AST & FORCING: MANDATORY HOMEWORK NO. 2

This is the 2nd mandatory homework assignment for the course Axiomatic Set Theory and Forcing. It is due on March 7, 2014, at the beginning of lecture.

February 27: Corrections to the hint in Ex. 2.(2), and to the formulation of the bonus problem in Ex. 3.

March 4: The unnecessary h' in the hint for Ex. 2.(2) has now been removed.

The purpose of this assignment is to introduce *Martin's Axiom*:

Definition 1. Let $\mathbb{P}$ be a forcing poset.

(0) An *antichain* in $\mathbb{P}$ is a non-$\emptyset$ set $\mathcal{A} \subseteq \mathbb{P}$ such that for all $p, q \in \mathcal{A}$, if $p \neq q$ then $p \perp q$.

(1) We say that $\mathbb{P}$ satisfies (or *has*) the *countable chain condition*[1] (ccc) if all antichains in $\mathbb{P}$ are countable. (The phrase "$\mathbb{P}$ is a ccc poset" is very common, too.)

(2) Let κ be an infinite cardinal. MA(κ) is the statement: For all posets $\mathbb{P}$ with the ccc, and all families $\mathcal{D}$ of dense subsets of $\mathbb{P}$ such that $|\mathcal{D}| \leq \kappa$, there is a filter G on $\mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

(3) MA is the statement: For all $\kappa < 2^{\aleph_0}$, MA(κ) holds.

Note that any *countable* poset has the ccc. In particular, the Cohen poset has the ccc.

MA stands for Martin's Axiom, and was introduced by Donald A. Martin. It is not an axiom like the other axioms of set theory, because it was never meant as a natural statement that is "obviously" true in any way. Rather, it is what is called a *forcing axiom*, an axiom that conveniently packages an array of forcing methods, and makes it possible to prove many statements that can be shown to be consistent, using forcing. MA itself can be shown to be relatively consistent with ZFC, though this is rather involved.

Exercise 1 (6 points)**.** (1) Prove from ZFC that MA($\aleph_0$) holds. Where, if at all, did you need to use that the posets involved have the ccc?

(2) Show that MA($2^{\aleph_0}$) is false. (Hint: Use the Cohen poset).

(3) Show that if we drop the requirement that posets be ccc, then MA($\aleph_1$) is false (provably in ZFC.) You're not allowed to invoke the Continuum Hypothesis in your solution of this problem!

Suggestion for (3): Define a poset consisting of finite functions p with $\mathrm{dom}(p) \subseteq \omega$ and $\mathrm{ran}(p) \subseteq \aleph_1$, and order them by $p \leq q$ iff $p \supseteq q$, where we identify p, q with their graphs.

Exercise 2 (6 points)**.** Let ${}^{\omega}\omega$ be the set of all (total) functions $\omega \to \omega$. We say that $F \subseteq {}^{\omega}\omega$ is a *dominating family* if for all $g : \omega \to \omega$ there is $f \in F$ such that $\{n \in \omega : f(n) \leq g(n)\}$ is finite.

(1) Prove that no countable family of functions is dominating.

[1] Arguably, it should be called the countable antichain condition, but the terminology stuck many years ago.

(2) Assume MA(κ) for some $\kappa < 2^{\aleph_0}$. Prove that there are no dominating families of size $\leq \kappa$. (Hint: Define a poset where conditions are pairs $\langle s, f\rangle$, where $s \in {}^{<\omega}\omega$ and $f \in {}^{\omega}\omega$, and $f \supseteq s$. Order them as follows: $\langle s, f\rangle \leq \langle t, g\rangle$ iff $s \supseteq t$ and $f(n) \geq g(n)$ for all $n > \mathrm{dom}(t)$. Fix a family $F \subseteq {}^{\omega}\omega$ of size $\leq \kappa$, and use the s part of conditions in an appropriately chosen filter to define a total function $h : \omega \to \omega$ which dominates F. If you use the hint, remember to prove that the above defines a a forcing poset, and that it has the ccc.)

Exercise 3 (3 points+Bonus)**.** Give the set ${}^{\omega}\omega$ the product topology, that is, the topology generated by the basic open sets

$$N_s = \{f : \omega \to \omega : f \supseteq s\},$$

where s ranges over the set ${}^{<\omega}\omega$. Equivalently, it is the topology induced by the metric

$$d(f, g) = 2^{-\min\{n\in\omega: f(n)\neq g(n)\}}$$

when $f \neq g$, and $d(f, g) = 0$ if $f = g$.

(1) Assume MA(κ) for some $\omega \leq \kappa < 2^{\aleph_0}$. Prove that if $\{U_\alpha : \alpha \in \kappa\}$ is a family of dense open sets in ${}^{\omega}\omega$ (in the topology defined above), then $\bigcap_{\alpha\in\kappa} U_\alpha$ is dense in ${}^{\omega}\omega$.

Hint: Consider a poset where $\mathbb{P} = {}^{<\omega}\omega$, and $p \leq q$ iff $p \supseteq q$. Find κ appropriate dense sets in $\mathbb{P}$ that guarantee that a filter that meets them all must define a total function $\omega \to \omega$ in the intersection of the U_α.

(2) **Bonus exercise for extra credit**: Not assigned, and not required. Worth an additional 3 points. Assume MA(κ) for κ as above. Prove that if (X, d) is a **separable** complete metric space, and $\{U_\alpha : \alpha < \kappa\}$ is a family of dense open sets in X, then $\bigcap_{\alpha<\kappa} U_\alpha$ is dense in X.

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