\mathcal{T}$. Further,

$$0 \longrightarrow \mathbb{K}^{\mathsf{n}_\mathcal{T}} \longrightarrow \mathcal{O}_\mathcal{T} \longrightarrow C(\underline{\mathsf{X}}_\mathcal{T}) \rtimes_\sigma \mathbb{Z} \longrightarrow 0$$

However,

$$\left.\begin{array}{r} C(\underline{\mathsf{X}}_\mathcal{T}) \rtimes_\sigma \mathbb{Z} \simeq C(\underline{\mathsf{X}}_\mathcal{U}) \rtimes_\sigma \mathbb{Z} \\ \mathsf{n}_\mathcal{T} = \mathsf{n}_\mathcal{U} \end{array}\right\} \not\Longrightarrow \mathcal{O}_\mathcal{T} \simeq \mathcal{O}_\mathcal{U}$$

## $K$-qualities of $\mathcal{O}_\tau$

The short exact sequence induces

$$\begin{array}{ccccc} \mathbb{Z}^{\mathsf{n}_\tau} & \longrightarrow & K_0(\mathcal{O}_\tau) & \longrightarrow & K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_\sigma \mathbb{Z}) \\ {\scriptstyle \mathsf{p}_\tau}\uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

for suitable $\mathsf{p}_\tau \in \mathbb{N}^{\mathsf{n}_\tau} \backslash \{\underline{0}\}$. Consequently, $\mathcal{O}_\tau$ has real rank zero but is not stably finite.

And again,

$$\left.\begin{array}{r} K_0(C(\underline{\mathsf{X}}_\tau) \rtimes_\sigma \mathbb{Z}) \simeq K_0(C(\underline{\mathsf{X}}_\upsilon) \rtimes_\sigma \mathbb{Z}) \\ \mathsf{n}_\tau = \mathsf{n}_\upsilon \\ \mathsf{p}_\tau = \mathsf{p}_\upsilon \end{array}\right\} \not\Longrightarrow$$

$$K_0(\mathcal{O}_\tau) \simeq K_0(\mathcal{O}_\upsilon)$$

**Theorem** [Carlsen/Eilers]

Let $\tau$ be a primitive, aperiodic, proper* and injective† substitution of constant length‡. For suitable $\mathsf{n}_\tau \times |\mathfrak{a}|$-matrix $\mathbf{E}_\tau$ we define

$$\widetilde{\mathbf{A}}_\tau = \begin{bmatrix} \mathbf{A}_\tau & 0 \\ \mathbf{E}_\tau & \mathbf{Id} \end{bmatrix}$$

$$H_\tau = \mathbb{Z}^{\mathsf{n}_\tau}/\mathsf{p}_\tau\mathbb{Z}$$

and have

$$K_0(\mathcal{O}_\tau) = \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus H_\tau, \widetilde{\mathbf{A}}_\tau)$$

as ordered group, where $\mathbb{Z}^{|\mathfrak{a}|} \oplus H_\tau$ is ordered by

$$(x, y) \geq 0 \Longleftrightarrow x \geq 0$$

*No loss of generality

†No loss of generality

‡Dispensable

## *What are* $\mathsf{n}_\tau, \mathsf{p}_\tau$ *?*

**Definition** $x \in \underline{\mathsf{X}}_\tau$ is right special if

$$\exists n : x_{[n,\infty[} = y_{[n,\infty[} \wedge x \neq y$$

**Theorem** [Queffélec]
If $\tau$ is a primitive and aperiodic substitution on $\mathfrak{a}$, the number of orbit classes of special words is nonzero, but finite.

**Answer**

$$\mathsf{n}_\tau = \#\{[x]_{\mathsf{orbit}} \mid x \text{ is right special}\}$$

Enumerate one-sided representatives of the right special orbits for $\tau$ as

$$x_1, \ldots, x_{\mathsf{n}_\tau}.$$

**Answer**

$$(\mathsf{p}_\tau)_i = \#\{y \in \underline{\mathsf{X}}_\tau \mid y_{[0,\infty[} = x_i\} - 1$$

## *What is* $\mathsf{E}_\tau$*?*

When $\tau$ is of constant length $\ell$, the right special elements are all of the form

$$\cdots \tau^{3k}(w)\tau^{2k}(w)\tau^{k}(w)a\tau^{k}(v)\tau^{2k}(v)\tau^{3k}(v)\cdots$$

where $w$ and $v$ are unique if $|w|, |v| < \ell$. Fix $v_i$ representing to the right orbit class $x_i$ and note that if we enumerate the corresponding $w$ as

$$w_1, \ldots, w_{m_i},$$

then $(\mathsf{p}_\tau)_i = m_i - 1$.

**Answer** With this setup,

$$(\mathbf{E})_{i,b} = \sum_{i=1}^{m_i-1} [\# \text{ of occurrences of } b \text{ in } w_i]$$

# *Computability, I*

*[Previous work by Barge/Diamond, related unpublished work by Barge/Diamond/Holton]*

There are efficient algorithms for computing $\mathsf{n}_\mathcal{T}, \mathsf{p}_\mathcal{T}$ and $\mathbf{E}_\mathcal{T}$:

```
CONCERNING tau GIVEN BY:
[a->bcada, b->bdbca, c->bccda, d->bddca]

COMPUTING ND_tau:
(a,c) <-- da,+ -- (a,c)
(b,d) <-- ca,+ -- (b,d)
(c,d) <-- a,- -- (c,d)

COMPUTING CONFIGURATION DATA [Pass to (tau)^2]
[0--0, 0--0, 2--2, 2--2, 4--4, 5--4, 6--6, 6--7]

COMPUTING p-VECTOR AND E-MATRIX [Pass to (tau)^2]
Enumerating: [dabddcabcada, cabccdabcada, abcada]
p_tau: [1, 1, 1]
E_tau: [[2, 4, 4, 2], [2, 4, 2, 4], [3, 5, 5, 5]]
```

This extends to all primitive and aperiodic substitutions as regards $\mathsf{n}_\mathcal{T}$ and $\mathsf{p}_\mathcal{T}$.

## *Computability, II*

Decidability of

$$K_0(\mathcal{O}_\tau) \simeq K_0(\mathcal{O}_\upsilon)$$

is open in general. C, but cf. Bratteli/Jorgensen/Kim/Roush.