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ON HIGHER ARITHMETIC INTERSECTION THEORY

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Introduction

The *Mordell's conjecture* states that there are a finite number of rational points on a non-singular algebraic curve C over $\mathbb{Q}$ of genus $g > 1$. The geometric analog of this conjecture was proved by Manin in 1963 (see [44]), using the Gauss-Manin connection. This suggested that the geometric tools where more developed than the arithmetic ones. *Arakelov theory* was introduced by Arakelov in [3], in order to give analogs of the algebraic geometry results in the field of arithmetic geometry. Arakelov defined a new notion of divisor class on the non-singular model of an algebraic curve defined over an algebraic number field. He then defined an intersection theory for these divisor classes, following the intersection theory of divisors in algebraic geometry. The idea is that one can compactify a curve defined over the ring of integers of a number field by considering Green functions on the associated complex curve. This initial work on arithmetic surfaces was expanded on by Deligne [16], Szpiro [56] and Faltings [19] among others. These studies provided results on arithmetic surfaces like the adjunction formula, the Hodge index theorem and the Riemann-Roch theorem. Mordell's conjecture was first proved by Faltings in [18]. A proof of the Mordell's conjecture using the tools of Arakelov theory, was given by Vojta in [58].

These studies were generalized to higher dimensions in [24] by Gillet and Soulé, who defined an intersection theory for arithmetic varieties. That paper was the starting point of a program aiming to obtain an *arithmetic intersection theory*, following the steps of the algebraic intersection theory, but suitable for arithmetic varieties. This program included, in its initial stages, the definition of arithmetic Chow groups equipped with an intersection product, the definition of the arithmetic K_0-group and the definition of characteristic classes leading to Riemann-Roch theorems.

The program should be continued with the development of higher arithmetic intersection theory, which should include the definition of higher arithmetic Chow groups with an intersection pairing, the definition of higher arithmetic K-theory, the definition of characteristic class maps between them and higher Riemann-Roch theorems.

We will now review the Arakelov program and explain the contribution of this thesis to its fulfill. We start by studying the algebraic analogues.

Algebraic intersection theory. Let X be an equidimensional algebraic variety and let $CH^p(X)$ be the *Chow group* of codimension p algebraic cycles. Different approaches may be used to equip it with a product structure

$$CH^p(X) \otimes CH^q(X) \dashrightarrow CH^{p+q}(X).$$

The first theory is based on the moving lemma. Given the class of two irreducible subvarieties, the method consists of finding representatives that intersect properly. This approach is valid for quasi-projective schemes over a field. Another approach due to Fulton and MacPherson is based on the deformation to the normal cone. In this case, the scheme need not be quasi-projective and is valid for schemes over the spectrum of a Dedekind domain.

Alternatively, in [23], Gillet and Soulé showed that the intersection theory can be developed by transferring the product of the *algebraic K-groups* of a regular noetherian scheme X to the Chow groups. This relies on the graded isomorphism

$$\bigoplus_{p\geq 0} K_0(X)^{(p)}_{\mathbb{Q}} \cong K_0(X)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}} \bigoplus_{p\geq 0} CH^p(X)_{\mathbb{Q}},$$

where the pieces $K_0(X)^{(p)}_{\mathbb{Q}}$ are the eigenspaces of the *Adams operations* Ψ^k on $K_0(X)_{\mathbb{Q}}$ and "ch" is the *Chern character.*

The commutation relation of the Chern character with push-forward maps is given by the *Grothendieck-Riemann-Roch theorem.* Let Td denote the Todd class of the tangent bundle over an algebraic variety. Let X, Y be regular schemes which are quasi-projective and flat over the spectrum S of a Dedekind domain and let $f : X \to Y$ be a flat and projective S-morphism. Then, the Grothendieck-Riemann-Roch theorem says that there is a commutative diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{Td}(X)\cdot \mathrm{ch}(\cdot)} & \bigoplus_{p\geq 0} CH^p(X)_{\mathbb{Q}} \\ \downarrow f_* & & \downarrow f_* \\ K_0(Y) & \xrightarrow{\mathrm{Td}(Y)\cdot \mathrm{ch}(\cdot)} & \bigoplus_{p\geq 0} CH^p(Y)_{\mathbb{Q}}. \end{array}$$

In [7], Bloch developed a theory of *higher algebraic Chow groups* for smooth algebraic varieties over a field. If X is such a variety, these groups are denoted by $CH^p(X, n)$, for $n, p \geq 0$. He proved that there is an isomorphism

$$\bigoplus_{p\geq 0} K_n(X)^{(p)}_{\mathbb{Q}} \cong K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X, n)_{\mathbb{Q}}.$$

Bloch also gave a product structure on $CH^*(X, *)$, which relied on the moving lemma.

This theory established itself as a candidate for motivic cohomology. Since then, there have been many other proposals for motivic cohomology, which apply to bigger classes of schemes. Under certain conditions, the new definitions agree with the higher Chow groups. For this reason, the Bloch Chow groups have remained as a basis and simple description of motivic cohomology for smooth schemes over certain fields.

Arithmetic Chow groups and arithmetic intersection theory. As mentioned above, the advent of arithmetic intersection theory in arbitrary dimensions is due to Gillet and Soulé in [24]. In loc. cit., an arithmetic variety is a regular, quasi-projective scheme flat over an arithmetic ring. Let X be an arithmetic variety. An *arithmetic cycle* on X is a pair (Z, g) where Z is an algebraic cycle and g is a Green current for Z, that is, a current on the complex manifold associated to X satisfying the relation

$$dd^c g + \delta_Z = [\omega],$$

with ω a smooth differential form and δ_Z the current associated to Z. Then, the *arithmetic Chow group* $\widehat{CH}^*(X)$ is defined as the quotient of the free abelian group generated by the arithmetic cycles by an appropriate equivalence relation.

If X is an arithmetic variety, let $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ be the complex conjugation, and let $E^{p,q}(X)$ denote the vector space of $\mathbb{C}$-value differential forms ω on $X(\mathbb{C})$ of type (p, q) that satisfy the relation $F_\infty^* \omega = (-1)^p \omega$. Denote by $\widetilde{E}^{p,p}(X)$ the quotient of $E^{p,p}(X)$ by $(\operatorname{im} \partial + \operatorname{im} \bar{\partial})$.

Gillet and Soulé proved the following properties:

(i) The groups $\widehat{CH}^p(X)$ fit into an exact sequence:

$$CH^{p-1,p}(X) \xrightarrow{\rho} \widetilde{E}^{p-1,p-1}(X) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0, \tag{1}$$

where $CH^{p-1,p}(X)$ is the term $E_2^{p-1,-p}(X)$ of the Quillen spectral sequence (see [48], §7) and ρ the Beilinson regulator (up to a constant factor).

(ii) There is a pairing

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \xrightarrow{\cdot} \widehat{CH}^{p+q}(X)_{\mathbb{Q}}$$

turning $\bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$ into a commutative graded unitary $\mathbb{Q}$-algebra.

(iii) If X, Y are projective and $f : X \to Y$ is a morphism, there exists a pull-back morphism

$$f^* : \widehat{CH}^p(Y) \to \widehat{CH}^p(X).$$

If f is proper, X, Y are equidimensional and $f_{\mathbb{Q}} : X_Q \to Y_{\mathbb{Q}}$ is smooth, there is a push-forward morphism

$$f_* : \widehat{CH}^p(X) \to \widehat{CH}^{p-\delta}(Y),$$

where $\delta = \dim X - \dim Y$. Moreover, the projection formula holds.

Gillet and Soulé continued the project in [25] and [26] defining *characteristic classes* for a hermitian vector bundle over an arithmetic variety X. In order to define the *arithmetic Chern character* "ch", they introduced the *arithmetic K_0-group*, $\widehat{K}_0(X)$, and showed that "ch" gives an isomorphism between $\widehat{K}_0(X)_{\mathbb{Q}}$ and $\bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$. Let us briefly review the definition of $\widehat{K}_0(X)$ and "ch".

Let X be an arithmetic variety. A *hermitian vector bundle* $\overline{E} = (E, h)$ over X is a locally free sheaf of finite rank on X together with a hermitian metric on the associated holomorphic bundle. Let $\overline{E}$ be a hermitian vector bundle over X. Then, there is a Chern character

$$\widehat{\mathrm{ch}}(\overline{E}) \in \bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$$

characterized by five properties: the functoriality, additivity, multiplicativity, compatibility with the Chern forms properties and a normalization condition. Moreover, for every exact sequence of hermitian vector bundles $\epsilon : 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$, the Chern character satisfies

$$\widehat{\mathrm{ch}}(\overline{E}) = \widehat{\mathrm{ch}}(\overline{S}) + \widehat{\mathrm{ch}}(\overline{Q}) - (0, \widetilde{\mathrm{ch}}(\epsilon)),$$

where $\widetilde{\mathrm{ch}}(\epsilon)$ is the secondary Bott-Chern class of ϵ. This leads to the following definition of $\widehat{K}_0(X)$. Let $\widehat{K}_0(X)$ be the group generated by pairs $(\overline{E}, \alpha)$, with $\alpha \in \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-1}(X, p)$, modulo the relation

$$(\overline{S}, \alpha') + (\overline{Q}, \alpha'') = (\overline{E}, \alpha' + \alpha'' + \widetilde{\mathrm{ch}}(\epsilon)),$$

for every exact sequence ϵ as above. This group fits in an exact sequence

$$K_1(X) \xrightarrow{\rho} \bigoplus_{p \geq 0} \widetilde{\mathcal{D}}^{2p-1}(X, p) \to \widehat{K}_0(X) \to K_0(X) \to 0, \tag{2}$$

with ρ the Beilinson regulator (up to a constant factor).

Then, the Chern character induces an isomorphism

$$\widehat{\mathrm{ch}} : \widehat{K}_0(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}.$$

As in the algebraic situation, this isomorphism relies on the graded decomposition of $\widehat{K}_0(X)$ given by the Adams operations. That is, $\widehat{K}_0(X)$ is endowed with a pre-λ-ring structure such that $\widehat{\mathrm{ch}}$ induces an isomorphism on the eigenspaces of $\widehat{K}_0(X)_{\mathbb{Q}}$ by the Adams operations:

$$\widehat{\mathrm{ch}} : \widehat{K}_0(X)_{\mathbb{Q}}^{(p)} \xrightarrow{\cong} \widehat{CH}^p(X)_{\mathbb{Q}}.$$

Gillet and Soulé, using the results of Bismut and his collaborators, proved an *arithmetic Grothendieck-Riemann-Roch theorem* (see [27] and [22]). Another approach to the Grothendieck-Riemann-Roch theorem is given by Faltings (see [19] and [20]).

Let $\widehat{\mathrm{Td}}$ denote the arithmetic Todd class of the tangent bundle over an arithmetic variety, let X, Y be arithmetic varieties and let $f : X \to Y$ be a projective, flat morphism of arithmetic varieties, which is smooth over the rational numbers. Then the arithmetic Grothendieck-Riemann-Roch theorem states that there is a commutative diagram:

$$\begin{array}{ccc}
\widehat{K}_0(X)_{\mathbb{Q}} & \xrightarrow{\widehat{\mathrm{Td}}(X)\cdot\widehat{\mathrm{ch}}(\cdot)} & \bigoplus_{p\geq 0} \widehat{CH}^p(X)_{\mathbb{Q}} \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
\widehat{K}_0(Y)_{\mathbb{Q}} & \xrightarrow{\widehat{\mathrm{Td}}(Y)\cdot\widehat{\mathrm{ch}}(\cdot)} & \bigoplus_{p\geq 0} \widehat{CH}^p(Y)_{\mathbb{Q}}.
\end{array}$$

In [13], Burgos gave an alternative definition of the arithmetic Chow groups. It consists of considering a different space of Green forms associated with an algebraic cycle, by using *Deligne-Beilinson cohomology*. For projective schemes, Burgos definition of arithmetic Chow groups agrees with the one given by Gillet and Soulé.

Let us briefly review his definition. Let X be an arithmetic variety and consider $(\mathcal{D}^*_{\log}(X,p), d_{\mathcal{D}})$ to be the Deligne complex of differential forms on the associated real variety $X_{\mathbb{R}}$ with logarithmic singularities along infinity (see [16] or [13]). The cohomology of this complex gives the Deligne-Beilinson cohomology groups of $X_{\mathbb{R}}$, $H^*_{\mathcal{D}}(X,\mathbb{R}(p))$. For any codimension p irreducible subvariety of X, consider also the Deligne-Beilinson cohomology with supports in Z:

$$H^*_{\mathcal{D},Z}(X,\mathbb{R}(p)) = H^*(s(\mathcal{D}^*_{\log}(X,p) \to \mathcal{D}^*_{\log}(X \setminus Z,p))).$$

There is an isomorphism

$$cl : \mathbb{R}[Z] \xrightarrow{\cong} H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)),$$

called the cycle class map.

Let $\widetilde{\mathcal{D}}^*_{\log}(X,p)$ denote the quotient of $\mathcal{D}^*_{\log}(X,p)$ by the image of $d_{\mathcal{D}}$. Just as a remark, at degree $2p-1$ the differential $d_{\mathcal{D}}$ is $-2\partial\bar{\partial} = (4\pi i)dd^c$. A *Green form* for a codimension p irreducible subvariety Z is an element $(\omega,\tilde{g}) \in \mathcal{D}^{2p}_{\log}(X,p) \oplus \widetilde{\mathcal{D}}^{2p-1}_{\log}(X \setminus Z,p)$, such that $\omega = d_{\mathcal{D}}\tilde{g}$ and

$$cl(Z) = [(\omega,\tilde{g})] \in H^{2p}_{\mathcal{D},Z}(X,\mathbb{R}(p)).$$

Then, an *arithmetic cycle* is now a couple $(Z,(\omega,\tilde{g}))$, with $(\omega,\tilde{g})$ a Green form for Z. The *arithmetic Chow group of* X, $\widehat{CH}^p(X)$, is defined as the quotient of the free abelian group generated by the arithmetic cycles by an equivalence relation given by the group of arithmetic rational cycles.

The arithmetic Chow groups defined by Burgos satisfy the analogous properties *(i)-(iii)* stated above for the arithmetic Chow group defined by Gillet and Soulé. In particular, the exact sequence (1) is written as:

$$CH^{p-1,p}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}^{2p-1}_{\log}(X,p) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0. \quad (3)$$

The Burgos definition of arithmetic Chow groups is the definition adopted in this study.

Later on, in [14], Burgos, Kramer and Kühn developed a formal theory of abstract arithmetic Chow rings, where the role of fibers at infinity is played by a complex of abelian groups that computes suitable cohomology theory. That is, the space of Green forms can be replaced by complexes with different properties in order to obtain arithmetic intersection theories enjoying suitable properties.

Higher arithmetic intersection theory. In a way, we could consider that the program of Arakelov intersection theory in the degree zero case is accomplished. To go further towards the goal of obtaining arithmetic analogues for the algebraic theories established, we would like to give the formalism of a higher intersection theory for arithmetic varieties. This should include the theory of higher arithmetic Chow groups, $\widehat{CH}^p(X,n)$, equipped with an intersection product, the definition of higher arithmetic K-groups, $\widehat{K}_n(X)$, characteristic class maps and Riemann- Roch theorems.

It has been suggested by Deligne and Soulé (see [16], Remark 5.4 and [54] §III.2.3.4) that the extension to higher degrees of the arithmetic K_0-group should be by means of extending the exact sequence (2) in order to obtain a long exact sequence

$$\cdots \to K_{n+1}(X) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n-1}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{K}_n(X) \xrightarrow{\zeta} K_n(X) \to \cdots$$

$$\cdots \to K_1(X) \xrightarrow{\rho} \widetilde{\mathcal{D}}_{\log}^{2p-1}(X,p) \xrightarrow{a} \widehat{K}_0(X) \xrightarrow{\zeta} K_0(X) \to 0.$$

The morphism ρ is the *Beilinson regulator*, that is, the Chern character taking values in real Deligne-Beilinson cohomology. Hence, the Archimedean component of the higher arithmetic K-groups should be handled by the Beilinson regulator:

$$\rho : K_n(X)_{\mathbb{Q}} \xrightarrow{\cong} \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}} \to \bigoplus_{p\geq 0} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)).$$

Analogously, higher arithmetic Chow groups may be defined in order to extend the exact sequence (3) into a long exact sequence:

$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H_{\mathcal{D}}^{2p-n}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$

$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \widetilde{\mathcal{D}}_{\log}^{2p-1}(X,p) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0.$$

The above long exact sequences can be obtained by considering the homotopy groups of the homotopy fiber of a simplicial representative of the Beilinson regulator.

Higher arithmetic Chow groups. Using these ideas, if X is proper, the arithmetic Chow groups have been extended in [30] by Goncharov.

Let ${}'\mathcal{D}^{2p-*}(X,p)$ be the Deligne complex of currents over X and let $E^{p,p}(X)(p)$ be the group of p-twisted differential forms of type (p,p). Denote by ${}'\widetilde{\mathcal{D}}^{2p-*}(X,p)$ the quotient of ${}'\mathcal{D}^{2p-*}(X,p)$ by the complex

$$\cdots \to 0 \to \cdots \to 0 \to E^{p,p}(X)(p) \to 0.$$

Let $Z^p(X,*)$ be the chain complex whose homology groups define $CH^p(X,*)$. Goncharov defined an explicit regulator morphism

$$Z^p(X,*) \xrightarrow{\mathcal{P}} {}'\widetilde{\mathcal{D}}^{2p-*}(X,p).$$

The *higher arithmetic Chow groups* of a regular complex variety X are given by the homology groups of the simple of the morphism $\mathcal{P}$:

$$\widehat{CH}^p(X,n) := H_n(s(\mathcal{P})).$$

For $n=0$, these groups agree with the ones given by Gillet and Soulé. However, this construction leaves the following questions open:

(1) Is the composition of the isomorphism $K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}$ with the morphism induced by $\mathcal{P}$ the Beilinson regulator?

(2) Can one define a product structure on $\bigoplus_{p,n} \widehat{CH}^p(X,n)$?

(3) Are there well-defined pull-back morphisms?

The main obstacle when we try to answer these questions is that we have to deal with the complex of currents, which does not behave well under pull-back or products. Moreover, the techniques on the comparison of regulators apply to morphisms defined for the class of quasi-projective varieties, which is not the case of $\mathcal{P}$.

Higher arithmetic K-theory. The first contribution in the direction of providing an explicit definition of higher arithmetic K-groups is the simplicial description of the Beilinson regulator given by Burgos and Wang in [15]. Let X be a complex manifold. Let $\widetilde{\mathbb{Z}}\widehat{C}_*(X)$ be the complex of cubes of hermitian vector bundles on X. Its homology groups with rational coefficients are the rational algebraic K-groups of X, i.e., there is an isomorphism $H_n(\widetilde{\mathbb{Z}}\widehat{C}_*(X),\mathbb{Q}) \cong K_n(X)_{\mathbb{Q}}$ (see [47]). In [15], Burgos and Wang defined a chain morphism

$$\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \to \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p).$$

Here, $\widetilde{\mathcal{D}}_{\mathbb{P}}^{2p-*}(X,p)$ is a complex built of differential forms on $X \times (\mathbb{P}^1)^{\cdot}$. It is quasi-isomorphic to the Deligne complex of differential forms on X with logarithmic singularities, $\mathcal{D}_{\log}^{2p-*}(X,p)$. Moreover, if X is compact, then there is an explicit inverse quasi-isomorphism $\widetilde{\mathcal{D}}_{\mathbb{P}}^{*}(X,p) \to \mathcal{D}^*(X,p)$ giving a morphism

$$\mathrm{ch} : \widetilde{\mathbb{Z}}\widehat{C}_*(X) \to \bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p).$$

A result of Burgos and Wang shows that this morphism induces the Beilinson regulator in cohomology with rational coefficients.

The idea of the construction of the morphism "ch" is the following. To every n-cube E on X there is an associated locally free sheaf, $\mathrm{tr}_n(E)$, on $X \times (\mathbb{P}^1)^n$ which gives a deformation of the initial n-cube E by split cubes. Then, if "ch" is the Chern form given by the Weil formulae, $\mathrm{ch}(\mathrm{tr}_n(E))$ is a differential form on $\mathcal{D}_{\log}^{2p-n}(X \times (\mathbb{P}^1)^n, p)$. If X is compact, one can integrate this form along $(\mathbb{P}^1)^n$ against suitable differential forms T_n obtaining a differential form on X.

Let $\widehat{S}.(X)$ be the Waldhausen simplicial set for algebraic K-theory of the category of hermitian vector bundles on X and $\mathcal{K}.(\cdot)$ the Dold-Puppe functor from chain complexes to simplicial abelian groups. Then, the composition

$$\widehat{S}.(X) \xrightarrow{Hurewicz} \mathcal{K}.(\mathbb{Z}\widehat{S}_*(X)) \xrightarrow{\mathcal{K}(\mathrm{Cub})} \mathcal{K}.(\widetilde{\mathbb{Z}}\widehat{C}_*(X)) \xrightarrow{\mathrm{ch}} \mathcal{K}.\Big(\bigoplus_{p\geq 0} \mathcal{D}^{2p-*}(X,p)\Big)$$

is a simplicial representative of the Beilinson regulator.

Let $\widehat{\mathcal{D}}^*(X,p)$ be the bête truncation of the complex $\mathcal{D}^*(X,p)$ at degree greater than or equal to $2p$, and let

$$\widehat{\mathrm{ch}} : \widehat{S}.(X) \xrightarrow{\widehat{\mathrm{ch}}} \mathcal{K}.\Big(\bigoplus_{p\geq 0} \widehat{\mathcal{D}}^{2p-*}(X,p)\Big),$$

be the morphism induced by "ch". Then, following the ideas of Deligne and Soulé, one defines the *higher arithmetic K-groups* by

$$\widehat{K}_n(X) = \pi_{n+1}(\text{Homotopy fiber of } |\mathcal{K}(\widehat{\mathrm{ch}})|).$$

In this way, the desired long exact sequence extending (2) is obtained.

Observe that this definition of higher arithmetic K-groups treats the degree zero case in a different way from the rest. That is, the role of the differential forms in the non-zero degree groups is played only by those differential forms in the kernel of the differential $d_{\mathcal{D}}$, whereas no restriction is imposed in the degree zero group.

In order to avoid this difference, Takeda, in [57], has given an alternative definition of the *higher arithmetic K-groups* of X, by means of homotopy groups modified by the representative of the Beilinson regulator "ch". We denote these higher arithmetic K-groups by $\widehat{K}_n^T(X)$. The main characteristic of these groups is that instead of extending the exact sequence (2) to a long exact sequence, for every n there is an exact sequence

$$K_{n+1}(X) \xrightarrow{\rho} \bigoplus_{p\geq 0} \widetilde{\mathcal{D}}^{2p-n-1}(X,p) \xrightarrow{a} \widehat{K}_n^T(X) \xrightarrow{\zeta} K_n(X) \to 0,$$

analogous to the exact sequence for $\widehat{K}_0(X)$.

The two definitions do not agree, but, as proved by Takeda in [57], they are related by the characteristic class "ch":

$$\widehat{K}_n(X) \cong_{can} \ker(\mathrm{ch} : \widehat{K}_n^T(X) \to \widehat{\mathcal{D}}^{2p-n}(X,p)).$$

Overview of the results

The results of this thesis contribute to the program of developing a higher arithmetic intersection theory. These results constitute chapters 3 and 5. Chapters 2 and 4 consist of the preliminary results needed for chapters 3 and 5, in the area of homotopy theory of simplicial sheaves and algebraic K-theory.

In chapter 3, we develop a higher intersection theory on arithmetic varieties, à la Bloch. That is, we modify the higher Chow groups defined by Bloch by an explicit construction of the Beilinson regulator in terms of algebraic cycles.

We construct a representative of the Beilinson regulator using the Deligne complex of differential forms instead of the Deligne complex of currents. The regulator that we obtain turns out to be a minor modification of the regulator described by Bloch in [8].

Next, we develop a theory of higher arithmetic Chow groups, $\widehat{CH}^p(X,n)$, for any arithmetic variety X over a field. These groups are the homology groups of the simple of a diagram of complexes which represents the Beilinson regulator. We prove that there is a commutative and associative product structure on $\widehat{CH}^*(X,*) = \bigoplus_{p,n} \widehat{CH}^p(X,n)$, compatible with the algebraic intersection product. Therefore, we provide an arithmetic intersection product for arithmetic varieties over a field.

The advantages of our definition over Goncharov's definition are the following: the construction is valid for quasi-projective arithmetic varieties over a field, and not only over projective varieties; we can prove that our regulator is the Beilinson regulator; the groups we obtain are contravariant with respect to arbitrary maps; we can endow them with a product structure. All these improvements are mainly due to the fact that we avoid using the complex of currents.

The higher algebraic Chow groups defined by Bloch give a simple description of the motivic cohomology groups for smooth algebraic varieties over a field. One should view the higher arithmetic Chow groups as a simple description of a yet to be defined arithmetic motivic cohomology theory, valid for arithmetic varieties over a field.

We next focused on the relation between the higher arithmetic Chow groups and higher arithmetic K-theory. In order to follow the algebraic ideas, we should have a decomposition of the groups $\widehat{K}_n(X)_{\mathbb{Q}}$ given by eigenspaces of Adams operations $\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}}$. By the nature of the definition of $\widehat{K}_n(X)$, either by considering the homotopy fiber, or the modified homotopy groups of Takeda, it is apparently necessary to have a description of the Adams operations in algebraic K-theory in terms of a chain morphism, compatible with the representative of the Beilinson regulator "ch".

In chapter 4, we obtain a chain morphism inducing Adams operations on higher algebraic K-theory over the field of rational numbers. This definition is of combinatory nature. This chain morphism is designed to commute with the Beilinson regulator "ch" given by Burgos and Wang. Hence, one can appreciate that it has been strongly inspired by the definition of the Beilinson regulator and follows the same logical pattern.

In chapter 5 it is shown that this chain morphism indeed commutes with the representative of the Beilinson regulator "ch" and we use this fact to define Adams operations on the rational higher arithmetic K-groups.

Further studies in this direction will focus on determining if the Adams operations induce a graded decomposition $\widehat{K}_n(X)_{\mathbb{Q}} = \bigoplus_{p\geq 0} \widehat{K}_n(X)_{\mathbb{Q}}^{(p)}$ such that there is an isomorphism $\widehat{CH}^p(X,n)_{\mathbb{Q}} \cong \widehat{K}_n(X)_{\mathbb{Q}}^{(p)}$, as is the case in the algebraic setting. Notice that the arithmetic analogues of the algebraic theories discussed here rely on an explicit description of a certain morphism in the algebraic context. This is the case for the Beilinson regulator, in order to define higher arithmetic K-groups or Chow groups, and for the Adams operations, in order to define Adams operations on the higher arithmetic K-groups. In our view, the main difficulty to prove that there is an isomorphism

$$\widehat{CH}^p(X,n)_{\mathbb{Q}} \cong \widehat{K}_n(X)_{\mathbb{Q}}^{(p)}$$

is that, for the moment, there is no explicit representative of the algebraic analogue.

The development of this study required tools to compare morphisms from algebraic K-groups to a suitable cohomology theory or to the K-groups themselves. Indeed, we construct a chain morphism that is proved to induce the Beilinson regulator, and we construct a chain morphism that is proved to induce the Adams operations on algebraic K-theory. In chapter 2, we study these comparisons at a general level, providing theorems giving sufficient conditions for two morphisms to agree. The theory underlying the proofs is the homotopy theory of simplicial sheaves.

These theorems provide an alternative proof that the regulator defined by Burgos and Wang in [15] induces the Beilinson regulator. Moreover, we prove that the Adams operations defined by Grayson in [31] agree for any regular noetherian scheme of finite Krull dimension with the Adams operations defined by Gillet and Soulé in [28]. In particular, this implies that the Adams operations defined by Grayson satisfy the usual identities of a λ-ring, a fact that was left unproved in Grayson's work.

Results

We now explain the structure of the work and detail the main results.

Chapter 1 is of a preliminary nature. We briefly give the background needed for the understanding of the central work of the thesis. It also has the purpose of fixing the notation and definitions that will be used frequently in the forthcoming chapters. In the first section we discuss simplicial model categories, focusing on the category of simplicial sets and on the cubical abelian groups. In the second section we fix the notation on multi-indices, and discuss general facts on (co)chain complexes. We also discuss the relationship between simplicial or cubical abelian groups and chain complexes. In the third section we give the definition of algebraic K-theory in terms of the Quillen Q-construction and the Waldhausen construction. We also introduce the chain complex of cubes, which computes algebraic K-theory with rational coefficients and plays a central role in the definition of the Adams operations. Finally, in the last section of this chapter, we recall the definition of Deligne-Beilinson cohomology and state the main properties used in the study.

In **Chapter 2** we give theorems for the comparison of characteristic classes in algebraic K-theory. For a class of maps, named *weakly additive*, we give a criterion to decide whether two of them agree. All group morphisms induced by a map of simplicial sheaves are in this class, but these are not the only ones.

As mentioned already, in [15], Burgos and Wang defined a variant of the Chern character morphism from higher K-theory to real absolute Hodge cohomology,

$$\mathrm{ch} : K_n(X) \to \bigoplus_{p\geq 0} H^{2p-n}_{\mathcal{H}}(X, \mathbb{R}(p)),$$

for every smooth complex variety X. They proved that this morphism agrees with the already defined Beilinson regulator map. The proof relies only on the properties satisfied by the morphisms and by real absolute Hodge cohomology, and not on their definition. Hence, it is reasonable to think that there may be an axiomatic theorem for characteristic classes on higher K-theory. The proof of Burgos and Wang makes use of the bisimplicial scheme $B.P.$, introduced by Schechtman in [51]. This implies that a delooping in K-theory is necessary and hence, the method only applies to maps inducing group morphisms.

We use the techniques on the generalized cohomology theory described by Gillet and Soulé in [28]. Roughly speaking, the idea is that any good enough map from K-theory to K-theory or to a cohomology theory is characterized by its behavior over the K-groups of the simplicial scheme $B.GL_N$.

We give several characterization theorems. As a main consequence of these, we give a characterization of the *Adams and lambda operations* on higher K-theory and of the *Chern character* and *Chern classes* on a suitable cohomology theory.

More explicitly, let $\mathbf{C}$ be the big Zariski site over a noetherian finite dimensional scheme S. Denote by $B.GL_{N/S}$ the simplicial scheme $B.GL_N \times_{\mathbb{Z}} S$ and let $Gr(N, k)$ be the Grassmanian scheme over S. Let $S.\mathcal{P}$ be the Waldhausen simplicial sheaf computing algebraic K-theory and let $\mathbb{F}.$ be a simplicial sheaf. Note that $S.\mathcal{P}$ is an H-space. Let Ψ^k_{GS} be the Adams operations on higher algebraic K-theory defined by Gillet and Soulé in [28]. The two main consequences of our uniqueness theorem are the following.

Theorem 1 (Corollary 2.4.4)**.** *Let $\rho : S.\mathcal{P} \to S.\mathcal{P}$ be an H-space map in the homotopy category of simplicial sheaves on $\mathbf{C}$. If for some $k \geq 1$ there is a commutative square*

$$\begin{array}{ccc} K_0(\mathrm{Vect}(B.GL_{N/S})) & \xrightarrow{\psi} & K_0(B.GL_{N/S}) \\ \downarrow{\scriptstyle \Psi^k} & & \downarrow{\scriptstyle \rho} \\ K_0(\mathrm{Vect}(B.GL_{N/S})) & \xrightarrow[\psi]{} & K_0(B.GL_{N/S}), \end{array}$$

then ρ agrees with the Adams operation Ψ^k_{GS}, for all schemes X over S.

Theorem 2 (Theorem 2.5.5)**.** *Let $\mathcal{F}^*$ be a cochain complex of sheaves of abelian groups in $\mathbf{C}$. Let*

$$S.\mathcal{P} \longrightarrow \prod_{j\in\mathbb{Z}} \mathcal{K}.(\mathcal{F}(j)[2j])$$

be an H-space map in the homotopy category of simplicial sheaves on $\mathbf{C}$. The induced morphisms

$$K_m(X) \to \prod_{j\in\mathbb{Z}} H^{2j-m}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j))$$

agree with the Chern character defined by Gillet in [21] for every scheme X, if and only if, the induced map

$$K_0(X) \to \prod_{j\in\mathbb{Z}} H^{2j}_{\mathrm{ZAR}}(X, \mathcal{F}^*(j))$$

is the Chern character for $X = Gr(N, k)$, for all N, k.

In particular:

- We prove that the Adams operations defined by Grayson in [31] agree with the ones defined by Gillet and Soulé in [28], for all noetherian schemes of finite Krull dimension. This implies that for this class of schemes, the operations defined by Grayson satisfy the usual identities of a λ-ring.

- We prove that the Adams operations defined in Chapter 4 agree with the ones defined by Gillet and Soulé in [28], for all noetherian schemes of finite Krull dimension.

- We give an alternative proof that the morphism defined by Burgos and Wang in [15] agrees with the Beilinson regulator.

Chapter 3 is devoted to the development of the theory of higher arithmetic Chow groups for arithmetic varieties. Since the theory of higher algebraic Chow groups given by Bloch, $CH^p(X,n)$, is only fully established for schemes over field, we have to restrict ourselves to arithmetic varieties over a field.

Let X be a complex algebraic manifold and let $H^*_{\mathcal{D}}(X,\mathbb{R}(p))$ denote the Deligne-Beilinson cohomology groups with real coefficients. For every $p \geq 0$, we define two cochain complexes, $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ and $\mathcal{D}^*_{\mathbb{A}}(X,p)_0$, constructed out of differential forms on $X \times (\mathbb{A}^1)^n$ with logarithmic singularities along infinity. The following isomorphisms are satisfied:

$$H^{2p-n}(\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0) \cong CH^p(X,n)_{\mathbb{R}},$$

and

$$H^r(\mathcal{D}^*_{\mathbb{A}}(X,p)_0) \cong H^r_{\mathcal{D}}(X,\mathbb{R}(p)), \qquad \text{for } r \leq 2p.$$

We show that the complex $\mathcal{D}^*_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ enjoys the same properties as the complex $Z^p(X,n)_0$ defined by Bloch in [7]. We actually use its cubical analog, defined by Levine in [41], due to its suitability for describing the product structure on $CH^*(X,*)$. The subindex 0 means the normalized chain complex associated to a cubical abelian group.

Moreover, there is a natural chain morphism

$$\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0$$

which induces, after composition with the isomorphism

$$K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}}$$

described by Bloch in [7], the *Beilinson regulator* (**Theorem 3.4.5**):

$$K_n(X)_{\mathbb{Q}} \cong \bigoplus_{p\geq 0} CH^p(X,n)_{\mathbb{Q}} \xrightarrow{\rho} \bigoplus_{p\geq 0} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

An analogous construction using projective lines instead of affine lines can be developed. We define a chain complex, $\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p)$, analogous to the complex $\mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0$ and a chain complex $\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P}}(X,p)$, analogous to the complex $\mathcal{D}^{2p-*}_{\mathbb{A}}(X,p)_0$. We also define a chain morphism

$$\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \xrightarrow{\rho} \mathcal{D}^{2p-*}_{\mathbb{P}}(X,p).$$

In this case, if X is proper, following the methods of Burgos and Wang in [15], section 6, integration along projective lines induces a chain morphism

$$\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P}}(X,p) \to \mathcal{D}^{2p-*}(X,p).$$

This gives a chain morphism

$$\widetilde{\mathcal{D}}^{2p-*}_{\mathbb{P},\mathcal{Z}^p}(X,p) \xrightarrow{\rho} \mathcal{D}^{2p-*}(X,p)$$

representing the Beilinson regulator. Observe that, when X is proper, this representative has the advantage of having as target precisely the Deligne complex of differential forms on X, and not a chain complex involving differential forms on $X \times (\mathbb{A}^1)^n$. This is needed in order to develop a theory of higher arithmetic Chow groups analogous to the higher arithmetic K-theory developed by Takeda in [57].

In the second part of this chapter we use the morphism ρ to define the higher arithmetic Chow group $\widehat{CH}^p(X,n)$, for any arithmetic variety X over a field. The formalism underlying our definition is the theory of diagrams of complexes and their associated simple complexes, developed by Beilinson in [5]. That is, one considers the diagram of chain complexes

$$\widehat{\mathcal{Z}}^p(X,*)_0 = \left(\begin{array}{ccccc} & H^{2p}_{\mathcal{D}}(X\times\mathbb{A}^*,\mathbb{R}(p))_0 & & \widehat{\mathcal{D}}^{2p-*}_{\mathbb{A}}(X,p)_0 & \\ \overset{f_1}{\nearrow} & & \underset{\sim}{\overset{g_1}{\nwarrow}} & & \overset{\rho}{\nearrow} \\ Z^p(X,*)_0 & & & \mathcal{D}^{2p-*}_{\mathbb{A},\mathcal{Z}^p}(X,p)_0 & \end{array} \right).$$

Then, the higher arithmetic Chow groups of X are given by the homology groups of the simple of the diagram $\widehat{\mathcal{Z}}^p(X,*)_0$:

$$\widehat{CH}^p(X,n) := H_n(s(\widehat{\mathcal{Z}}^p(X,*)_0)).$$

The following properties are shown:

- **Theorem 3.6.11**: Let $\widehat{CH}^p(X)$ denote the arithmetic Chow group defined by Burgos. Then, there is a natural isomorphism

$$\widehat{CH}^p(X) \xrightarrow{\cong} \widehat{CH}^p(X,0).$$

- **Proposition 3.6.7**: There is a long exact sequence

$$\cdots \to \widehat{CH}^p(X,n) \xrightarrow{\zeta} CH^p(X,n) \xrightarrow{\rho} H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X,n-1) \to \cdots$$

$$\cdots \to CH^p(X,1) \xrightarrow{\rho} \mathcal{D}^{2p-1}_{\log}(X,p)/\operatorname{im} d_{\mathcal{D}} \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0.$$

- **Proposition 3.6.15** (*Pull-back*): Let $f : X \to Y$ be a morphism between two arithmetic varieties over a field. Then, there is a pull-back morphism

$$\widehat{CH}^p(Y,n) \xrightarrow{f^*} \widehat{CH}^p(X,n),$$

for every p and n, compatible with the pull-back maps on the groups $CH^p(X,n)$ and $H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$.

- **Corollary 3.6.19** (*Homotopy invariance*): Let $\pi : X \times \mathbb{A}^m \to X$ be the projection on X. Then, the pull-back map

$$\pi^* : \widehat{CH}^p(X,n) \to \widehat{CH}^p(X\times\mathbb{A}^m,n), \quad n \geq 1$$

is an isomorphism.

▶ **Theorem 3.9.7** (*Product*): There exists a product on

$$\widehat{CH}^*(X,*) := \bigoplus_{p\geq 0, n\geq 0} \widehat{CH}^p(X,n),$$

which is associative, graded commutative with respect to the degree n and commutative with respect to the degree p.

Finally, we briefly discuss an alternative approach for the definition of higher arithmetic Chow groups, which follows the ideas of Takeda in [57], for the definition of the higher arithmetic K-groups of a proper arithmetic variety. To this end, we use the definition of the regulator by means of projective lines, restricting ourselves to proper arithmetic varieties over a field.

The following two questions remain open:

▷ Do the groups constructed here agree with the definition of higher arithmetic Chow groups of Goncharov?

▷ Can we extend the definition to arithmetic varieties over an arithmetic ring?

In **Chapter 4**, we construct a representative of the Adams operations on higher algebraic K-theory. Let X be any scheme and let $\mathcal{P}(X)$ be the exact category of locally free sheaves of finite rank on X. The algebraic K-groups of X, $K_n(X)$, are defined as the Quillen K-groups of the category $\mathcal{P}(X)$.

These groups can be equipped with a λ-ring structure. Then, the *Adams operations* on each $K_n(X)$ are obtained from the λ-operations by a polynomial formula on the λ-operations. In the literature there are several definitions of the Adams operations on the higher K-groups of a scheme X. By means of the homotopy theory of simplicial sheaves (as recalled in chapter 2), Gillet and Soulé defined Adams operations for any noetherian scheme of finite Krull dimension. Grayson, in [31], constructed a simplicial map inducing Adams operations on the K-groups of any category endowed with a suitable tensor product, symmetric power and exterior power. In particular, he constructed Adams operations for the algebraic K-groups of any scheme X. Following the methods of Schechtman in [51], Lecomte, in [40], defined Adams operations for the rational K-theory of any scheme X equipped with an ample family of invertible sheaves. They are induced by map in the homotopy category of infinite loop spectra.

Our aim is to construct an explicit chain morphism which induces the Adams operations on rational algebraic K-theory. It is our hope that this construction will improve our understanding of the eigenvalue spaces for the Adams operations.

Consider the chain complex of cubes associated to the category $\mathcal{P}(X)$. McCarthy in [47], showed that the homology groups of this complex, with rational coefficients, are isomorphic to the rational algebraic K-groups of X (see section 1.3.3).

We first attempted to find a homological version of Grayson's simplicial construction, but this seems particularly difficult from the combinatorial point of view.

The current approach is based on a simplification obtained by using the *transgressions of cubes by affine or projective lines*, at the price of having to reduce to regular noetherian schemes. This was Burgos and Wang's idea in [15], in order to define a chain morphism representing Beilinson's regulator.

In order to commute with the representative of the Beilinson regulator "ch", the desired morphism should be of the form

$$E \mapsto \Psi^k(\mathrm{tr}_n(E)),$$

with Ψ^k a description of the k-th Adams operation at the level of vector bundles. Unfortunately, for the known choices of Ψ^k, this map does not define a chain morphism. The key obstruction is that while, for any two hermitian vector bundles $\overline{E}, \overline{F}$, we have the equality

$$\mathrm{ch}(\overline{E} \oplus \overline{F}) = \mathrm{ch}(\overline{E}) + \mathrm{ch}(\overline{F}),$$

it is not true that for any two vector bundles E, F, we have the equality

$$\Psi^k(E \oplus F) = \Psi^k(E) \oplus \Psi^k(F).$$

It is true, however, at the level of $K_0(X)$.

The root of the problem is that the map

$$E \mapsto \mathrm{tr}_n(E)$$

is not a chain morphism. However, adding to this map a collection of cubes which have the property of being split in all directions, we obtain a chain morphism. The fact that the added cubes are split in all directions implies that they are cancelled after applying "ch". Therefore, we will still have commutativity of Ψ^k with "ch".

With this strategy, we first assign to a cube on X a collection of cubes defined either on $X \times (\mathbb{P}^1)^*$ or on $X \times (\mathbb{A}^1)^*$, which have the property of being split in all directions (**Proposition 4.3.17**). These cubes are called *split cubes.* This gives a morphism which we call the transgression morphism.

Then, by a purely combinatorial formula on the Adams operations of locally free sheaves, we give a formula for the Adams operations on split cubes (**Corollary 4.2.39**). The key point is to use Grayson's idea of considering the secondary Euler characteristic class of the Koszul complex associated to a locally free sheaf of finite rank.

The composition of the transgression morphism with the Adams operations for split cubes gives a chain morphism representing the Adams operations for any regular noetherian scheme of finite Krull dimension (**Theorem 4.4.2**).

The two constructions, with projective lines or with affine lines, are completely analogous. One may choose the more suitable one in each particular case. For instance, to define Adams operations on the K-groups of a regular ring R, one may consider the definition with affine lines so as to remain in the category of affine schemes. On the other hand, if for instance our category of schemes is the category of projective regular schemes, then the construction with projective lines may be the appropriate one.

The main application of our construction of Adams operations is the definition of a (pre)-λ-ring structure on the rational arithmetic K-groups of an arithmetic variety X.

In **Chapter 5**, we give a pre-λ-ring structure to both definitions of higher arithmetic K-groups tensored by the rational numbers $\mathbb{Q}$, $\widehat{K}_n(X)_{\mathbb{Q}}$ and $\widehat{K}_n^T(X)_{\mathbb{Q}}$. It is compatible with the λ-ring structure on the algebraic K-groups, $K_n(X)$, defined by Gillet and Soulé in [28], and with the canonical λ-ring structure on $\bigoplus_{p\geq 0}\widetilde{\mathcal{D}}^{2p-*}(X,p)$, given by the graduation by p (see lemma 1.3.28). Moreover, for $n=0$ we recover the λ-ring structure of $\widehat{K}_0(X)\otimes\mathbb{Q}$.

More concretely, we construct *Adams operations*

$$\Psi^k : \widehat{K}_n(X)_{\mathbb{Q}} \to \widehat{K}_n(X)_{\mathbb{Q}}, \qquad k\geq 0,$$

which, since we have tensored by $\mathbb{Q}$, induce λ-operations on $\widehat{K}_n(X)_{\mathbb{Q}}$.

In order to deal with $\widehat{K}_n^T(X)_{\mathbb{Q}}$, we introduce the *modified homology groups*, which are the analogue in homology of the modified homotopy groups. Then, the homology groups modified by "ch" give a homological description of $\widehat{K}_n(X)_{\mathbb{Q}}$ (**Theorem 5.3.11**).

In this chapter we show that the construction of Adams operations of chapter 4 commutes strictly with "ch" (**Theorem 5.4.11**), and we deduce the pre-λ-ring structure for $\widehat{K}_n(X)_{\mathbb{Q}}$ and $\widehat{K}_n^T(X)_{\mathbb{Q}}$ (**Corollary 5.4.14 and Corollary 5.4.16**).

For the time being, we have not been able to prove that it is a λ-ring.

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