Classifying graph C^*-algebras

Søren Eilers
eilers@math.ku.dk

Department of Mathematical Sciences
University of Copenhagen

NBFAS 05.03.10

Program

Finitely many ideals

Observation (cf. Jordan-Hölder)

When the C^*-algebra A has finitely many ideals a finite decomposition series

$$0 = I_0 \lhd I_1 \lhd \cdots \lhd I_n = A, \qquad I_j/I_{j-1} \text{ simple}$$

exists with $(I_1/I_0, I_2/I_1, \ldots, I_n/I_{n-1})$ unique up to isomorphism and permutation.

Of course, the decomposition series does **not** determine A. But suppose the I_j/I_{j-1} are all classifiable by K-theory, is the same then true for A?

$\mathbb{B}(H)$: A C^*-algebra with one non-trivial ideal

$\mathbb{K}$ is AF

The compacts form an AF algebra, i.e. for any finite set $a_1, \ldots a_\ell$ and $\epsilon > 0$ there is a finite-dimensional algebra $F \subseteq \mathbb{K}$ with $\|a_i - f_i\| < \epsilon$ for some $f_i \in F$.

$\mathbb{B}(H)/\mathbb{K}$ is purely infinite

The Calkin algebra is purely infinite, i.e. for any $x, y \in \mathbb{B}(H)/\mathbb{K}$ with $x \neq 0$ there exist elements a, b such that

$$y = axb$$

Further properties

Real rank zero

$\mathbb{B}(H)$, $\mathbb{K}$ and $\mathbb{B}(H)/\mathbb{K}$ have real rank zero, i.e. for any self-adjoint element a and any $\epsilon > 0$ there is a self-adjoint element f with finite spectrum such that $\|a - f\| < \epsilon$.

Separability and nuclearity

$\mathbb{K}$ is separable and nuclear. Neither of $\mathbb{B}(H)$ and $\mathbb{B}(H)/\mathbb{K}$ are.

Graph algebras

Graph algebras

Any countable graph $G = (E^0, E^1)$ defines a C^*-algebra $C^*(G)$ given as a universal C^*-algebra by **projections** $\{p_v : v \in E^0\}$ and **partial isometries** $\{s_e : e \in E^1\}$ subject to the *Cuntz-Krieger relations*:

1. $p_v p_w = 0$ when $v \neq w$
2. $(s_e s_e^*)(s_f s_f^*) = 0$ when $e \neq f$
3. $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$
4. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every v with $0 < |\{e \mid s(e) = v\}| < \infty$.

Singular vertices

When $\{e \mid s(e) = v\} = \emptyset$ we say that v is a **sink**. When $|\{e \mid s(e) = v\}| = \infty$ we say that v is an **infinite emitter**. In either case, we say that v is **singular**.

C^*-equivalence of graphs

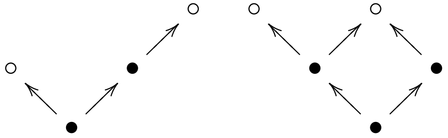
For which pairs of graphs do we have

$$C^*(G) \otimes \mathbb{K} \simeq C^*(H) \otimes \mathbb{K}?$$

Subcase: *AF*

Theorem (Kumjian-Pask-Raeburn)

$C^*(G)$ is AF precisely when G has no cycles, i.e. is a forest.

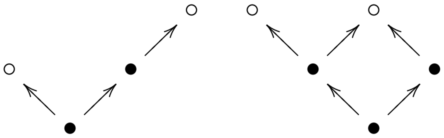


Subsubcase: Finite forest

Theorem

The following are equivalent for finite forests G and H

- $C^*(G) \otimes \mathbb{K} \simeq C^*(H) \otimes \mathbb{K}$
- *G and H have the same number of leaves*



Subsubcase: A matroid tree

Consider the case where $G = G[n_i]$ is given by a sequence of integers n_i describing an infinite tree

$$\bullet \overset{n_1}{\Longrightarrow} \bullet \overset{n_2}{\Longrightarrow} \bullet \overset{n_3}{\Longrightarrow} \bullet \overset{n_4}{\Longrightarrow} \cdots$$

Theorem

The following are equivalent

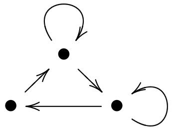
- $C^*(G[n_i]) \otimes \mathbb{K} \simeq C^*(G[m_i]) \otimes \mathbb{K}$
- $\exists j : x | \prod_{i=1}^{j} n_i \iff \exists j : x | \prod_{i=1}^{j} m_i$

Subcase: Purely infinite

Theorem (Cuntz-Krieger, an Huef-Raeburn)

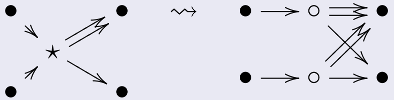
When G is a finite and strongly connected graph then the following are equivalent

1. $C^*(G)$ *has finitely many ideals*
2. $C^*(G)$ *is simple*
3. $C^*(G)$ *has real rank zero*
4. $C^*(G)$ *is purely infinite*
5. G *is not a cycle*



Theorem (Franks, Cuntz, Rørdam)

The relation induced on the class of finite and strongly connected graphs by stable isomorphism of the associated graph C^*-algebra is the smallest equivalence relation containing

Edge expansion	
State splitting	
Cuntz splice	

Unifying invariant

Theorem

A graph C^*-algebra is separable and nuclear.

Theorem (Kumjian-Pask-Raeburn)

A simple graph C^*-algebra is either AF or purely infinite.

Theorem (Elliott, Kirchberg-Phillips)

$K_*(-)$ is a complete invariant for stable isomorphism of graph C^*-algebras which are simple, or AF.

Theorem (Hong-Szymański)

$C^*(G)$ has real rank zero precisely when no cycle in G is unique.

Corollary

If $C^*(G)$ has finitely many ideals, then $C^*(G)$ has real rank zero.

Sets of vertices

Hereditary

$F^0 \subseteq E^0$ is **hereditary** when $s(e) \in F^0 \Rightarrow r(e) \in F^0$

Saturated

$F^0 \subseteq E^0$ is **saturated** when for any non-singular $v \notin F^0$ there is an edge e with $r(e) = v$, $s(e) \notin F^0$.

Breaking vertex

An infinite emitter v is a **breaking vertex** for F^0 if

$$0 < |\{e \in E^1 \mid r(e) = v, s(e) \notin F^0\}| < \infty$$

Ideal structure

Theorem

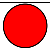
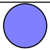
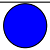
When $C^*(G)$ has real rank zero there is a one-to-one correspondance between the ideals of $C^*(G)$ and pairs (F^0, B^0) chosen such that

- F^0 *is hereditary*
- F^0 *is saturated*
- B^0 *is a set of breaking vertices for* F^0

Theorem

The ideal corresponding to $(F^0, \emptyset)$ is stably isomorphic to $C^*(H)$ where H is the subgraph of G with F^0 as vertex set.

Color coding

G	$C^*(G)$	Legend
Cofinal tree	Simple AF algebra	
Finite, strongly connected graph (not a cycle)	Simple Cuntz-Krieger algebra	
Graph with a cycle, no unique cycles, and only trivial hereditary and saturated subsets	Simple purely infinite algebra	

K-theory

When G is presented by an adjacency matrix in block form

$$\begin{bmatrix} A & \alpha \\ * & * \end{bmatrix}$$

with singular vertices in the last row and column blocks, then

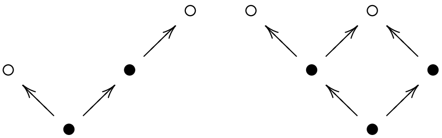
$$K_0(C^*(G)) = \operatorname{cok} \begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix} \qquad K_1(C^*(G)) = \ker \begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix}$$

Subsubcase: Finite forest

Theorem

The following are equivalent for finite forests G and H

- $C^*(G) \otimes \mathbb{K} \simeq C^*(H) \otimes \mathbb{K}$
- *G and H have the same number of leaves*



K-theory

$$K_0(C^*(G)) = \mathbb{Z}^{\#\text{leaves}} \qquad K_1(C^*(G)) = 0$$

Subsubcase: A matroid tree

Consider the case where $G = G[n_i]$ is given by a sequence of integers n_i describing an infinite tree

$$\bullet \overset{n_1}{\Longrightarrow} \bullet \overset{n_2}{\Longrightarrow} \bullet \overset{n_3}{\Longrightarrow} \bullet \overset{n_4}{\Longrightarrow} \cdots$$

Theorem

- $C^*(G[n_i]) \otimes \mathbb{K} \simeq C^*(G[m_i]) \otimes \mathbb{K}$
- $\exists j : x | \prod_{i=1}^{j} n_i \iff \exists j : x | \prod_{i=1}^{j} m_i$

K-theory

$$K_0(C^*(G[n_i])) = \lim(\mathbb{Z} \xrightarrow{n_1} \mathbb{Z} \xrightarrow{n_2} \mathbb{Z} \xrightarrow{n_2} \cdots)$$

$$K_1(C^*(G[n_i])) = 0$$

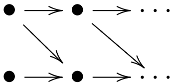
Theorem (Franks, Cuntz, Rørdam)

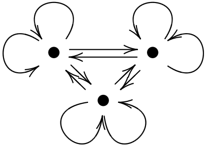
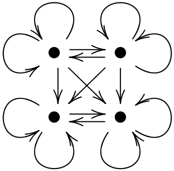
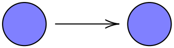
The relation induced on the class of finite and strongly connected graphs by stable isomorphism of the associated graph C^*-algebra is the smallest equivalence relation containing

Edge expansion	$\bullet \rightarrow \bullet \quad \rightsquigarrow \quad \bullet \rightarrow \circ \rightarrow \bullet$
State splitting	
Cuntz splice	

K-theory

$$K_0(C^*(G_A)) = \operatorname{cok}(A^t - 1)$$
$$K_1(C^*(G_A)) = \ker(A^t - 1) = \operatorname{cok}(A^t - 1)/\operatorname{tor}(\operatorname{cok}(A^t - 1))$$

G	$K_0(G)$	$K_0(G)_+$	Ideals
	$\mathbb{Z}^2$	$\{(x,y) \mid x \geq 0, y \geq 0\}$	
	$\mathbb{Z}^2$	$\{(x,y) \mid x + \frac{\sqrt{5}-1}{2} y \geq 0\}$	

G	$K_0(G)$	$K_0(G)_+$	Ideals
	$\mathbb{Z}^2$	$\mathbb{Z}^2$	
	$\mathbb{Z}^2$	$\mathbb{Z}^2$	

Theorem (Drinen-Tomforde, Carlsen-E-Tomforde)

For $C^*(G)$ given by $\begin{bmatrix} A & \alpha & 0 & 0 \\ * & * & 0 & 0 \\ X & \xi & B & \beta \\ * & * & * & * \end{bmatrix}$ *the six-term exact sequence in K-theory becomes*

$$\begin{array}{ccccc}
\operatorname{cok}\begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix} & \xrightarrow{I} & \operatorname{cok}\begin{bmatrix} A^t-1 & X^t \\ \alpha^t & \xi^t \\ 0 & B^t-1 \\ 0 & \beta^t \end{bmatrix} & \xrightarrow{P} & \operatorname{cok}\begin{bmatrix} B^t - 1 \\ \beta^t \end{bmatrix} \\
\begin{bmatrix} X^t \\ \xi^t \end{bmatrix}\Big\uparrow & & & & \Big\downarrow 0 \\
\ker\begin{bmatrix} B^t - 1 \\ \beta^t \end{bmatrix} & \xleftarrow[P]{} & \ker\begin{bmatrix} A^t-1 & X^t \\ \alpha^t & \xi^t \\ 0 & B^t-1 \\ 0 & \beta^t \end{bmatrix} & \xleftarrow[I]{} & \ker\begin{bmatrix} A^t - 1 \\ \alpha^t \end{bmatrix}
\end{array}$$

Filtrated K-theory

$\mathfrak{K}(A)$:

The collection of all six term exact sequences

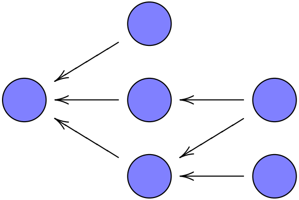
$$\begin{array}{ccccc} K_0(J/I) & \longrightarrow & K_0(K/I) & \longrightarrow & K_0(K/J) \\ \uparrow & & & & \downarrow \\ K_1(K/J) & \longleftarrow & K_1(K/I) & \longleftarrow & K_1(J/I) \end{array}$$

whenever $I \lhd J \lhd K \lhd A$.

Remark

Each subquotient may occur several times, in which case the K-groups of the various six-term exact sequences are identified. Thus the invariant is also called the “K-web”.

Cuntz-Krieger



Theorem (Restorff)

When G and H are finite graphs with no unique cycles, no sinks, and no sources, then the following are equivalent

- $C^*(G) \otimes \mathbb{K} \simeq C^*(H) \otimes \mathbb{K}$
- $\mathfrak{K}(C^*(G)) \simeq \mathfrak{K}(C^*(H))$

Fundamental question

$\mathfrak{K}(A)_+$:

As above, but with each K_0-group

$$K_0(J/I) \longrightarrow K_0(K/I) \longrightarrow K_0(K/J)$$

considered as an **ordered** group.

Working conjecture

$\mathfrak{K}(-)_+$ is a complete invariant for stable isomorphism of all graph C^*-algebras with finitely many ideals.

One ideal

Theorem (E-Tomforde)

$\mathfrak{K}(-)_+$:

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

is a complete invariant up to stable isomorphism for the class of graph algebras with precisely one non-trivial ideal.

UCT approach

Theorem (Kirchberg)

Any $\alpha \in KK_X(A, B)^{-1}$ induces a stable isomorphism between A and B when these are (non-simply) purely infinite and nuclear with $\mathrm{Prim}(A) = \mathrm{Prim}(B) = X$.

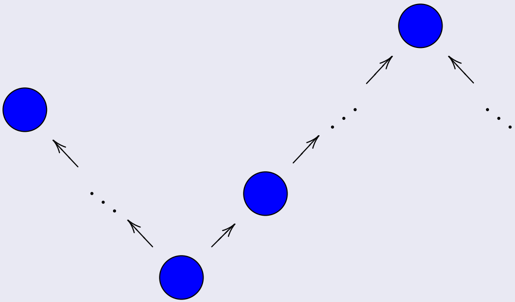
Theorem (Meyer-Nest)

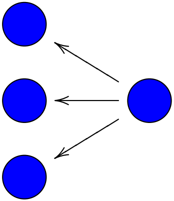
When A, B are in the bootstrap class and $\mathrm{p.dim}(\mathfrak{K}(A)) \leq 1$ we have a UCT

$$0 \longrightarrow \mathrm{Ext}(\mathfrak{K}(A), \mathfrak{K}(B)) \longrightarrow KK_X(A, B) \longrightarrow \mathrm{Hom}(\mathfrak{K}(A), \mathfrak{K}(B)) \longrightarrow 0$$

Corollary (Meyer-Nest, Köhler-NN)

$\mathfrak{K}(-)$ is a complete invariant for purely infinite graph algebras of the form



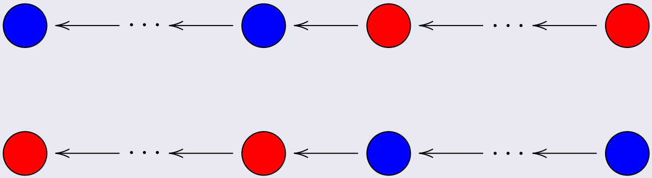


Problem

For a certain purely infinite C^*-algebra A with 7 ideals, $\operatorname{p.dim}(\mathfrak{K}(A)) > 1$. Consequently, $\mathfrak{K}(-)$ is **not** a complete invariant for all nuclear, purely infinite C^*-algebras in the bootstrap class with real rank zero.
However, the K-theory of this example is not obtainable by graph algebras.

Theorem (E-Restorff-Ruiz)

$\mathfrak{K}(-)_+$ is a complete invariant for the class of graph algebras with finite linear ideal lattices of the form:



Theorem (E-Restorff-Ruiz)

$\mathfrak{K}(-)_+$ is a complete invariant for the class of graph algebras with finite linear ideal lattices when for all subquotients we have

$$K_0(I_j/I_{j-1}) = \mathbb{Z}^k \qquad K_1(I_j/I_{j-1}) = 0$$