Non-simple C^*-algebras associated to minimal dynamics*

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*This is a provocative title!

A substitution

$$\omega : \{1,2,3,4,5\} \longrightarrow \{1,2,3,4,5\}^{\sharp}$$

given by

$$\begin{aligned} \omega(1) &= 123514 \\ \omega(2) &= 124 \\ \omega(3) &= 13214 \\ \omega(4) &= 14124 \\ \omega(5) &= 15214 \end{aligned}$$

The *fixed point* u

$\cdots 12351212414124.1235141241321415 21412 \cdots$

satisfies $\omega(u) = u$. And it makes sense to define

$$\underline{\mathsf{X}}_{\omega} = \overline{\{\sigma^n(u) \mid n \in \mathbb{Z}\}}$$

A dynamical system

The definition

$$\underline{\mathsf{X}}_\tau = \overline{\{\sigma^n(u) \mid n \in \mathbb{Z}\}}$$

makes sense for a general *primitive* substitution τ, provided that ones allows $\tau^m(u) = u$.

The dynamical system $(\underline{\mathsf{X}}_\tau, \sigma)$ will be minimal (all orbits dense).

Problem How does one determine from τ and υ whether

$$\underline{\mathsf{X}}_\tau \simeq \underline{\mathsf{X}}_\upsilon \qquad \text{[conjugacy]}$$

or

$$\underline{\mathsf{X}}_\tau \sim_{FE} \underline{\mathsf{X}}_\upsilon \qquad \text{[flow equivalence]?}$$

Some substitutions

$$\tau_1(\aleph) = \aleph\beth\aleph \qquad \tau_1(\beth) = \beth\aleph\aleph\beth$$

$$\tau_2(\alpha) = \alpha\beta \qquad \tau_2(\beta) = \alpha\beta\gamma\delta\epsilon \qquad \tau_2(\gamma) = \alpha\beta$$

$$\tau_2(\delta) = \gamma\delta\epsilon \qquad \tau_2(\epsilon) = \alpha\beta\gamma\delta\epsilon$$

$$\tau_3(1) = 1212345$$

$$\tau_3(2) = 1212345123451234 5$$

$$\tau_3(3) = 1212345 \qquad \tau_3(4) = 1234512345$$

$$\tau_3(5) = 12123451234512345$$

$$\tau_4(a) = aabaababab \qquad \tau_4(b) = aabababaababab$$

Abelianization

To a substitution τ one associates the $|\mathfrak{a}| \times |\mathfrak{a}|$-matrix $\mathbf{A}_\tau$ given by

$$(\mathbf{A}_\tau)_{a,b} = \# \text{ of occurrences of } b \text{ in } \tau(a)$$

When τ is aperiodic, primitive and proper*,

$$\varinjlim (\mathbb{Z}^{|\mathfrak{a}|} \xrightarrow{\mathbf{A}_\tau} \mathbb{Z}^{|\mathfrak{a}|} \xrightarrow{\mathbf{A}_\tau} \cdots)$$

as an ordered group, is an invariant for conjugacy and flow equivalence.

Theorem [Giordano/Putnam/Skau^2/Durand/Host]

A complete invariant of strong orbit equivalence!

*No loss of generality

Special words

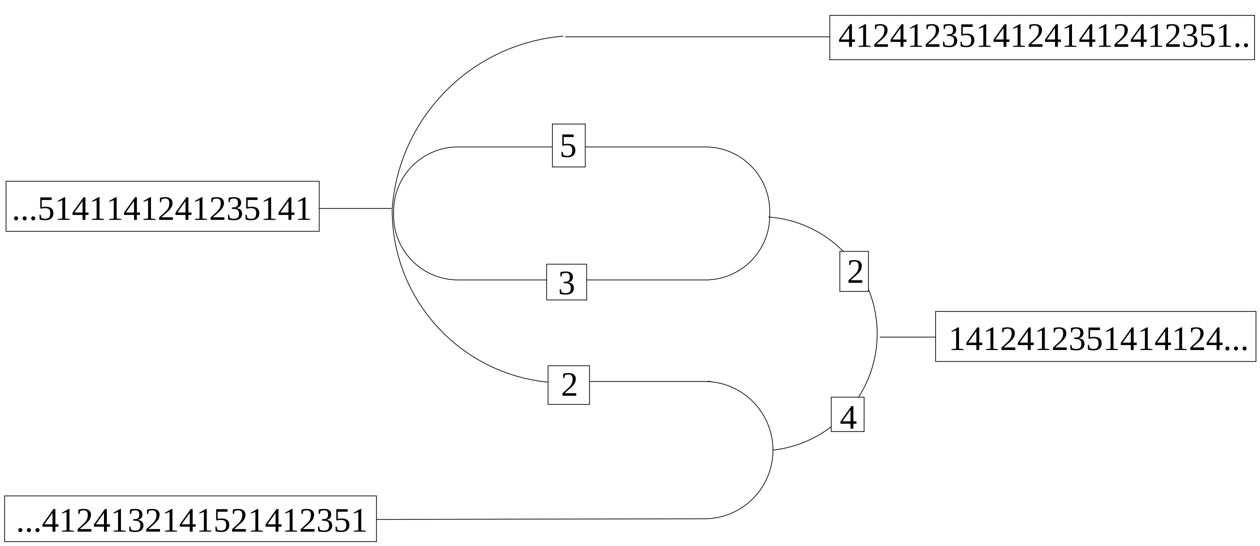
Consider

$$\pi : \mathfrak{a}^{\mathbb{Z}} \longrightarrow \mathfrak{a}^{\mathbb{N}_0}$$

and its restrictions. Most $x \in \underline{\mathsf{X}}_{\mathcal{T}}$ have the property that one tail determines the other, as in

$$\pi(x) = \pi(y) \Longrightarrow x = y$$

But there is always (up to orbit equivalence) a finite number of exceptions to this rule, as in



What is $\mathbf{E}_\tau$*?*

One may arrange that all special words for τ have the form

$$\cdots \tau^3(v)\tau^2(v)\tau(v)vu.w\tau(w)\tau^2(w)\tau^3(w)\cdots$$

with $\tau(u) = vuw$. Denote the rightmost letter of u by a. Represent all (adjusted/cofinal) special words this way. Then

$$(\mathbf{E}_\tau)_{j,b} = \left(\sum_{k=1}^{p_j+1} e_{\tau,a_k^j,w_k^j}(b)\right) - e_{\tau,\tilde{a}^j,\tilde{w}^j}(b)$$

with

$$e_{\tau,a,w}(b) = \max(0, \#[b,\tau(a)] - \#[b,aw])$$

For the susbtitution υ the exact sequence

$$0 \longrightarrow \mathbb{Z}^{\mathsf{n}_\upsilon}/\mathsf{p}_\upsilon\mathbb{Z} \longrightarrow K_0(\mathcal{O}_\upsilon) \xrightarrow{\rho_*} K_0(C(\underline{\mathsf{X}}_\upsilon) \rtimes_\sigma \mathbb{Z}) \longrightarrow 0$$

becomes

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z}[\tfrac{1}{3}] \xrightarrow{[-2\ 1]} \mathbb{Z}[\tfrac{1}{3}] \longrightarrow 0$$

But for υ^{-1} we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \mathbb{Z} \oplus \mathbb{Z}[\tfrac{1}{3}] \xrightarrow{[0\ 1]} \mathbb{Z}[\tfrac{1}{3}] \longrightarrow 0$$

Ultimate example

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C^*-algebras considered by Matsumoto

For any shift space $\underline{\mathsf{X}}$ we define $\mathcal{O}_{\underline{\mathsf{X}}}$ as the universal C^*-algebra given by generators S_a, $a \in \mathfrak{a}$ and relations

(i) $\sum_{a\in\mathfrak{a}} S_a S_a^* = 1$

(ii) $[S_v S_v^*, S_w^* S_w] = 0,\ v, w \in \mathfrak{a}^\sharp$

(iii) $\{S_v^* S_v\}_{v\in\mathfrak{a}^\sharp}$ relate mutually as do the indicator functions of

$$\{x \in \pi(\underline{\mathsf{X}}) \mid vx \in \pi(\underline{\mathsf{X}})\}$$

where $\pi : \mathfrak{a}^{\mathbb{Z}} \longrightarrow \mathfrak{a}^{\mathbb{N}_0}$

Key results by Matsumoto

- $\mathcal{O}_{\underline{\mathsf{X}}} \otimes \mathbb{K}$ is a flow invariant

- You know $K_*(\mathcal{O}_{\underline{\mathsf{X}}})$ *as a group* if you know the relations $\sim_l$ on $\pi(\underline{\mathsf{X}})$ defined by

$$x \sim_l y$$
$$\Longleftrightarrow$$
$$\forall v \in \mathfrak{a}^\sharp, |v| \leq l : vx \in \pi(\underline{\mathsf{X}}) \Longleftrightarrow vy \in \pi(\underline{\mathsf{X}})$$

  and the actions

$$a : [x]_{l+1} \mapsto [ax]_l, a \in \mathfrak{a}$$

- General simplicity criteria under property (I):

$$\forall x \in \pi(\underline{\mathsf{X}}) \forall l \in \mathbb{N} \exists y \in \pi(\underline{\mathsf{X}}) : \begin{cases} y \neq x \\ y \sim_l x \end{cases}$$

Properties of $\mathcal{O}_\mathcal{T}$

Definition $\mathcal{O}_\mathcal{T} = \mathcal{O}_{\underline{\mathsf{X}}_\mathcal{T}}$

- $\mathcal{O}_\mathcal{T}$ is nonsimple, and has a maximal ideal isomorphic to $\mathbb{K}^{\mathsf{n}_\mathcal{T}}$ for $\mathsf{n}_\mathcal{T} \in \mathbb{N}$. Further,

$$0 \longrightarrow \mathbb{K}^{\mathsf{n}_\mathcal{T}} \longrightarrow \mathcal{O}_\mathcal{T} \xrightarrow{\rho} C(\underline{\mathsf{X}}_\mathcal{T}) \rtimes_\sigma \mathbb{Z} \longrightarrow 0$$

- The short exact sequence induces

$$\begin{array}{ccccc} \mathbb{Z}^{\mathsf{n}_\mathcal{T}} & \longrightarrow & K_0(\mathcal{O}_\mathcal{T}) & \xrightarrow{\rho_*} & K_0(C(\underline{\mathsf{X}}_\mathcal{T}) \rtimes_\sigma \mathbb{Z}) \\ {\scriptstyle \mathsf{p}_\mathcal{T}}\uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

for $\mathsf{p}_\mathcal{T} \in \mathbb{N}^{\mathsf{n}_\mathcal{T}}$.

- The order on $K_0(\mathcal{O}_\mathcal{T})$ is given by

$$g \geq 0 \Longleftrightarrow \rho_*(g) \geq 0$$

Complete desciption

Theorem [CE]

Let τ be a primitive, aperiodic, proper* and elementary† substitution. For suitable $\mathsf{n}_\tau \times |\mathfrak{a}|$-matrix $\mathbf{E}_\tau$ we define

$$\widetilde{\mathbf{A}}_\tau = \begin{bmatrix} \mathbf{A}_\tau & 0 \\ \mathbf{E}_\tau & \mathbf{Id} \end{bmatrix}$$

$$H_\tau = \mathbb{Z}^{\mathsf{n}_\tau} / \mathsf{p}_\tau \mathbb{Z}$$

and have

$$K_0(\mathcal{O}_\tau) = \varinjlim(\mathbb{Z}^{|\mathfrak{a}|} \oplus H_\tau, \widetilde{\mathbf{A}}_\tau)$$

as an ordered group, where $\mathbb{Z}^{|\mathfrak{a}|} \oplus H_\tau$ is ordered by

$$(x, y) \geq 0 \Longleftrightarrow x \geq 0$$

The constituent quantities n_τ, p_τ and $\widetilde{\mathbf{A}}_\tau$ are computable.

*No loss of generality

†No loss of generality