Classification of symbolic dynamical systems

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Based on joint work with Béal, Berstel, Perrin; Boyle, Carlsen

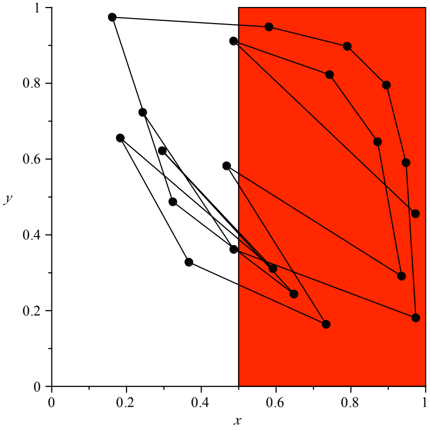
Content

See also

- M.-P. Béal, S. Eilers, J. Berstel, and D. Perrin: *Symbolic Dynamics*. Chapter for "Handbook in Automata Theory". ArXiV 2010.
- M. Boyle, T.M. Carlsen, and S. Eilers: *Flow equivalence of sofic systems*. ArXiV 2011 (sorry!).
- D. Lind, B. Marcus: *Introduction to symbolic dynamics and coding*. Cambridge University Press, 1995.

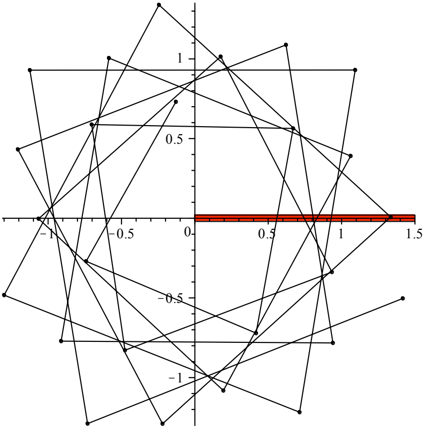
Outline

Baker's map



$$101110100101001111100\cdots$$

Irrational rotation



$$000100010010001001000100 0\cdots$$

Symbolic dynamics

Let $\mathfrak{a}$ be a finite set and equip $\mathfrak{a}^{\mathbb{Z}}$ with the product topology based on the discrete topology on $\mathfrak{a}$.

Definition

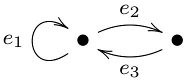
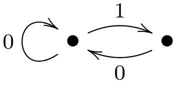
A **shift space** is a subset X of $\mathfrak{a}^{\mathbb{Z}}$ which is closed and closed under the **shift map**

$$\sigma : \mathfrak{a}^{\mathbb{Z}} \to \mathfrak{a}^{\mathbb{Z}} \qquad \sigma((x_i)) = (x_{i+1})$$

Definition

A shift space is **irreducible** if some orbit $\{\sigma^n(x) \mid n \in \mathbb{N}\}$ is dense.

3 constructions

Name	Input	Description	Example
$\mathsf{X}^{(W)}$	List of words W	Sequences not containing words from W	$W = \{11\}$
X_G	Graph G	Infinite paths on G	
$\mathsf{L}_\mathcal{A}$	Automaton $\mathcal{A}$	Words recognized by $\mathcal{A}$	

Forbidden word shifts

Let W be a set of finite words on $\mathfrak{a}$.

Definition

$\mathsf{X}^{(W)}$ is the shift space $\{x \in \mathfrak{a}^{\mathbb{Z}} \mid \forall i < j : x_i \cdots x_j \notin W\}$

Example

With $\mathfrak{a} = \{0, 1\}$ and $W = \{11\}$ the shift space $\mathsf{X}^{(W)}$ contains elements such as

$$\cdots 0100001000100010000101010100100100010010\cdots$$

Lemma

For any shift space X, $X = \mathsf{X}^{(W)}$ where W is chosen as the complement of the language

$$\mathcal{L}(X) = \{x_i \cdots x_j \mid x \in X, i < j\}$$

Edge shifts

Let a graph $G = (V, E, r, s)$ be given with

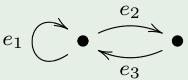
- Vertices V
- Edges E enumerated $\{e_1, \dots e_n\}$
- Range and source maps $r, s : E \to V$.

Definition

X_G is the shift space $\mathsf{X}^{(W)}$ with alphabet E and

$$W = \{e_i e_j \mid r(e_i) \neq s(e_j)\}$$

Example

With $G =$ 

, X_G contains elements such as

$$\cdots e_1 e_1 e_2 e_3 e_2 e_3 e_2 e_1 e_1 e_2 e_3 e_2 e_3 e_1 e_2 e_3 e_2 e_3 e_1 e_1 e_1 e_1 e_1 e_1 e_2 \cdots$$

Labeled edge shifts

Convention

All automata $\mathcal{A} = (V, E, r, s, \mathfrak{a}, \lambda)$ are finite and all states are both initial and final. Thus, they are given by the underlying graph (V, E, r, s) and a labelling map $\Lambda : E \to \mathfrak{a}$

Definition

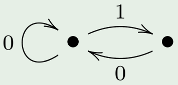
We denote by $\mathsf{X}_{\mathcal{A}}$ the edge shift associated to the underlying graph of $\mathcal{A}$ and by

$$\lambda : \mathsf{X}_{\mathcal{A}} \to \mathfrak{a}^{\mathbb{Z}}$$

the *labeling map* induced by Λ. The shift *recognized by* $\mathcal{A}$ is $\mathsf{L}_{\mathcal{A}} = \lambda(\mathsf{X}_{\mathcal{A}})$.

Labeled edge shifts

Example

With $\mathcal{A} =$ 

the shift space $\mathsf{X}_{\mathcal{A}}$ contains elements such as

$$\cdots 01000010001000100001010101001001000100010 \cdots$$

Outline

Definition

Let $X \subseteq \mathfrak{a}^{\mathbb{Z}}$ and $Y \subseteq \mathfrak{b}^{\mathbb{Z}}$. $\phi : X \to Y$ is the (m, n) *sliding block code* given by a map

$$\Phi : \mathfrak{a}^{n+1+m} \to \mathfrak{b}$$

when

$$\phi(x)_i = \Phi(x_{i-m} \cdots x_{i+n})$$

Lemma

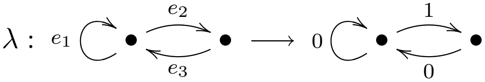
The following are equivalent:

- ϕ *is continuous and shift-commuting*
- ϕ *is a sliding block code*

Definition

X and Y are *conjugate* when there is a bijective sliding block code $\phi : X \to Y$

With $\mathcal{A}$ as above,



becomes a conjugacy. Indeed, the labeling map is always a $(0,0)$ sliding block code induced by Λ. And in this case it has a $(1,0)$ block inverse μ given by

$$00 \mapsto e_1 \qquad 01 \mapsto e_2 \qquad 10 \mapsto e_3$$

For instance,

$$\mu \circ \lambda(\cdots e_1 e_2 e_3 e_1 e_1 e_1 e_2 e_3 e_1 \dots) =$$
$$\mu(\cdots 010000100 \dots) =$$
$$\cdots e_2 e_3 e_1 e_1 e_1 e_2 e_3 e_1 \dots$$

Multiplicity set

Definition

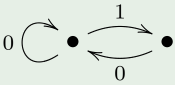
With a given map $\lambda : \mathsf{X}_{\mathcal{A}} \to \mathsf{L}_{\mathcal{A}}$ we set

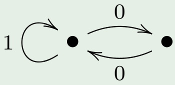
$$\begin{aligned} \widetilde{\mathsf{L}_{\mathcal{A}}} &= \{x \in \mathsf{L}_{\mathcal{A}} \mid |\lambda^{-1}(\{x\})| > 1\} \\ \widetilde{\mathsf{X}_{\mathcal{A}}} &= \lambda^{-1}(\widetilde{\mathsf{L}_{\mathcal{A}}}) \end{aligned}$$

and restrict λ to

$$\widetilde{\lambda} : \widetilde{\mathsf{X}_{\mathcal{A}}} \to \widetilde{\mathsf{L}_{\mathcal{A}}}$$

Example

With $\mathcal{A} =$ 

and $\mathcal{B} =$ 

we get $\widetilde{\mathsf{L}_{\mathcal{A}}} = \emptyset$ and $\widetilde{\mathsf{L}_{\mathcal{B}}} = \{0^\infty\}$.

Shifts of finite type

Definition

A shift space is a *shift of finite type (SFT)* if is has the form $\mathsf{X}^{(W)}$ with W finite.

Lemma

The following are equivalent:

- X *is an SFT*
- $X \simeq \mathsf{X}_G$ *for some graph* G

Sofic shifts

Definition

A shift space is *sofic* if is has the form $\mathsf{X}^{(W)}$ with W recognizable.

Lemma

The following are equivalent:

- X *is sofic*
- $X \simeq \mathsf{X}_{\mathcal{A}}$ *for some automaton* $\mathcal{A}$

Theorem

When X *is irreducible and sofic, there is a unique deterministic automaton* $\mathcal{A}$ *with fewest possible vertices such that* $X \simeq \mathsf{X}_{\mathcal{A}}$. $\mathcal{A}$ *is called the Fischer cover of* X.

Classification

The SFT classification problem

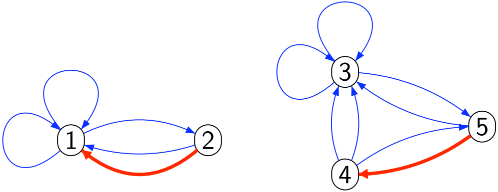
Let X and Y be irreducible shifts of finite type given by graphs G and H, respectively. Determine in terms of G and H when X and Y are conjugate.

The sofic classification problem

Let X and Y be irreducible sofic shifts given by Fischer automata $\mathcal{A}$ and $\mathcal{B}$, respectively. Determine in terms of $\mathcal{A}$ and $\mathcal{B}$ when X and Y are conjugate.

Outline

State splitting



$$\begin{bmatrix}2 & 1\\2 & 0\end{bmatrix}=\begin{bmatrix}2 & 0 & 1\\1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix} \qquad \begin{bmatrix}2 & 0 & 1\\2 & 0 & 1\\1 & 1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\1 & 1 & 0\end{bmatrix}$$

Theorem (Williams)

Let X_G and X_H be two irreducible SFTs given by graphs with adjacency matrices A and B, respectively. The following conditions are equivalent.

(i) *X_G and X_H are conjugate.*

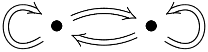
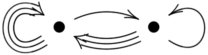
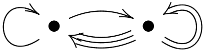
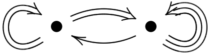
(ii) *There exist nonnegative integral matrices D_i and E_i with*

$$A = D_0E_0, E_0D_0 = D_1E_1, \cdots, E_nD_n = B$$

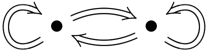
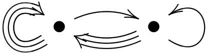
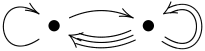
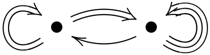
Arsenal of invariants

Real numbers (entropy), power series (zeta function), ordered abelian groups (Dimension group), finitely generated abelian groups (Bowen-Franks groups), C^*-algebras (Cuntz-Krieger algebra),. . .

4 examples

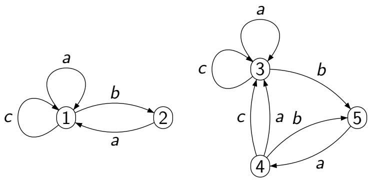
A	G	$h(\mathsf{X}_G)$	$BF(\mathsf{X}_G)$
$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$		4	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$		4	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$		$\frac{3+\sqrt{13}}{2}$	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$		4	$(\mathbb{Z}, 0)$

4 examples

A	G	$h(\mathsf{X}_G)$	$BF(\mathsf{X}_G)$
$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$		4	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$		4	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$		$\frac{3+\sqrt{13}}{2}$	$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$		4	$(\mathbb{Z}, 0)$

Outline

Symbolic state splitting



$$\begin{bmatrix} a+c & b \\ a & 0 \end{bmatrix} = \begin{bmatrix} a+c & 0 & b \\ 0 & a & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a+c & 0 & b \\ a+c & 0 & b \\ 0 & a & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a+c & 0 & b \\ 0 & a & 0 \end{bmatrix}$$

Theorem (Nasu)

Let X and Y be two irreducible sofic shifts and let $\mathcal{A}, \mathcal{B}$ be their Fischer automata given by alphabetic adjacency matrices A and B. The following conditions are equivalent.

(i) $X \simeq Y$

(ii) *There exist nonnegative integral matrices D_i and alphabetic matrices E_i with*

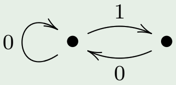
$$A = D_0E_0, E_0D_0 = D_1E_1, \cdots, E_nD_n = B$$

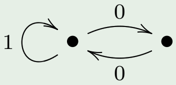
(iii)
$$\begin{array}{ccc} \mathsf{X}_{\mathcal{A}} & \xrightarrow{\simeq} & \mathsf{X}_{\mathcal{B}} \\ {\scriptstyle \lambda_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \lambda_{\mathcal{B}}} \\ \mathsf{L}_{\mathcal{A}} & \xrightarrow{\simeq} & \mathsf{L}_{\mathcal{B}} \end{array}$$

Arsenal of invariants

Invariants of $\mathsf{X}_{\mathcal{A}}$, invariants of λ, the syntactic graph, ...

Example

With $\mathcal{A} =$ 

and $\mathcal{B} =$ 

we get $\widetilde{\mathsf{L}_\mathcal{A}} = \emptyset$ and $\widetilde{\mathsf{L}_\mathcal{B}} = \{0^\infty\}$.

Hence

$$\begin{array}{ccc} \mathsf{X}_\mathcal{A} & \xrightarrow{\simeq} & \mathsf{X}_\mathcal{B} \\ {\scriptstyle \lambda_\mathcal{A}}\downarrow & & \downarrow{\scriptstyle \lambda_\mathcal{B}} \\ \mathsf{L}_\mathcal{A} & \xrightarrow{\simeq} & \mathsf{L}_\mathcal{B} \end{array}$$

is impossible and $\mathsf{L}_\mathcal{A} \not\simeq \mathsf{L}_\mathcal{B}$.

Outline

Symbol expansion

Fix $a \in \mathfrak{a}$ and $\star \notin \mathfrak{a}$ and define $\eta : \mathfrak{a}^{\mathbb{Z}} \longrightarrow (\mathfrak{a} \cup \{\star\})^{\mathbb{Z}}$ as the map inserting a $\star$ after each a:

$$\cdots babbbaba \cdots \qquad \mapsto \qquad \cdots ba \star bbba \star ba \star \cdots$$

Definition

The "$a \mapsto a\star$" symbol expansion of a shift space X is the shift space $X_{a \mapsto a\star} = \eta(X)$.

Flow equivalence

Associated to any shift space there is a **suspension flow** given by product topology on

$$SX = \frac{X \times \mathbb{R}}{(x,t) \sim (\sigma(x), t+1)}$$

Definition

X and Y are *flow equivalent* (written $X \simeq_{fe} Y$) when SX and SY are homeomorphic in a way preserving direction in $\mathbb{R}$.

Theorem (Parry-Sullivan)

Flow equivalence is the coarsest equivalence relation containing conjugacy and $X \sim X_{a \to a\star}$

Flow classification

Lemma

If $X \simeq_{fe} Y$ and X is SFT, sofic or irreducible, then so is Y.

The SFT flow classification problem

Let X and Y be irreducible shifts of finite type given by graphs G and H, respectively. Determine in terms of G and H when X and Y are flow equivalent.

The sofic flow classification problem

Let X and Y be irreducible sofic shifts given by Fischer automata $\mathcal{A}$ and $\mathcal{B}$, respectively. Determine in terms of $\mathcal{A}$ and $\mathcal{B}$ when X and Y are flow equivalent.

Outline

Flow classifcication of SFTs

Theorem (Franks)

Let X_G and X_H be two irreducible SFTs given by graphs with adjacency matrices A and B, respectively. The following conditions are equivalent.

(i) $\mathsf{X}_G \simeq_{fe} \mathsf{X}_H$

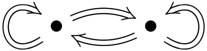
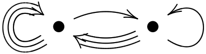
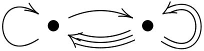
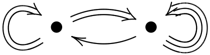
(ii)

$$\mathbb{Z}^m/(1-A)\mathbb{Z}^m \simeq \mathbb{Z}^n/(1-B)\mathbb{Z}^n$$

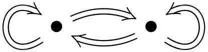
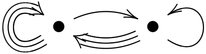
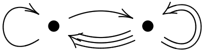
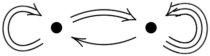
and

$$\operatorname{sgn}\det(1-A) = \operatorname{sgn}\det(1-B)$$

4 examples

A	G	$BF(\mathsf{X}_G)$
$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$		$(\mathbb{Z}, 0)$

4 examples

A	G	$BF(\mathsf{X}_G)$
$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$		$(\mathbb{Z}_3, -)$
$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$		$(\mathbb{Z}, 0)$

Outline

Flow classification of sofics

Theorem

Let X and Y be two irreducible sofic shifts and let $\mathcal{A}, \mathcal{B}$ be their Fischer automata. The following conditions are equivalent.

(i) $X \simeq_{fe} Y$

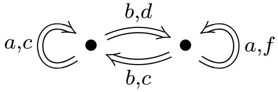
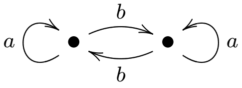
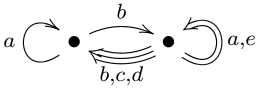
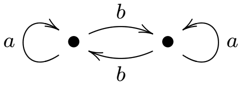
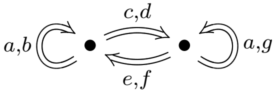
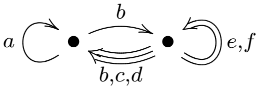
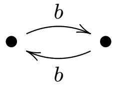
(ii)
$$\begin{array}{ccc} S\mathsf{X}_{\mathcal{A}} & \xrightarrow{\sim_+} & S\mathsf{X}_{\mathcal{B}} \\ {\scriptstyle S\lambda_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle S\lambda_{\mathcal{B}}} \\ S\mathsf{L}_{\mathcal{A}} & \xrightarrow{\sim_+} & S\mathsf{L}_{\mathcal{B}} \end{array}$$

Theorem (Boyle-Carlsen-E)

Let X and Y be two irreducible sofic shift spaces with Fischer automata $\mathcal{A}$ and $\mathcal{B}$, respectively, and assume that $\widetilde{\mathsf{X}_{\mathcal{A}}}$ and $\widetilde{\mathsf{X}_{\mathcal{B}}}$ are both closed. Then X and Y are flow equivalent exactly when the following conditions hold:

(1) $\mathsf{X}_{\mathcal{A}} \simeq_{fe} \mathsf{X}_{\mathcal{B}}$

(2)
$$\begin{array}{ccc} S\widetilde{\mathsf{X}_{\mathcal{A}}} & \xrightarrow{\sim_+} & S\widetilde{\mathsf{X}_{\mathcal{B}}} \\ {\scriptstyle S\widetilde{\lambda_{\mathcal{A}}}}\downarrow & & \downarrow{\scriptstyle S\widetilde{\lambda_{\mathcal{B}}}} \\ S\widetilde{\mathsf{L}_{\mathcal{A}}} & \xrightarrow{\sim_+} & S\widetilde{\mathsf{L}_{\mathcal{B}}} \end{array}$$

$\lambda : \mathsf{X}_{\mathcal{A}} \to \mathsf{L}_{\mathcal{A}}$	$\widetilde{\lambda} : \widetilde{\mathsf{X}_{\mathcal{A}}} \to \widetilde{\mathsf{L}_{\mathcal{A}}}$
	
	
	
	

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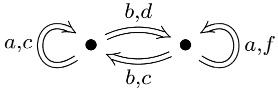
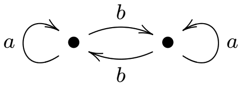
<tr><td>
</td><td>
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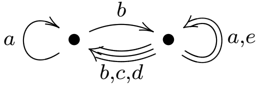
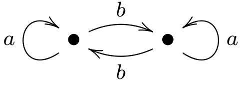
<tr><td>
</td><td>
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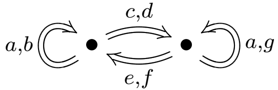
<tr><td>
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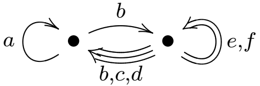
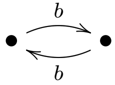
<tr><td>
</td><td>
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Deciding flow equivalence

Theorem (Boyle-Carlsen-E)

When $\mathsf{X}_{\mathcal{A}}$ is an AFT 2-sofic shift, $\widetilde{\mathsf{X}_{\mathcal{A}}}$ is an SFT, and the flow class of $\widetilde{\lambda} : \widetilde{\mathsf{X}_{\mathcal{A}}} \to \widetilde{\mathsf{L}_{\mathcal{A}}}$ may be described as the $\mathbb{Z}_2$-equivariant flow class of $\widetilde{\mathsf{X}_{\mathcal{A}}}$. There are procedures for doing this in several cases.