Counting LEGO® buildings

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Introduction

It is commonly believed that the number of ways to combine six 2×4 LEGO blocks of the same color is

$$102981500$$

This number was computed by engineers at LEGO several decades ago and has been repeated by LEGO several times, for instance in [1, p. 15]. Consequently, the number can be found in several "fun fact" books and on more than 250 home pages in, among other languages, Persian, Japanese and English[1].

However, the number is wrong – very wrong. In fact, as we shall see below, the correct number is

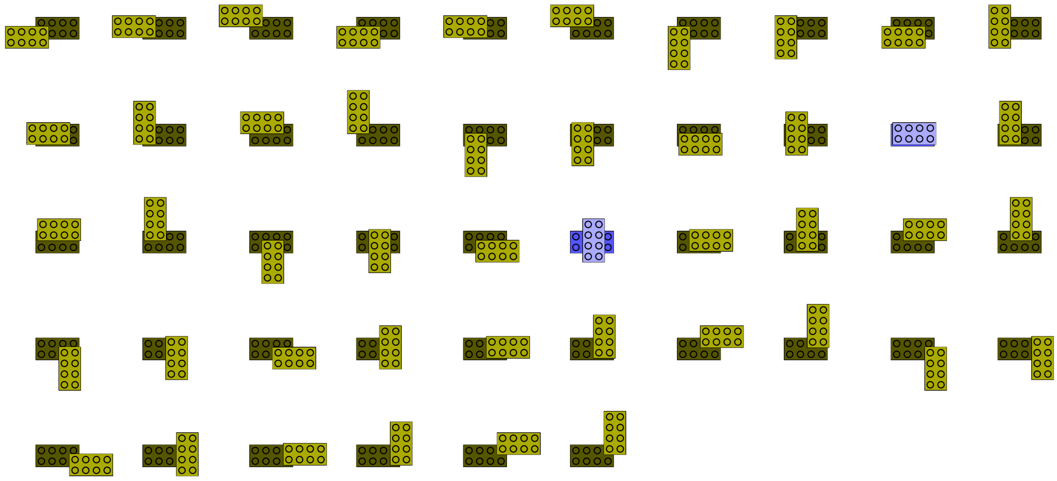
$$915103765$$

This probably does not reflect an error on the part of the engineers at LEGO, but rather that over the years, it has been misreported and finally forgotten what they computed. It is definitely not the number of ways to combine six 2×4 LEGO blocks, and in fact it seems that it would have been impossible or at least very difficult to compute the correct number with the computational tools available when the computation was done. Today, with a bit of patience, the correct number can be computed on most home computers by a rather simple computer program.

Counting high configurations

To first convince ourselves that the number 102981500 is too small, let us see where it comes from. Consider first how many ways two 2×4 LEGO blocks can be combined. If one fixes the position of the lower LEGO block, the second can be put on top in 46 ways, namely

[1] Google search, October 2004



However, only the blue combinations here are unique. All of the other combinations have doubles which can be achieved by rotating the lower LEGO block 180°, so we see that the number of different combinations of two LEGO blocks of the considered size is

$$2 + \frac{44}{2} = 24.$$

This computation can be generalized to compute the number of ways to combine n 2×4 LEGO blocks in to a tower **of height** n. For instance, to see how many ways to put 6 LEGO blocks on top of each other to produce a tower of height 6, we may again fix the lowest LEGO block in a certain position. Each of the five times we put another LEGO block on top, we have 46 choices, leading to the number

$$46 \cdot 46 \cdot 46 \cdot 46 \cdot 46 = 46^5 = 205962976$$

but again, we will produce doubles in all but a few configurations. Indeed, the combinations are only unique in the situation where all blocks are put on top of the previous one in one of the two blue configurations above, so the number of ways to produce unique combinations is

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32.$$

All in all, we have 32 unique combinations among the 205962976, and all of the others are doubled, so the total number is

$$32 + (205962976 - 32)/2 = 102981504 \tag{1}$$

It stands to reason that this is the number computed by the engineers at LEGO; the difference at the last digit probably being a consequence of lack of precision in the computational tools used at the time. But of course there are many more ways to combine six 2×4 LEGO blocks to buildings of lower height, since the LEGO blocks may be combined with much more flexibility which for instance allows six 2×4 blocks to be attached to the top of a 2×4 block, and 6 to be attached below. Thus, to do LEGO justice, we need to count **all** buildings and include those of height 2, of height 3, of height 4 and height 5.

It may seem surprising that LEGO did not do so at the time, since the purpose of the calculation must have been to convince the public that this toy was a flexible one. But at least to my knowledge, there was no way the engineers *could* have computed the number. For the mathematics of the computations is so irregular that it is very difficult to come up with a formula for the number of "low" configurations, as opposed to the ones of maximal height which were indeed considered. The engineers at LEGO developed and used the formula

$$\frac{46^{n-1} + 2^{n-1}}{2}$$

for the number of ways to build a tower of n 2×4 LEGO blocks of height n. With a lot more work, they probably could have seen that the number of ways to build a tower of n 2×4 LEGO blocks of height $n-1$ (putting two blocks at precisely one of the stories of the building) is

$$\frac{37065n - 89115}{4477456} 46^n + (n - \frac{1}{2})2^n$$

and computed that the number of buildings of height 5 with 6 LEGO blocks is

$$282010252. \tag{2}$$

But I am aware of no method which can produce a formula for buildings of even lower height, or indeed of the total number of combinations. It would seem to involve a completely different method of counting. There are many such methods known, but I have been unable to find one which applies to this situation[2].

Thus, it would seem that the engineers at LEGO would have been left with the very tedious, error-prone, and mathematically unsatisfactory option of going essentially though all possibilities and counting them[3].

[2]Although a professional mathematician, I do not specialize in the field of combinatorics, to which the problem belongs

[3]It was this observation, rather than the number itself, that raised my suspicion concerning the number 102981500

Today, we are in the fortunate situation that we can leave this tedium to computers. Writing a computer program going through all combinations requires care in making sure that no combination is counted twice, but may be done in a couple of hundred lines in a programming language like Java. However, the problem remains computationally demanding — to compute the number above I had to let my (admittedly, rather old) home computer work for about a week. Hence I am not capable at present to compute how many combinations there are for, say, ten 2×4 LEGO blocks.

The program computed the following number of buildings with six 2×4 LEGO blocks, grouped by their height:

Height	**Number**
2	7946227
3	162216127
4	359949655
5	282010252
6	102981504
Total	915103765

Note the consistency with (1) and (2).

References

[1] LEGO Company Profile 2004, LEGO, http://www.lego.com/info/pdf/compprofileeng.pdf.

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